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Numerical semigroups and triple cyclic covers of curves [1]

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Abstract

We construct some triple cyclic covers of any curves and calculate the Weierstrass semigroups of ramification points on the triple covers.

1 Introduction

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0\backslash H$ is finite. The cardinality of $\mathbb{N}_0\backslash H$ is called the *genus* of H, denoted by $g(H)$. In this paper H always stands for a numerical semigroup. A *curve* means a complete non-singular irreducible algebraic curve over an algebraically closed field k of characteritic 0. For a pointed curve (C, P) we set

$$H(P) = \{n \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = nP\},$$

where $k(C)$ is the field of rational functions on C. Then $H(P)$ is a numerical semigroup, which is called the *Weierstrass semigroup* of P. Here $g(H(P))$ is equal to the genus $g(C)$ of the curve C. For positive integers $a_1, \ldots, a_s$ we denote by $\langle a_1, \ldots, a_s\rangle$ the monoid generated by $a_1, \ldots, a_s$. For any integer $t \geqq 2$ we set $d_t(H) = \{h' \in \mathbb{N}_0 \mid th' \in H\}$, which is a numerical semigroup. We have the following.

Theorem 1.1 *Let t be an integer which is larger than or equal to two. Let $\pi : C \longrightarrow C'$ be a cyclic covering of degree t with a totally ramification point P over P'. Then $d_t(H(P)) = H(P')$.*

We are devoted to the case $t = 3$. A numerical semigroup H is said to be *of triple covering type*, which is abbreviated to *TC* if there exists a triple cyclic covering $\pi : C \longrightarrow C'$ with a ramification point P such that $H = H(P)$. We are interested in numerical semigroups which are TC.

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[1]This paper is an extended abstract and the details will appear elsewhere.
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2 Weierstrass semigroups on triple cyclic covers of $\mathbb{P}^1$

Let $\pi : C \longrightarrow \mathbb{P}^1$ be a triple cyclic covering with a ramification point P. Since we have $d_3(H(P)) = \mathbb{N}_0$, the Weierstrass semigroup $H(P)$ is either $\mathbb{N}_0$ or $\langle 2, 3\rangle$ or a 3-semigroup, where for a positive integer m an m-semigroup H means a numerical semigroup whose minimum positive integer in H is m. The following is a well-known fact:

Remark 2.1 *The converse holds, namely* $\mathbb{N}_0$, $\langle 2, 3\rangle$ *and any* 3*-semigroup are TC.*

3 Weierstrass semigroups on cyclic covers of $\mathbb{P}^1$ with degree 6

A 6-semigroup H is *cyclic* if it is the Weierstrass semigroup of a total ramification point on a cyclic cover of $\mathbb{P}^1$ with degree 6. We have the following:

Remark 3.1 *A cyclic* 6*-semigroup is TC.*

Let H be an m-semigroup. For $1 \leqq i \leqq m-1$ we set $s_i = \min\{h \in H \mid h \equiv i \bmod m\}$. The set $S(H) = \{m, s_1, \ldots, s_{m-1}\}$ becomes a set of generators for H, which is called the *standard basis* for H.

Example $S(\langle 6, 7\rangle) = \{6, 7, 14, 21, 28, 35\}$.

We have the following necessary and sufficient condition for a 6-semigroup to be cyclic.

Theorem 3.2 (Komeda-Ohbuchi [1]) *Let H be a* 6*-semigroup with*

$$S(H) = \{6\} \cup \{6m_i + i \mid 1 \leqq i \leqq 5\}.$$

Then the following are equivalent:
i) *H is cyclic.*
ii) *We have the three inequalities*

$$m_2 + m_5 \geqq m_3 + m_4,\ m_1 + m_5 \geqq m_2 + m_4 \text{ and } m_1 + m_4 \geqq m_2 + m_3.$$

Example Let $H = \langle 6, 9, 10\rangle$. Then we have $S(H) = \{6, 9, 10, 19, 20, 29\}$. Hence,

$$m_1 = 3,\ m_2 = 3,\ m_3 = 1,\ m_4 = 1 \text{ and } m_5 = 4,$$

which implies that H is cyclic, hence TC.

4 Weierstrass semigroups on triple cyclic covers of any pointed curves

We have the following:

Lemma 4.1 *Let H be an m-semigroup with $S(H) = \{m, s_1, \ldots, s_{m-1}\}$. Let n be an integer with $n \geqq \max\{c(H) - m + 1, 3m\}$ and $n \not\equiv 0$ mod 3 where we set*

$$c(H) = \min\{c \in \mathbb{N}_0 \mid c + \mathbb{N}_0 \subseteqq H\}.$$

Then the following holds:
i) *We have*

$$S(3H+n\mathbb{N}_0) = \{3m, 3s_1, \cdots, 3s_{m-1}, n, 2n\} \cup \{n+3s_1, 2n+3s_1, \cdots, n+3s_{m-1}, 2n+3s_{m-1}\}.$$

ii) *We obtain $g(3H + n\mathbb{N}_0) = 3g(H) + n - 1$.*

Example We have

$$g(3\langle 3, 4\rangle + 10\mathbb{N}_0) = g(\langle 9, 12, 10\rangle) = 3g(\langle 3, 4\rangle) + 10 - 1 = 18.$$

Lemma 4.2 *Let C be a curve and D a divisor on C such that $3D$ is linearly equivalent to a reduced divisor R. We give an $\mathcal{O}_C$-Algebra structure on*

$$\mathcal{V}_2(D) = \mathcal{O}_C \oplus \mathcal{O}_C(-D) \oplus \mathcal{O}_C(-2D).$$

Then we get a triple cyclic covering

$$\pi : \tilde{C} = \mathrm{Spec}(\mathcal{V}_2(D)) \longrightarrow C$$

whose branch locus is R.

The above lemma follows from Miranda [2]. In the case $t = 2$, i.e., the case of double coverings the following result is known:

Theorem 4.3 (Komeda-Ohbuch [1]) *Let (C, P) be a pointed curve and set $H = H(P)$, which is an m-semigroiup. Let d be an integer with $2d - 1 \geqq \max\{c(H) - m + 2, 2m\}$ Assume that $2d-1 \in H$. Then we get a double covering $\pi : \mathrm{Spec}(\mathcal{O}_C \oplus \mathcal{O}_C(-dP)) \longrightarrow C$, with a ramification point $\tilde{P}$ over P satisfying $H(\tilde{P}) = 2H(P) + (2d - 1)\mathbb{N}_0$.*

In our case we get the following:

Theorem 4.4 *Let (C, P) be a pointed curve and set $H = H(P)$, which is an m-semigroup. Let d be an integer with $3d - 1 \geqq \max\{c(H) - m + 2, 3m\}$. Assume that $3d - 1 \in H$. Then we get a triple cyclic covering*

$$\pi : \tilde{C} = \mathrm{Spec}(\mathcal{O}_C \oplus \mathcal{O}_C(-dP) \oplus \mathcal{O}_C(-2dP)) \longrightarrow C$$

with a ramification point $\tilde{P}$ over P satisfying $H(\tilde{P}) = 3H(P) + (3d - 1)\mathbb{N}_0$.

Corollary 4.5 *Let H be an m-semigroup such that $H = H(P)$ for some pointed curve (C, P). Let n be an integer with $n \equiv 2 \bmod 3$ and $n \geqq \max\{c(H)-m+2, 3m\}$. Assume that $n \in H$. Then the numerical semigroup $3H + n\mathbb{N}_0$ is TC.*

Example Let a and b be integers with $2 \leqq a < b$ and $(a, b) = 1$. Let d be an integer with $3d - 1 \geqq \max\{(a-1)(b-2)+1, 3a\}$. Assume that $3d - 1 \in \langle a, b\rangle$. Then the numerical semigroup $3\langle a, b\rangle + (3d-1)\mathbb{N}_0$ is TC, because there is a pointed curve (C, P) such that $H(P) = \langle a, b\rangle$.

References

[1] J. Komeda and A. Ohbuchi, *On double coverings of a pointed non-singular curve with any Weierstrass semigroup*, Tsukuba J. Math. Soc. **31** (2007) 205–215.

[2] R. Miranda, *Triple covers in algebraic geometry*, Amer. J. Math. **107** (1985) 1123–1158.