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Lattices of surjective weak weight preserving homomorphisms of digraphs

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abstract We introduced an extension of homomorphisms of general weighted directed graphs and investigated the semigroups of surjective homomorphims and synthesize graphs to obtain a generator of pricipal left (or right) ideal in the semigroup[11]. This study is originally motivated by reducing the redundancy in concurrent systems, for example, Petri nets. [10]. We have got the result that for a given graph our homomorphism G has freeness determined by the connection and the cycles in G.

In a general weighted directed graphs $(V_i, E_i, W_i)(i = 1, 2)$, a usual graph homomorphism $\phi : V_1 \to V_2$ satisfies $W_2(\phi(u), \phi(v)) = W_1(u, v)$ to preserve adjacency of the graphs. Whereas we extend this definition slightly and our homomorphism is defined by the pair (ϕ, ρ) based on the similarity of the edge connection. (ϕ, ρ) satisfies $W_2(\phi(u), \phi(v)) = \rho(u)\rho(v)W_1(u, v)$, where $\phi : V_1 \to V_2, \rho : V_1 \to \boldsymbol{R}_+$ and $\boldsymbol{R}_+$ is the set of positive real numbers.

In this paper we investigate whether for a w-homomorphism $(\phi,\ \rho)$ from a given digraph G, ρ is uniquely determined or not. As a result, it is uniquely determined if undirected graph $\bar{G}$ obtained from G has no even cycles and no isolated vertices. Additionally we overview the lattice structure of graphs, which are ordered by surjective w-homomorphisms.

1 Preliminaries

We introduced an extension of homomorphisms of general weighted directed graphs[11]. Here we overview the extension and give new examples of them with free parameters.

1.1 Graphs and Morphisms

In this section we summarize definitions of weighted digraphs, w-homomorphisms and compositions. We denote the set of positive real numbers by $\boldsymbol{R}_{>0}$ and the set of nonnegative real numbers by $\boldsymbol{R}_{\geq 0}$.

DEFINITION 1.1 A *weighted directed graph* (weighted digraph, for short) is a 3-tuple $(V,\ E,\ W)$ where
(1) V is a finite set of vertices,
(2) $E\,(\subset V \times V)$ is a set of edges,
(3) $W : (V \times V) \to \boldsymbol{R}_{\geq 0}$ is a *weight function.* □

According to custom, $(u, v) \in E \iff W(u, v) \neq 0$.

DEFINITION 1.2 Let $G_i = (V_i, E_i, W_i)$ $(i = 1, 2)$ be the weighted digraphs. Then a pair $(\phi,\ \rho)$ is called a *(weak weight preserving) homomorphism* (for short, *w-homomorphism*) from G_1 to G_2 if the maps $\phi : V_1 \to V_2$, $\rho : V_1 \to \boldsymbol{R}_{>0}$ satisfy the condition that for any $u, v \in V_1$,

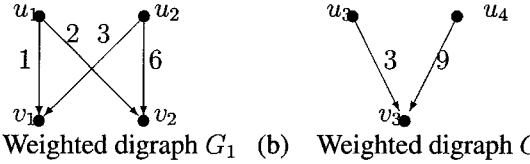
$$W_2(\phi(u), \phi(v)) = \rho(u)\rho(v)W_1(u, v). \tag{1.1}$$

Especially if $\rho = 1_{V_1}$, i.e., $\rho(u) = 1$ for any $u \in V_1$, then w-homomorphism is called a *strictly weight preserving homomorphism* (*s-homomorphism*, for short). □

The w-homomorphism $(\phi,\ \rho)$ is called *injective* (resp. *surjective*) if ϕ is injective (resp. surjective). In particular, it is called a *w-isomorphism* from G_1 to G_2 if it is injective and surjective. Then G_1 is said to be *w-isomorphic* to G_2 and we write $G_1 \simeq_{\mathrm{w}} G_2$. Moreover, in case of $G_1 = G_2 = G$, a w-isomorphism is called an *w-automorphism* of G. By $\mathbf{Aut}_{\mathrm{w}}(G)$ we denote the set of all the w-automorphisms of G. Similarily s-isomorphism $\simeq_{\mathrm{s}}$ s-automorphism and $\mathbf{Aut}_{\mathrm{s}}(G)$ are defined.

EXAMPLE 1.1 Let $G_i = (V_i, E_i, W_i)$ $(i = 1, 2)$ be the weighted digraphs depicted in Figure 1, $W_i : V_i \to \boldsymbol{R}_{>0}$ be the weight functions. That is,

$V_1 = \{u_1, u_2, v_1, v_2\}, V_2 = \{u_3, u_4, v_3\}.$
$W_1(u_1, v_1) = 1, W_1(u_1, v_2) = 2, W_1(u_2, v_1) = 3, W_1(u_2, v_2) = 6.$
$W_2(u_3, v_3) = 3, W_2(u_4, v_3) = 9.$ Any other edges are of weight 0.



(a) Weighted digraph G_1 (b) Weighted digraph G_2

Figure 1. Weighted Digraph G_1 and G_2.

Let ϕ be the following function from V_1 to V_2.

$$\phi = \begin{pmatrix} u_1 & u_2 & v_1 & v_2 \\ u_3 & u_4 & v_3 & v_3 \end{pmatrix},$$

Then the following equations hold.

$$\begin{aligned} 3 &= \rho(u_1)\rho(v_1) \times 1 \\ 3 &= \rho(u_1)\rho(v_2) \times 2 \\ 9 &= \rho(u_2)\rho(v_1) \times 3 \\ 9 &= \rho(u_2)\rho(v_2) \times 6 \end{aligned}$$

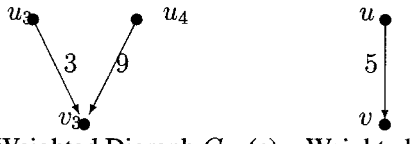
Solving the these equations, we have the solution (ϕ, ρ), a w-homomorphism from G_1 to G_2, with one parameter $r \in \boldsymbol{R}_{>0}$.

$$\rho = \begin{pmatrix} u_1 & u_2 & v_1 & v_2 \\ 3/(2r) & 3/(2r) & 2r & r \end{pmatrix}.$$

□

EXAMPLE 1.2 Let $G_i = (V_i, E_i, W_i)\,(i = 2, 3)$ be the weighted digraphs depicted in Figure 2, $W_i : V_i \to \boldsymbol{R}_{>0}$ the weight functions.That is,

$$V_2 = \{u_3, u_4, v_3\}, V_3 = \{u, v\}.$$
$$W_2(u_3, v_3) = 3, W_2(u_4, v_3) = 9.$$
$$W_3(u, v) = 5. \quad \text{Any other edges are of weight 0.}$$



(b) Weighted Digraph G_2 (c) Weighted Digraph G_3

Figure 2. Weighted digraphs G_2 and G_3.

Let ψ be the following function from V_2 to V_3.

$$\psi = \begin{pmatrix} u_3 & u_4 & v_3 \\ u & u & v \end{pmatrix},$$

Then the following equations hold.

$$5 = \sigma(u_3)\sigma(v_3) \times 3$$
$$5 = \sigma(u_4)\sigma(v_3) \times 9$$

Solving the these equations, we have the solution (ψ, σ), a w-homomorphism from G_2 to G_3, with one parameter $s \in \boldsymbol{R}_{>0}$.

$$\psi = \begin{pmatrix} u_3 & u_4 & v_3 \\ u & u & v \end{pmatrix}. \quad \sigma = \begin{pmatrix} u_3 & u_4 & v_3 \\ 5/(3s) & 5/(9s) & s \end{pmatrix}.$$

□

1.2 Composition of the w-homomorphisms

We define the composition of the w-homomorphisms. In this manuscript, we denote the composition $\psi \circ \phi$ of maps by $\phi\psi$.

DEFINITION 1.3 Let $G_i = (V_i, E_i, W_i)\,(i = 1, 2, 3)$ be weighted digraphs, $(\phi, \rho) : G_1 \to G_2$ and $(\psi, \sigma) : G_2 \to G_3$ be w-homomorphisms. Then the composition of these w-homomorphisms are defined by the semidirect product

$$(\phi, \rho)(\psi, \sigma) \stackrel{\text{def}}{=} (\phi, \rho) \rtimes (\psi, \sigma) = (\phi\psi, \rho \otimes (\phi\sigma)),$$

where $\rho \otimes (\phi\sigma) : V \to Q(R), u \mapsto \rho(u)\sigma(\phi(u))$. The set $Q(R)^V$ of maps from V to $Q(R)$ forms abelian group under the operation $\otimes$: $(f \otimes g)(v) = f(v)g(v)$. □

Indeed, checking the validity of the definition.

$$\begin{aligned}&W_3(\psi(\phi(u)),\psi(\phi(v)))\\&=\sigma(\phi(u))\sigma(\phi(v))W_2(\phi(u),\phi(v))\\&=\sigma(\phi(u))\sigma(\phi(v))\rho(u)\rho(v)W_1(u,v)\\&=\sigma(\phi(u))\rho(u)\sigma(\phi(v))\rho(v)W_1(u,v)\\&=((\phi\sigma)\otimes\rho)(u)((\phi\sigma)\otimes\rho)(v)W_1(u,v)\end{aligned}$$

hold.

EXAMPLE 1.3 Let $G_i = (V_i, E_i, W_i)\,(i = 1, 2, 3)$ be weighted digraphs depicted in Figures 1.1 and 1.2. The following (ϕ, ρ) is the w-homomorphism from G_1 to G_2 in Example 1.1. (ψ, σ) is a w-homomorphism from G_2 to G_3 in Example 1.2.

$$\phi=\begin{pmatrix}u_1 & u_2 & v_1 & v_2\\ u_3 & u_4 & v_3 & v_3\end{pmatrix},\ \rho=\begin{pmatrix}u_1 & u_2 & v_1 & v_2\\ 3/(2r) & 3/(2r) & 2r & r\end{pmatrix},$$
$$\psi=\begin{pmatrix}u_3 & u_4 & v_3\\ u & u & v\end{pmatrix},\quad \sigma=\begin{pmatrix}u_3 & u_4 & v_3\\ 5/(3s) & 5/(9s) & s\end{pmatrix}.$$

Let

$$\xi=\phi\psi=\begin{pmatrix}u_1 & u_2 & v_1 & v_2\\ u & u & v & v\end{pmatrix}.$$

Then if (ξ, τ) is a w-homomorphism from G_1 to G_3, the following equations must hold.

$$\begin{aligned}5&=\tau(u_1)\tau(v_1)\times 1\\5&=\tau(u_1)\tau(v_2)\times 2\\5&=\tau(u_2)\tau(v_1)\times 3\\5&=\tau(u_2)\tau(v_2)\times 6\end{aligned}$$

Therefore τ is represented as below with one positive real parameter t

$$\tau=\begin{pmatrix}u_1 & u_2 & v_1 & v_2\\ 5/(2t) & 5/(6t) & 2t & t\end{pmatrix}.$$

While calculating $(\phi\psi, (\phi\sigma)\otimes\rho)$

$$\begin{aligned}(\phi\sigma)\otimes\rho&=\begin{pmatrix}u_1 & u_2 & v_1 & v_2\\ \sigma(u_3) & \sigma(u_4) & \sigma(v_3) & \sigma(v_3)\end{pmatrix}\otimes\begin{pmatrix}u_1 & u_2 & v_1 & v_2\\ 3/(2r) & 3/(2r) & 2r & r\end{pmatrix}\\&=\begin{pmatrix}u_1 & u_2 & v_1 & v_2\\ 5/(3s) & 5/(9s) & s & s\end{pmatrix}\otimes\begin{pmatrix}u_1 & u_2 & v_1 & v_2\\ 3/(2r) & 3/(2r) & 2r & r\end{pmatrix}\\&=\begin{pmatrix}u_1 & u_2 & v_1 & v_2\\ 5/(2rs) & 5/(6rs) & 2rs & rs\end{pmatrix}\end{aligned}$$

Thus we can check that the direct solution (ξ, τ) and the composition $(\phi\psi, (\phi\sigma)\otimes\rho)$ are identical. □

For weighted digraphs G_1 and G_2, we write $G_1 \sqsupseteq G_2$ if there exists a surjective w-homomorphism from G_1 to G_2. Since in Definition 1.3, ϕ and ψ are sujective, $\phi\psi$ is also. Therefore $G_1 \sqsupseteq G_2 \sqsupseteq G_3$ holds. The relation $\sqsupseteq$ forms a pre-order (a relation satisfying the reflexive law and the transitive law) as shown below. Of course, the pre-order $\sqsupseteq$ is regarded as an order up to w-isomorphism.

PROPOSITION 1.1 [11] Let G_1, G_2, G_3 be weighted digraphs. Then,
(1) $G_1 \sqsupseteq G_1$.
(2) $G_1 \sqsupseteq G_2$ and $G_2 \sqsupseteq G_1 \iff G_1 \simeq_{\mathrm{w}} G_2$.
(3) $G_1 \sqsupseteq G_2$ and $G_2 \sqsupseteq G_3$ imply $G_1 \sqsupseteq G_3$. □

2 Freeness of w-homomorphism

Suppose that there exists two w-homomorphisms (ϕ_1, ρ_1) and (ϕ_2, ρ_2) from G_1 to G_2 for given two digraphs G_1 and G_2. As we have seen in the examples in the previous section, even though $\phi_1 = \phi_2$ holds, $\rho_1 = \rho_2$ is not necessarily true. Here we investigate whether for a given w-homomorphisms (ϕ, ρ), ρ is uniquely determined or not.

DEFINITION 2.1 Let $G = (V, E, W)$ be a weighted digraph. We call $\bar{G} = (V, \bar{E})$ a unweighted undirected graph obtained from G, if

$$v_i v_j \in \bar{E} \iff W(v_i, v_j) > 0 \text{ or } W(v_j, v_i) > 0,$$

where $v_i v_j$ is an undirected edge, i.e. we identify $v_i v_j$ with $v_j v_i$. □

Let (ϕ, ρ) be a w-homomorphism from G_1 to G_2. To determine ρ, we must solve the equation of the form.

$$W_2(\phi(v_i), \phi(v_j)) = \rho(v_i)\rho(v_j)W_1(v_i, v_j), \ (i \leqq j)$$

Put $x_i = \log \rho(v_i)$, $x_j = \log \rho(v_j)$, $w_{ij} = \log(W_2(\phi(v_i), \phi(v_j))) - \log(W_1(v_i, v_j))$. The equation above is written in the form:

$$x_i + x_j = w_{ij}$$

Note that when both $W_1(v_i, v_j) > 0$ and $W_1(v_j, v_i) > 0$ imply $w_{ij} = w_{ji}$, two equations $x_i + x_j = w_{ij}$ and $x_j + x_i = w_{ji}$ are identical.

So let n and m be the numbers of vertices and edges in the undirected graph $\bar{G}_1 = (V, \bar{E})$. Then these equations can be represented as $M\boldsymbol{x} = \boldsymbol{w}$, where M is $m \times n$ matrix whose elements are 0 or 1, the row vector $\boldsymbol{x}$ consists of n variables, the row vector $\boldsymbol{w}$ consists of m real numbers. It is easily seen that ρ is uniquely determined if the rank $r = rank(M)$ of M is equal to n. Otherwise, ρ is not uniquely determined, and has $n - r$ free parameters. So (ϕ, ρ) or ρ is said to be of freeness $n - rank(M)$.

DEFINITION 2.2 Let $G = (V, E, W)$ be a weighted digraph with $V = \{v_1, v_2, \ldots, v_n\}$ of ordered vertices and $\bar{G} = (V, \bar{E})$ be a undirected graph obtained from G. The $m \times n$ matrix $M_{\mathrm{E}}(G)$ is called the *edge matrix* of G, if its (k, i) and (k, j)-components are 1 when $v_i v_j$ $(i \leq j)$ is the k-th smallest edge in $\bar{E}$, otherwise 0. □

EXAMPLE 2.1 Consider w-automorphisms of digraphs depicted in Figure 3.
(1) Let (ϕ, ρ) be the automorphism of the loop L depicted in Figure 3(a). $2 = \rho(v)\rho(v) \times 2$, $x = \log \rho(v)$. Therefore $\rho(v) = 1$.

$$\left[\, 1 \,\right] \left[\, x_1 \,\right] = \left[\, \log 1 \,\right] = \left[\, 0 \,\right]$$

(2) (ϕ, ρ) is the automorphism of the digraph C_2 depicted in Figure 3 (b).

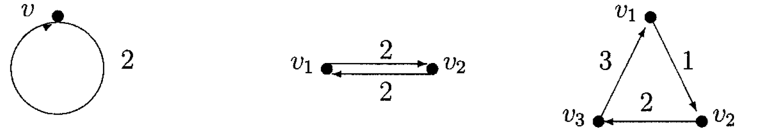
$$\phi = \begin{pmatrix} v_1 & v_2 \\ v_2 & v_1 \end{pmatrix}, \quad \rho = \begin{pmatrix} v_1 & v_2 \\ t & 1/t \end{pmatrix}.$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \log(2/2) \\ \log(2/2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(3) (ϕ, ρ) is the automorphism of the digraph C_3 depicted in Figure 3 (c).

$$\phi = \begin{pmatrix} v_1 & v_2 & v_3 \\ v_2 & v_3 & v_1 \end{pmatrix}, \quad \rho = \begin{pmatrix} v_1 & v_2 & v_3 \\ 2/3 & 3 & 1/2 \end{pmatrix}.$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \log 2 \\ \log(1/3) \\ \log(3/2) \end{bmatrix}$$



(a) Loop L (b) Digraph C_2 (c) Digraph C_3

Figure 3. Loop L.

□

FACT 1 *If a undirected graph G is connected and has n vertices and m edges, then $n \leq m + 1$.*

FACT 2 *If G is a tree with n vertices and m edges, then $n = m + 1$.*

FACT 3 *If a undirected graph G with n vertices and m edges is connected and $n = m+1$, then G is a tree.*

THEOREM 2.1 Let $G = (V, E, W)$ be a digraph and the undirected graph $(V, \bar{E})$ be a tree with n vertices. Then the edge matrix $M_{\mathrm{E}}(G)$ is an $(n-1) \times n$ matrix. and any w-homomorphism from G is of freeness 1.

Proof) We prove that the row vectors of $M = M_{\mathrm{E}}(G)$ by induction on the number $m = n - 1$ of edges. If $m = 1$, then M has the only one nonzero row. Assume $m > 1$ and the claim is true for a tree with $m - 1$ edges. Let v_n be a terminal vertex in G and G' be the subgraph containing all element of $V - \{v_n\}$. $M' = M_{\mathrm{E}}(G')$ has $m - 1$ independent rows.

M has the last row which cannot be represented as a linear combination of other rows. Therefore the rank of M is m.

$$M = \left(\begin{array}{ccccc|c} & & M' & & & 0 \\ \hline 0 & \dots & 1 & \dots & 0 & 1 \end{array}\right)$$

□

THEOREM 2.2 Let $G = (V, E, W)$ be a digraph and the undirected graph $\bar{G} = (V, \bar{E})$ be an n-cycle. The edge matrix $M = M_{\mathrm{E}}(G)$ is an $n \times n$ matrix. If n is odd, then M is of rank n. If n is even, then M is of rank $n - 1$.

So a w-homomorphism from G is of freeness 0 if n is odd, of freeness 1 if n is even.

THEOREM 2.3 Let $G = (V, E, W)$ be a digraph and the undirected graph $\bar{G} = (V, \bar{E})$ be a connected graph with n vertices. Let G be a connected digraph with n vertices The rank of the edge matrix $M_{\mathrm{E}}(G)$ is $n-1$ or n. So a w-homomorphism from G is of freeness 0 or 1.

COROLLARY 2.1 Let $G = (V, E, W)$ be a digraph and the undirected graph $\bar{G} = (V, \bar{E})$ be a connected graph with n vertices. If $\bar{G}$ has an odd (resp. even) cycle, then the rank of the edge matrix $M_{\mathrm{E}}(G)$ is n (resp.$n - 1$) and w-homomorphism from G is of freeness 0 (resp.1).

THEOREM 2.4 Let $G = (V, E, W)$ be a digraph, $\bar{G} = (V, \bar{E})$ be the undirected graph and $V_1, V_2, \ldots, V_N$ be distinct connecting components with $V = V_1 + V_2 + \cdots + V_N$ and K be the number of isolated vertices. Let G_i be the subgraph of G containing all elements of V_i and $M_i = M_{\mathrm{E}}(G_i)$. Then, the rank of $W = M_{\mathrm{E}}(G)$ is the sum of $rank(M_i)$.

$$W = \left(\begin{array}{ccc|c} M_1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & \mathbf{0} \\ 0 & \cdots & M_{N-K} & \\ \hline & \mathbf{0} & & \mathit{0} \end{array}\right)$$

□

3 Ideals in the semigroup $\mathcal{S}$

In this section we define the set $\mathcal{S}$ of all surjective w-homomorphisms between two weighted digraphs and a (extra) zero element 0. Introducing the multiplication by the composition, $\mathcal{S}$ forms a semigroup.

For a surjective w-homomorphim $x : G_1 \to G_2$, G_1 is called the domain of x, denoted by $Dom(x)$, and G_2 is called the image(or range) of x, denoted by $Im(x)$. Especially $Dom(0) = Im(0) = \emptyset$. The multiplication of $x = (\phi, \rho)$ and $y = (\psi, \sigma)$ is defined by

$$x \cdot y \stackrel{\text{def}}{=} \begin{cases} (\phi\psi, (\phi\rho) \otimes \sigma) & \text{if} \quad Im(x) = Dom(y). \\ 0 & \text{otherwise.} \end{cases}$$

3.1 Green's equivalences on the semigroup $\mathcal{S}$

Regarding to a general semigroup S without an identity, for convenience of notation, $S^1 = S \cup \{1\}$ is the monoid obtained from a semigroup S by adjoining an (extra) identity 1, that is, $1 \cdot s = s \cdot 1 = s$ for all $s \in S$ and $1 \cdot 1 = 1$.

In general, Green's equivalences $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}, \mathcal{D}$ on a semigroup S, which are well-known and important equivalence relations in the development of semigroup theory, are defined as follows:

$$\begin{aligned} &x\mathcal{L}y \iff S^1x = S^1y, \\ &x\mathcal{R}y \iff xS^1 = yS^1, \\ &x\mathcal{J}y \iff S^1xS^1 = S^1yS^1, \\ &\mathcal{H} = \mathcal{L} \cap \mathcal{R}, \\ &\mathcal{D} = (\mathcal{L} \cup \mathcal{R})^*, \end{aligned}$$

where $(\mathcal{L} \cup \mathcal{R})^*$ means the reflexive and transitive closure of $\mathcal{L} \cup \mathcal{R}$. S^1x (resp. xS^1) is called the *principal left* (resp. *right*) *ideal generated by* x and S^1xS^1 the *principal (two-sided) ideal generated by* x. Then, the following facts are generally true[7, 2].

FACT 4 *The following relations are true.*

$$\begin{aligned} &(1)\, \mathcal{D} = \mathcal{LR} = \mathcal{RL} \\ &(2)\, \mathcal{H} \subset \mathcal{L} \ (\textit{resp.}\, \mathcal{R}) \ \subset \mathcal{D} \subset \mathcal{J} \end{aligned}$$

FACT 5 *An $\mathcal{H}$-class is a group if and only if it contains an idempotent e, that is $e^2 = e$.*

Now we consider the case of $S = \mathcal{S}$ in the rest of the maniscript. The following lemma is obviously true.

LEMMA 3.1 [11] Let $x : G_1 \to G_2$, $y : G_3 \to G_4 \in \mathcal{S}$. Then,
(1) $x\mathcal{S}^1 \subset y\mathcal{S}^1 \Longrightarrow G_1 = G_3$ and $G_2 \sqsubseteq G_4$.
(2) $\mathcal{S}^1x \subset \mathcal{S}^1y \Longrightarrow G_3 \sqsubseteq G_1$ and $G_2 = G_4$.
(3) $x\mathcal{S}^1 = y\mathcal{S}^1 \Longrightarrow G_1 = G_3$ and $G_2 \simeq_{\mathrm{w}} G_4$.
(4) $\mathcal{S}^1x = \mathcal{S}^1y \Longrightarrow G_1 \simeq_{\mathrm{w}} G_3$ and $G_2 = G_4$. □

Remark that any reverse implications above are not necessarily true.

PROPOSITION 3.1 [11] The following conditions are equivalent.
(1) H is an $\mathcal{H}$-class and a group.
(2) $H = \mathbf{Aut}_{\mathrm{w}}(G)$ for some weighted digraph G. □

PROPOSITION 3.2 [11] On the semigroup $\mathcal{S}$, $\mathcal{J} = \mathcal{D}$. □

3.2 Intersection of principal ideals

The aim here is that for given $x, y \in \mathcal{S}$ we find a elements z such that $\mathcal{S}^1x \cap \mathcal{S}^1y = \mathcal{S}^1z$ (resp. $x\mathcal{S}^1 \cap y\mathcal{S}^1 = z\mathcal{S}^1$). $x\mathcal{S}^1 \cap y\mathcal{S}^1 = \{0\}$ (resp. $\mathcal{S}^1x \cap \mathcal{S}^1y = \{0\}$) is a trivial case($z = 0$). We should only consider the non-trivial case.

PROPOSITION 3.3 (Intersection of Principal Left Ideals) [11] Let $G_i = (V_i, E_i, W_i)(i = 1, 2, 3)$ be weighted digraphs, $x = (\phi_1, \rho_1) : G_1 \to G_3$, $y = (\phi_2, \rho_2) : G_2 \to G_3$ be elements of $\mathcal{S}$. Then there exists $z \in \mathcal{S}$ such that $\mathcal{S}^1x \cap \mathcal{S}^1y = \mathcal{S}^1z$. □

COROLLARY 3.1 (Diamond Property I) [11] Let G_1, G_2, G_3 be weighted digraphs with $G_i \sqsupseteq G_3\, (i = 1, 2)$. Then there exists a unique least weighted digraph G up to w-isomorphism such that $G \sqsupseteq G_i\, (i = 1, 2)$. □

PROPOSITION 3.4 (Intersection of Principal Right Ideals) [11] Let $G_i = (V_i, E_i.W_i)(i = 0,1,2)$ be weighted digraphs, $x = (\phi_1, \rho_1) : G_0 \to G_1$, $y = (\phi_2, \rho_2) : G_0 \to G_2$ be elements of $\mathcal{S}$. Then there exists $z \in \mathcal{S}$ such that $x\mathcal{S}^1 \cap y\mathcal{S}^1 = z\mathcal{S}^1$. □

COROLLARY 3.2 (Diamond Property II) [11] Let $G_i\,(i = 0,1,2)$ be weighted digraphs with $G_0 \sqsupseteq G_i\,(i = 1,2)$. Then, there exists a unique maximum weighted digraph G up to isomorphism such that $G_i \sqsupseteq G\,(i = 1,2)$. □

We define the notion of irreducible forms of a weighted digraph with respect to $\sqsupseteq$.

DEFINITION 3.1 (Irreducible) A weighted digraph G is called a $\sqsupseteq$-*irreducible* if $G \sqsupseteq G'$ implies $G \simeq_w G'$ for any weighted digraph G'. Then G is called an $\sqsupseteq$-irreducible form. □

COROLLARY 3.3 [11] Let G, G' and G'' be weighted digraphs with $G \sqsupseteq G'$ and $G \sqsupseteq G''$. Then one has: If G' and G'' are $\sqsupseteq$-irreducible, then $G' \simeq_w G''$. □

3.3 Lattice structures of $\simeq_w$-classes of weighted digraphs

As an application of the theory of principal ideals developed in the previous section, we deal with lattice structures of equivalence classes ($\simeq_w$-classes) of digraphs divided by the w-isomorphism relation $\simeq_w$. By $[G]$ we denote the $\simeq_w$-class of a graph G. The set of all $\simeq_w$-class is an ordered set because $\sqsupseteq$ is well-defined and Lemma 1.1 holds.

Let G_{irr} be an $\sqsupseteq$-irreducible form and $L([G_{irr}]) = \{[G] \mid G \sqsupseteq G_{irr}\}$ through this section. By Corollary 3.3, the class $[G_{irr}]$ is the least element of $L([G_{irr}])$ because any other $\simeq_w$-class in $L([G_{irr}])$ cannot contain an $\sqsupseteq$-irreducible form.

PROPOSITION 3.5 (conditional LUB and GLB) The following claims hold.

(1) Let $[G_1], [G_2], [G_3]$ be $\simeq_w$-classes with $[G_i] \sqsupseteq [G_3]\,(i = 1,2)$. There exists the minimum $[G]$ such that $[G] \sqsupseteq [G_i] \sqsupseteq [G_3]\,(i = 1,2)$, denoted by $\mathrm{lub}([G_1],[G_2];[G_3])$.

(2) Let $[G_0], [G_1], [G_2]$ be $\simeq_w$-classes with $[G_0] \sqsupseteq [G_i]\,(i = 1,2)$. There exists the maximum $[G]$ such that $[G_0] \sqsupseteq [G_i] \sqsupseteq [G]\,(i = 1,2)$, denoted by $\mathrm{glb}([G_0];[G_1],[G_2])$.

Proof) Immediate from Corollary 3.1 and Corollary 3.2. □

PROPOSITION 3.6 The following claims hold.

(1) Let $[G_1], [G_2], [G_3], [G'_3]$ be $\simeq_w$-classes with $[G_i] \sqsupseteq [G_3]$ and $[G_i] \sqsupseteq [G'_3](i = 1,2)$. If $[G_3] \sqsupseteq [G'_3]$, then $\mathrm{lub}([G_1],[G_2];[G_3]) \sqsupseteq \mathrm{lub}([G_1],[G_2];[G'_3])$.

(2) Let $[G_0], [G'_0], [G_1], [G_2]$ be $\simeq_w$-classes with $[G_0] \sqsupseteq [G_i]$ and $[G'_0] \sqsupseteq [G_i](i = 1,2)$. If $[G_0] \sqsupseteq [G'_0]$, then $\mathrm{glb}([G_0];[G_1],[G_2]) \sqsupseteq \mathrm{glb}([G'_0];[G_1],[G_2])$.

Proof) (1) Put $[G] = \mathrm{lub}([G_1],[G_2];[G_3])$, $G' = \mathrm{lub}([G_1],[G_2];[G'_3])$. By Proposition 3.3, there exist surjective w-homomorphisms $z : G \to G_3$, $z' : G' \to G'_3$ and $u : G_3 \to G'_3$ such that $\mathcal{S}^1x \cap \mathcal{S}^1y = \mathcal{S}^1z$ and $\mathcal{S}^1xu \cap \mathcal{S}^1yu = \mathcal{S}^1z'$. Since $zu \in \mathcal{S}^1xu$ and $zu \in \mathcal{S}^1yu$ hold, $zu \in \mathcal{S}^1z'$ and thus $zu = vz'$ for some $v : G \to G'$ and $v \in \mathcal{S}^1$.

(2) By the left-right duality of (1). □

COROLLARY 3.4 Let $[G_1], [G_2]$ be elements in $L([G_{irr}])$. There exists the unique least (resp. greatest) $\simeq_w$ class $[G_U]$ (resp. $[G_L]$) such that $[G_U] \sqsupseteq [G_i]\,(i = 1, 2)$ (resp. $[G_i] \sqsupseteq [G_L]\,(i = 1, 2)$), denoted by $\text{lub}([G_1], [G_2])$ (resp. $\text{glb}([G_1], [G_2])$).

Proof) By Proposition 3.6, $[G_U] = \text{lub}([G_1], [G_2]; [G_{irr}])$ is least. Again, $[G_L] = \text{glb}([G_U]; [G_1], [G_2])$ is greatest. □

From this proposition we get the following theorem.

THEOREM 3.1 The ordered set $(L([G_{irr}]), \sqsupseteq)$ forms a lattice with the least element $[G_{irr}]$.

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