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Search for Eulerian Recurrent Lengths by Using Constraint Solvers

Shuji JIMBO
Graduate School of Natural Science and Technology, Okayama University

jimbo-s@okayama-u.ac.jp

Abstract

The Eulerian recurrent length of a graph G, $e(G)$, is the maximum of the shortest subcycle length of Eulerian circuits of G. Upper and lower bounds on the Eulerian recurrent length of complete graphs was provided by the author as $n-4 \leqq e(K_n) \leqq n-3$ for odd integers $n \geqq 15$. In this article, the method of proving the inequality is improved.

KEYWORDS. graph theory, Eulerian circuits, computer experiments, integer programming, Eulerian recurrent length.

1 Introduction

Let $C = v_0 \to v_1 \to v_2 \to \cdots \to v_{m-1} \to v_0$ be a circuit of a graph, where m is the length of C. For any integer k, we regard v_k as the vertex $v_{k \bmod m}$ in C for convenience of discussion, where $k \bmod m$ is the minimum nonnegative integer of the form $k - qm$ with integer q. We call a subwalk $W = v_i \to v_{i+1} \to \cdots \to v_j$ in C a subcycle if $v_i = v_j$ and W is a cycle of K_n. The shortest subcyle length of C is denoted by $s(C)$. Let G be a Eulerian graph. We call $e(G) = \max\{s(C) \mid C$ is an Eulerian circuit of $G\}$ the Eulerian recurrent length (ERL) of G. Let K_n denote the complete graph with n vertices, where n is an odd positive integer with $n \geqq 3$. Then, K_n is a Eulerian graph. We abbreviate $e(K_n)$ as $e(n)$ in this paper.

We have proved the following fact. Let k be an arbitrary integer greater than 330. Let G denote a given four-regular Eulerian graph. Then, the problem to determine whether $e(G) \geqq k$ or not is NP-complete[1]. We also determined the ERL of complete bipartite graphs as follows. Let $K_{m,n}$ denote the complete bipartite graph of vertex classes with m and n vertices. Let m and n be even integers with $0 < n < m$. Then, $e(K_{n,n}) = 2n-4$ and $e(K_{m,n}) = 2n$ hold[4][2]. We also determine the ERL of complete graphs K_n, namely $e(n)$, for particular small integers n by computer verification. Equation $e(3) = e(5) = 3$ holds, and, for any $n \in \{7, 9, 11, 13\}$, equation $e(n) = n-3$ holds. For complete graphs K_n with odd integer $n \geqq 15$, we have had the following upper and lower bounds on $e(n)$:

$$n - 4 \leqq e(K_n) \leqq n - 3.$$

The left inequality above can be obtained by showing a construction method of a Eulerian circuit C of K_n with a shortest subcycle of length $n-4$. The construction method is based on a decomposition of $E(K_n)$, the edge set of K_n, into Hamiltonian cycles. The Eulerian circuit C can be obtained by aligning the Hamiltonian cycles in appropriate order, and by joining them[2]. We will describe the method of proving the right inequality above by solving integer programming problems in the previous article[2]. Then, we will give an improvement of the proof by a modification to the integer programming problem.

2 Main arguments

For a circuit C of length m and an integer i, $C(i)$ denotes the vertex on position i mod m in C, and is called the i-th vertex on C. Hence, $C = C(0) \to C(1) \to \cdots \to C(m-1) \to C(0)$ holds, and any two edges $C(i)C(i+1)$ and $C(j)C(j+1)$ are different if $i < j < i+m$. For any integer i, $N_C(i)$ denotes the unique integer k such that

$$i < k \leqq i+m, \qquad C(k) = C(i), \quad \text{and} \quad C(j) \neq C(i) \quad \text{for any } j \text{ with } i < j < k.$$

Hence, for any integer i, $N_C^{-1}(i)$ denotes the unique integer k such that $N_C(k) = i$.

Suppose that C is an Eulerian circuit of a complete graph K_n. An edge $e = C(i)C(i+1)$ is negative if either

$$N_C^{-1}(i+1) < N_C^{-1}(i) \quad \text{and} \quad N^{(}_{C}i) < N_C(i+1)$$

or

$$N_C^{-1}(i) < N_C^{-1}(i+1) \quad \text{and} \quad N^{(}_{C}i+1) < N_C(i)$$

holds, and positive otherwise. A quadruple (i, j, k, l) of integers is a (non-contact) position reversal, or PR, on C, if $i < j < k = N_C(j) < l = N_C(i)$ holds. The following proposition readily follows from the definitions above.

Proposition 1 *For any Eulerian circuit C of K_n with $n \geqq 7$, the number of PR's on C is not less than that of negative edges on C, where two PR's (i, j, k, l) and (i', j', k', l') are regarded as identical if $i \equiv i' \pmod{m}$, $j \equiv j' \pmod{m}$, $k \equiv k' \pmod{m}$, and $l \equiv l' \pmod{m}$ hold.*

For a position reversal $r = (i, j, k, l)$ on C, i and l are called the position reversal head and the position reversal tail of r, respectively.

Let $n \geqq 7$ be an odd integer. Let m denote $n(n-1)/2$, the number of edges of K_n. We assume that there exists an Eulerian circuit C with $s(C) = n-2$ of K_n to prove that $e(n) \leqq n-3$ by contradiction. We have the following two lemmas by tedious arguments[3].

Lemma 1 *For any integer i,*

$$n - 2 \leqq N_C(i) - i \leqq n + 3$$

holds.

Lemma 2 *For any integer i, if i is a position reversal head on C, then the following equations hold:*

$$\begin{aligned} N_C(i) - i &= n + 3, \\ N_C(i+1) - (i+1) &= n - 2, \quad \textit{and} \\ N_C(i+2) - (i+2) &= n - 1. \end{aligned}$$

Furthermore, for any integer i, if i is a position reversal tail on C, then the following equations hold:

$$\begin{aligned} i - N_C^{-1}(i) &= n+3, \\ (i-1) - N_C^{-1}(i-1) &= n-2, \quad \text{and} \\ (i-2) - N_C^{-1}(i-2) &= n-1. \end{aligned}$$

For an integer p and a positive integer μ, $S_\mu(p)$ denotes the set $\{p, p+1, \ldots, p+\mu-1\}$. Then, $A_\mu^C(i)$, $B_\mu^C(i)$, and $C_\mu^C(i)$ denote the number of left positions of negative edges on C, position reversal heads on C, and position reversal tails on C in $S_\mu(i)$, respectively. The following lemma follows from Proposition 1 immediately.

Lemma 3 *Let n be an integer greater than or equal to 7. Let C be an Eulerian circuit of K_n. Let m denote $n(n-1)/2$, the length of C. Let μ be a positive integer less than m. Then, the following two inequalities hold:*

$$\sum_{i=0}^{m-1} \left(B_\mu^C(i) - A_\mu^C(i)\right) \geqq 0, \quad \text{and} \tag{1}$$

$$\sum_{i=0}^{m-1} \left(B_\mu^C(i) + C_\mu^C(i) - 2A_\mu^C(i)\right) \geqq 0. \tag{2}$$

Corollary 1 *Let n be an integer greater than or equal to 7. Let C be an Eulerian circuit of K_n. Let m denote $n(n-1)/2$, the length of C. Let μ be a positive integer less than m. Then, there is an integer $p \in \{0, 1, 2, \ldots, m-1\}$ such that*

$$B_\mu^C(p) - A_\mu^C(p) \geqq 0. \tag{3}$$

There also is $q \in \{0, 1, 2, \ldots, m-1\}$ such that

$$B_\mu^C(q) + C_\mu^C(q) - 2A_\mu^C(q) \geqq 0. \tag{4}$$

Suppose that an integer p and a positive integer μ are given. Let $x(i) = (p+i-2) - N_C^{-1}(p+i-2)$ and $y(i) = N_C(p+i-2) - (p+i-2)$ for each $i \in \{0, 1, 2, \ldots, \mu+3\}$. Then, the following conditions must hold by definition:

(a) $\forall i \in \{0, 1, \ldots, \mu+3\}$, $\forall j \in \{0, 1, \ldots, \mu+3\}$, $i \neq j \Rightarrow (i - x(i) \neq j - x(j) \wedge i + y(i) \neq j + y(j))$,

(b) $\forall i \in \{0, 1, \ldots, \mu+2\}$, $|x(i+1) - x(i) - 1| \neq 1 \wedge |y(i) - y(i+1) - 1| \neq 1$,

(c) $\forall i \in \{0, 1, \ldots, \mu+3\}$, $\forall j \in \{0, 1, \ldots, \mu+3\}$, $i \neq j \Rightarrow (|(i - x(i)) - (j - x(j))| \neq 1 \vee |(i + y(i)) - (j + y(j))| \neq 1)$,

(d1) $\forall i \in \{0, 1, \ldots, \mu-2\}$, $\exists j \in \{0, 1, 2, 3, 4, 5\}$, $x(i+j) = j$, and

(d2) $\forall i \in \{0, 1, \ldots, \mu-2\}$, $\exists j \in \{0, 1, 2, 3, 4, 5\}$, $y(i+j) = 5 - j$.

Let X and Y denote $(x(0), x(1), \ldots, x(\mu+3))$ and $(y(0), y(1), \ldots, y(\mu+3))$, respectively. Let $M_\mu(X,Y)$ denote the number of left positions of negative edges in $\{p, p+1, \ldots, p+\mu-1\}$. Notice that, for each $i \in \{0,1,2,\ldots,\mu+3\}$, $p+i-2$ is a left position of a negative edge on C, if and only if the following condition holds:

$$(x(i) < x(i+1) \land y(i) < y(i+1)) \quad \lor \quad (x(i) > x(i+1) \land y(i) > y(i+1)).$$

Let $R_\mu(X,Y)$ denote the number of position reversal heads in $\{p, p+1, \ldots, p+\mu-1\}$. Notice that, for each $i \in \{0,1,2,\ldots,\mu+3\}$, $p+i-2$ is a position reversal head on C, if and only if the following condition holds:

$$y(i) = 5 \land y(i+1) = 0 \land y(i+2) = 1.$$

Let $R'_\mu(X,Y)$ denote the number of position reversal tails in $\{p, p+1, \ldots, p+\mu-1\}$. Notice that, for each $i \in \{0,1,2,\ldots,\mu+3\}$, $p+i-2$ is a position reversal tail on C, if and only if the following condition holds:

$$x(i) = 5 \land x(i-1) = 0 \land x(i-2) = 1.$$

The problem finding the maximum value of $R_\mu(X,Y) - M_\mu(X,Y)$ under the conditions (a), (b), (c), (d1) and (d2) can be formulated as an integer programming problem. By solving the problem, we have:

- There is no position i on C such that $R_7(X,Y) > M_7(X,Y)$.
- For any position i on C, $M_7(X,Y) \geqq 2$ holds.

Those conditions are too tight for C. Additional arguments therefore lead us to contradiction. Thus, we have proved that there is no Eulerian circuit C of K_n with $s(C) = n-2$ for any odd integer $n \geqq 15$[3]. However, a modification to the integer programming problem saves the additional arguments above as described in the next paragraph.

The problem finding the maximum value of $R_\mu(X,Y) + R'_\mu(X,y) - 2M_\mu(X,Y)$ under the conditions (a), (b), (c), (d1), and (d2) can be formulated as an integer programming problem. By solving the problem, we have:

- There is no position i on C s.t. $R_7(X,Y) + R'_7(X,Y) \geqq M_7(X,Y)$.

It directly follows from the statement above that there is no Eulerian circuit C of K_n with $s(C) = n-2$. Thus, the proof have been simplified. We consider that complexity of the integer programming problem is substituted for a part of arguments in the previous proof.

3 Concluding Remarks

We has described an improvement of the previous proof of the upper bound on the Eulerian recurrent length of complete graphs K_n for odd integers $n \geqq 7$. The improvement has been brought by a modification to the integer programming problem used for deriving a contradiction.

The gap between the upper and lower bound on the Eulerian recurrent length of K_n, $e(n)$, for odd $k \geqq 15$ is still remains as $n-4 \leqq e(n) \leqq n-3$. We conjecture that

$e(n) = n - 4$ holds for any odd integer $n \geqq 15$. Currently, we try to verify that $e(15) = 11$ by computer. However, we estimate the number of primitive checks on graphs required by computer experiments for the verification according to naive methods at about 3^{45} or more. The computer experiments, therefore, should result in failure in the current computation environment.

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