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ALMOST SYMMETRIC NUMERICAL SEMIGROUPS AND ALMOST GORENSTEIN SEMIGROUP RINGS GENERATED BY FOUR ELEMENTS

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The theory of numerical semigroups is important to commutative ring theory via their associated semigroup ring as well as to the theory of algebraic curves via the Weierstrass semigroup of a point on a compact Riemann surface. Among the numerical semigroups, symmetric and almost symmetric semigroups play a central role. In this article, we are mostly interested in almost symmetric semigroups generated by 4 elements.

Recently, Moscariello in [Mo] proposed the notion of RF (Row Factorization) matrix and our mail tool is RF matrices and minimal free resolutions.

The main results are the following:

(1) Give a new proof of Komeda's theorem on pseudo-symmetric (= almost symmetric of type 2) numerical semigroups.
(2) Characterize the minimal free resolution of the semigroup ring of an almost symmetric numerical semigroup of type 3.
(3) Classify those numerical semigroups H, for which $H + n$ is almost symmetric of type 2 (resp. 3).

1. Basic concepts

In this section we fix notation and recall the basic definitions and concepts which will be used in this paper.

Pseudo-Frobenius numbers and Apery sets. A submonoid $H \subset \mathbb{N}$ with $0 \in H$ and $\mathbb{N} \setminus H$ is finite is called a numerical semigroup. Any numerical semigroup H induces a partial order on $\mathbb{Z}$, namely $a \leq_H b$ if and only if $b - a \in H$.

There exist finitely many positive integers $n_1, \dots, n_e$ belonging to H such that each $h \in H$ can be written as $h = \sum_{i=1}^{e} \alpha_i n_i$ with non-negative integers α_i. Such a presentation of h is called a a factorization of h, and the set $\{n_1, \dots, n_e\} \subset H$ is called a set of generators of H. If $\{n_1, \dots, n_e\}$ is a set of generators of H, then we write $H = \langle n_1, \dots, n_e \rangle$. The set

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of generators $\{n_1, \ldots, n_e\}$ is called a minimal set of generators of H, if none of the n_i can be omitted to generate H. A minimal set of generators of H is uniquely determined.

Now let $H = \langle n_1, \ldots, n_e \rangle$ be a numerical semigroup. We assume that $n_1, \ldots, n_e$ are minimal generators of H, that $\gcd(n_1, \ldots, n_e) = 1$ and that $H \neq \mathbb{N}$, unless otherwise stated.

The assumptions imply that the set $G(H) = \mathbb{N} \setminus H$ of gaps is a finite non-empty set. Its cardinality will be denoted by $g(H)$. The largest gap is called the Frobenius number of H, and denoted $\mathrm{F}(H)$.

An element $f \in \mathbb{Z} \setminus H$ is called a pseudo-Frobenius number, if $f + n_i \in H$ for all i. Of course, the Frobenius number is a pseudo-Frobenius number as well and each pseudo-Frobenius number belongs to $G(H)$. The set of pseudo-Frobenius numbers will be denoted by $\mathrm{PF}(H)$.

We also set $\mathrm{PF}'(H) = \mathrm{PF}(H) \setminus \{\mathrm{F}(H)\}$. The cardinality of $\mathrm{PF}(H)$ is called the type of H, denoted $t(H)$. Note that for any $a \in \mathbb{Z} \setminus H$, there exists $f \in \mathrm{PF}(H)$ such that $f - a \in H$.

Symmetric, pseudo-symmetric and almost symmetric numerical semigroups. For each $h \in H$, the element $\mathrm{F}(H) - h$ does not belong to H. Thus the assignment $h \mapsto \mathrm{F}(H) - h$ maps each element $h \in H$ with $h < \mathrm{F}(H)$ to a gap of H. If each gap of H is of the form $\mathrm{F}(H) - h$, then H is called symmetric. This is the case if and only if for each $a \in \mathbb{Z}$ one has: $a \in H$ if and only if $\mathrm{F}(H) - a \notin H$. It follows that a numerical semigroup is symmetric if and only if $g(G) = |\{h \in H \; h < \mathrm{F}(H)\}|$, equivalently if $2g(H) = F(H) + 1$. A symmetric semigroup is also characterized by the property that its type is 1. Thus we see that a symmetric semigroup satisfies $2g(H) = \mathrm{F}(H) + t(H)$, while in general $2g(H) \geq \mathrm{F}(H) + t(H)$. If equality holds, then H is called almost symmetric. The almost symmetric semigroups (AS semigroups) of type 2 are called pseudo-symmetric. It is quite obvious that a numerical semigroup is pseudo-symmetric if and only if $\mathrm{PF}(H) = \{\mathrm{F}(H)/2, \mathrm{F}(H)\}$. From this one easily deduces that if H is pseudo-symmetric, then $a \in H$ if and only if $\mathrm{F}(H) - a \notin H$ and $a \neq \mathrm{F}(H)/2$.

Less obvious is the following nice result of Nari [Na] which provides a certain symmetry property of the pseudo-Frobenius numbers of H.

Lemma 1.1 ([Na]). Let $\mathrm{PF}(H) = \{f_1, f_2, \ldots, f_{t-1}, \mathrm{F}(H)\}$ with $f_1 < f_2 \ldots < f_{t-1}$. Then H is AS if and only if

$$f_i + f_{t-i} = \mathrm{F}(H) \quad \text{for} \quad i = 1, \ldots, t.$$

Numerical semigroup rings. Many of the properties of a numerical semigroup ring are reflected by algebraic properties of the associated semigroup ring. Let H be a numerical semigroup, minimally generated by $n_1, \dots, n_e$. We fix a field K. The semigroup ring $K[H]$ attached to H is the K-subalgebra of the polynomial ring $K[t]$ which is generated by the monomials t^{n_i}. In other words, $K[H] = K[t^{n_1}, \dots, t^{n_e}]$. Note that $K[H]$ is a 1-dimensional Cohen-Macaulay domain. The symmetry of H has a nice algebraic counterpart, as shown by Kunz [Ku]. He has shown that H is symmetric if and only if and only if $K[H]$ is a Gorenstein ring. Recall that a positively graded Cohen–Macaulay K-algebra R of dimension d with graded maximal ideal $\mathfrak{m}$ is Gorenstein if and only if $\dim_K \mathrm{Ext}^d_R(R/\mathfrak{m}, R) = 1$. In general the K-dimension of the finite dimensional K-vector space is called the CM-type (Cohen-Macaulay type) of R. Kunz's theorem follows from the fact that the type of H coincides with the CM-type of $K[H]$.

We are now going to define Apery sets. Let $a \in H$. Then we let

$$\mathrm{Ap}(a, H) = \{h \in H \mid h - a \notin H\}.$$

This set is called the Apery set of a in H. It is clear that $|\mathrm{Ap}(a, H)| = a$ and that 0 and all n_i belong to $\mathrm{Ap}(a, H)$. For every a the largest element in $\mathrm{Ap}(a, H)$ is $a + \mathrm{F}(H)$.

If $a \in H$, then $K[H]/(t^a)$ is a 0-dimensional K-algebra with K-basis $t^h + (t^a)$ with $h \in \mathrm{Ap}(a, H)$. The elements $t^{f+a} + (t^a)$ with $f \in PF(H)$ form a K-basis of the socle of $K[H]/(t^a)$. This shows that indeed the type of H coincides with the CM-type of $K[H]$.

The canonical module of $\omega_{K[H]}$ of $K[H]$ can be identified with the fractionary ideal of $K[H]$ generated by the elements $t^{-f} \in Q(K[H])$ with $f \in \mathrm{PF}(H)$. Consider the exact sequence of graded $K[H]$-modules

$$0 \to K[H] \to \omega_{K[H]}(-\mathrm{F}(H)) \to C \to 0,$$

where $K[H] \to \omega_{K[H]}(-\mathrm{F}(H))$ is the $K[H]$-module homomorphism which sends 1 to $t^{-\mathrm{F}(H)}$ and where C is the cokernel of this map. One immediately verifies that H is AS if and only if $\mathfrak{m}C = 0$, where $\mathfrak{m}$ denotes the graded maximal ideal of $K[H]$. Motivated by this observation Goto et al [GTT] call a Cohen–Macaulay local ring with canonical module ω_R almost Gorenstein, if there exists an exact sequence

$$0 \to R \to \omega_R \to C \to 0.$$

with C an Ulrich module. If $\dim C = 0$, C is a Ulrich module if and only if $\dim C = 0$. Thus it can be seen that H is AS if and only if $K[H]$ is almost Gorenstein (in the graded sense). Henceforth we will write AS for almost symmetric and AG for almost Gorenstein.

In this paper we are interested in the defining relations of $K[H]$. Let $S = K[x_1, \dots, x_e]$ be the polynomial ring over K in the indeterminates $x_1, \dots, x_e$. Let $\pi : S \to K[H]$ be the surjective K-algebra homomorphism with $\pi(x_i) = t^{n_i}$ for $i = 1, \dots, n$. We denote by I_H the kernel of π. If we assign to each x_i the degree n_i, then with respect to this grading, I_H is a homogeneous ideal, generated by binomials. A binomial $\phi = \prod_{i=1}^{e} x_i^{\alpha_i} - \prod_{i=1}^{e} x_i^{\beta_i}$ belongs to I_H if and only if $\sum_{i=1}^{e} \alpha_i n_i = \sum_{i=1}^{e} \beta_i n_i$. With respect to this grading $\deg\phi = \sum_{i=1}^{e} \alpha_i n_i$.

Now, we put $H = \langle n_1, \dots, n_e \rangle$ and define the invariant α_i for each n_i.

Definition 1.2. For every i, $1 \le i \le e$, we define α_i to be the minimal positive integer such that

$$\alpha_i n_i = \sum_{j=1, j \neq i}^{e} \alpha_{ij} n_j.$$

Note that the coefficients α_{ij} may not be uniquely determined.

It is easy to see the following from the minimality of α_i.

Lemma 1.3. For every $1 \le i, k \le e$, $i \neq k$, $(\alpha_i - 1)n_i \in \mathrm{Ap}(n_k, H)$.

Combining these properties, we get the following, which will play an important role for the structure of AS semigroups.

Corollary 1.4. If H is AS, then for every k and $i \neq k$, either $\mathrm{F}(H) + n_k - (\alpha_i - 1)n_i \in H$ or $(\alpha_i - 1)n_i = f + n_k$ for some $f \in \mathrm{PF}'(H)$.

We give a short review on unique factorization of elements in H on the minimal generators of I_H.

Definition 1.5. Let H be a numerical semigroup minimally generated by $\{n_1, \dots, n_e\}$.

(1) We say that $h = \sum_i a_i n_i$ has UF (Unique Factorization) if this expression is unique. It is obvious that h does not have UF if and only if $h \ge_H \deg(\phi)$ for some $\phi \in I_H$.
(2) We put $\mathrm{NUF}(H) = \{h \in H \mid h \text{ does not have UF } \} = \{\deg(\phi) | \ \phi \in I_H\}$. This is an ideal of H.
(3) We put $\mathrm{mNUF}(H) = \{h \in \mathrm{NUF}(H) \mid h \text{ is minimal with respect to } \le_H\}$. Note that if $\phi \in I_H$ and $\deg(\phi) \in \mathrm{mNUF}(H)$, then ϕ is a minimal generator of I_H. But the converse is not true in general. Hence $\sharp\, \mathrm{mNUF}(H) \le \mu(I_H)$.

Lemma 1.6. Let $\phi = m_1 - m_2$ be a minimal generator of I_H, where m_1, m_2 are monomials on the $X_i's$. Then the following holds:

(1) Let i, j so that $X_i | m_1$ and $X_j | m_2$. Then $\deg\phi - n_i - n_j \notin H$ and hence for some $f \in \mathrm{PF}(H)$, $\deg(\phi) \leq_H f + n_i + n_j$.

(2) $\deg(\phi) = f + n_i + n_j$ for some $f \in \mathrm{PF}'(H)$ if and only if $\mathrm{F}(H) + n_i + n_j - \deg(\phi) \notin H$.

2. The Moscariello matrix RF(f) for $f \in \mathrm{PF}(H)$

A. Moscariello introduced the notion of RF (row factorization) matrices in his paper and we think this notion is very useful to describe the classification of AS semigroups.

Definition 2.1. ([Mo]) Let $f \in \mathrm{PF}(H)$. Then an $e \times e$ matrix $A = (a_{ij})$ is an RF-matrix for f, (short for row-factorization matrix) if $a_{ii} = -1$ for every i, $a_{ij} \in \mathbb{N}$ if $i \neq j$ and for every $i = 1, \ldots, e$,

$$\sum_{j=1}^{e} a_{ij} n_j = f.$$

The matrix A is denoted by RF(f). Note that RF(f) need not be determined uniquely.

The most important property of the RF-matrix RF(f) is the following.

Lemma 2.2. ([Mo], Lemma 4) Let $f, f' \in \mathrm{PF}(H)$ with $f + f' = \mathrm{F}(H)$. If we put $\mathrm{RF}(f) = A = (a_{ij})$ and $\mathrm{RF}(f') = B = (b_{ij})$, then either $a_{ij} = 0$ or $b_{ji} = 0$ for every pair $i \neq j$. In particular, if $F(H)$ is even, and we put $\mathrm{RF}(\mathrm{F}(H)/2) = (a_{ij})$, then either $a_{ij} = 0$ or $a_{ji} = 0$ for every $i \neq j$.

Proof. By our assumption, $f + n_i = \sum_{k \neq i} a_{ik} n_k$ and $f' + n_j = \sum_{l \neq j} b_{jl} n_l$. If $a_{ij} \geq 1$ and $b_{ji} \geq 1$, then summing up these equations, we get

$$\mathrm{F}(H) = f + f' = (b_{ji} - 1) n_i + (a_{ij} - 1) n_j + \sum_{s \neq i,j} (a_{is} + b_{js}) n_s \in H,$$

a contradiction! $\square$

Example 2.3. A nice property of RF(f) is that we can get generators of I_H from the set of matrices $\{\mathrm{RF}(f) \mid f \in \mathrm{PF}(H)\}$ by 1.6. Namely, take any 2 rows $\boldsymbol{a}_i, \boldsymbol{a}_j$ of RF(f) and write $\boldsymbol{a}_i - \boldsymbol{a}_j$ as $\boldsymbol{b}_+ - \boldsymbol{b}_-$, which corresponds to an element of I_H. We will explain this by 2 examples. In the following, we use variables x, y, z, w instead of $X_1, \ldots, X_4$.

(1) Let $H = \langle 12, 17, 31, 40 \rangle$ with $\mathrm{PF}(H) = \{45, 90\}$. Since $90 = 2 \cdot 45$, we know that H is pseudo-symmetric. We compute

$$\mathrm{RF}(45) = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & -1 & 2 & 0 \\ 3 & 0 & -1 & 1 \\ 0 & 5 & 0 & -1 \end{pmatrix},$$

and in this case $I_H = (z^5 - x^3yw, y^6 - z^2w, xz^2 - y^2w, w^2 - xy^4, x^4 - yz)$. The generators of I_H corresponds to $\boldsymbol{a}_1 - \boldsymbol{a}_3, \boldsymbol{a}_4 - \boldsymbol{a}_2, \boldsymbol{a}_2 - \boldsymbol{a}_1, \boldsymbol{a}_1 - \boldsymbol{a}_4, \boldsymbol{a}_3 - \boldsymbol{a}_1$, respectively.

(2) Let $H = \langle 18, 21, 23, 26\rangle$ with $\mathrm{PF}(H) = \{31, 66, 97\}$ and $I_H = (xw - yz, y^5 - x^2z^3, xz^4 - y^4w, z^5 - y^3w^2, x^2y^2 - w^3, x^3y - zw^2, x^4 - z^2w)$. We can check that H is AS of type 3 since $31 + 66 = 97$ and we compute

$$\mathrm{RF}(31) = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 2 \\ 3 & 0 & -1 & 0 \\ 2 & 1 & 0 & -1 \end{pmatrix}, \mathrm{RF}(66) = \begin{pmatrix} -1 & 4 & 0 & 0 \\ 1 & -1 & 3 & 0 \\ 0 & 3 & -1 & 1 \\ 0 & 0 & 4 & -1 \end{pmatrix}.$$

We see that the equations $x^2y^2 - w^3, x^3y - zw^2, x^4 - z^2w$ are obtained from RF(31), $y^5 - x^2z^3, xz^4 - y^4w, z^5 - y^3w^2$ from RF(66) and $xw - yz$ from both matrices.

Moscariello proves that if for some j one has that $a_{ij} = 0$ for every $i \neq j$, then$f = \mathrm{F}(H)/2$. But his result can be improved a little more.

Lemma 2.4. Assume $e = 4$. Assume $f \in \mathrm{PF}(H), f \neq \mathrm{F}(H)$ and put $A = (a_{ij}) = \mathrm{RF}(f)$. Then for every j, there exists i such that $a_{ij} > 0$. Namely, any column of A should contain some positive component.

Combining Lemma 2.2 and Lemma 2.4, we get the following Corollary.

Corollary 2.5. Assume H is AS and let $f \in \mathrm{PF}'(H)$. Then every row or column of $\mathrm{RF}(f)$ has at least one positive (resp. 0) entry .

We can restate the structure theorem of Komeda by using $\mathrm{RF}(\mathrm{F}(H)/2)$.

Theorem 2.6. ([Ko]) Let $H = \langle n_1, n_2, n_3, n_4\rangle$ be pseudo-symmetric. [1]

(1) For a suitable permutation of $\{1,2,3,4\}$, $\mathrm{F}(H)/2 + n_k$ has UF for every k (that is, $\mathrm{RF}(\mathrm{F}(H)/2)$ is uniquely determined) and $\mathrm{RF}(\mathrm{F}(H)/2)$ is in the following form
$$\mathrm{RF}(\mathrm{F}(H)/2) = \begin{pmatrix} -1 & \alpha_2 - 1 & 0 & 0 \\ 0 & -1 & \alpha_3 - 1 & 0 \\ \alpha_1 - 1 & 0 & -1 & \alpha_4 - 1 \\ \alpha_1 - 1 & \alpha_2 - 1 - \alpha_{12} & 0 & -1 \end{pmatrix}$$
(2) $\mathrm{F}(H) + n_2$ has UF and we have $n_2 = \alpha_1\alpha_4(\alpha_3 - 1) + 1$.
(3) Every generator of I_H is obtained from $\mathrm{RF}(\mathrm{F}(H)/2)$ as in the Example 2.3. Namely, $I_H = (x_2^{\alpha_2} - x_1x_3^{\alpha_3-1}, x_1^{\alpha_1} - x_2^{\alpha_{12}}x_4, x_3^{\alpha_3} - x_1^{\alpha_1-1}x_2x_4^{\alpha_4-1}, x_3^{\alpha_3-1}x_4 - x_1^{\alpha_1-1}x_2^{\alpha_2-\alpha_{12}}, x_4^{\alpha_4} - x_2^{\alpha_2-1-\alpha_{12}}x_3)$. (The difference of the 1st and the 3rd rows does not give a minimal generator of I_H.)

[1]Komeda uses the terminology "almost symmetric" for pseudo-symmetric

Remark 2.7. The generators of I_H in [Ko] or [BFS] are obtained after the permutation $(1,2,3,4) \to (3,1,4,2)$. Namely, if we put

$$\mathrm{RF}(\mathrm{F}(H)/2) = \begin{pmatrix} -1 & 0 & 0 & \alpha_4 - 1 \\ \alpha_{21} & -1 & \alpha_3 - 1 & 0 \\ \alpha_1 - 1 & 0 & -1 & 0 \\ 0 & \alpha_2 - 1 & \alpha_3 - 1 & -1 \end{pmatrix},$$

then we get their equations.

Using $\mathrm{RF}(f)$, we can have a different proof of Moscariello's Theorem.

Theorem 2.8. [Mo] If $H = \langle n_1, \dots, n_4 \rangle$ is AG, then $\mathrm{type}(H) \leq 3$.

We will not present the proof here but we list the lemma which we use to prove this theorem [2].

Lemma 2.9. We denote by $\boldsymbol{e}_i$ the i-th unit vector of $\mathbb{Z}^4$. Assume $e = 4$ and H is AS.

(1) There are 2 rows in $\mathrm{RF}(\mathrm{F}(H)/2)$ of the form $(\alpha_i - 1)\boldsymbol{e}_i - \boldsymbol{e}_k$.
(2) If $f \neq f' \in \mathrm{PF}(H)$ with $f + f' = \mathrm{F}(H)$, then there are 4 rows in $\mathrm{RF}(f)$ and $\mathrm{RF}(f')$ of the form $(\alpha_i - 1)\boldsymbol{e}_i - \boldsymbol{e}_k$.
(3) Assume $\mathrm{type}(H) = 3$ and $\mathrm{PF}(H) = \{f, f', \mathrm{F}(H)\}$ with $f + f' = \mathrm{F}(H)$. Then for every i, there is $k \neq i$ such that either $f + n_k = (\alpha_i - 1)n_i$ or $f' + n_k = (\alpha_i - 1)n_i$.

The following question is asked in [Mo].

Question 2.10. Is $\mathrm{type}(H)$ bounded for a given e if H is AS ? If this is the case, what is the upper bound?

3. On the free resolution of $K[H]$.

Let as before $H = \langle n_1, \dots, n_e \rangle$ be a numerical semigroup and $K[H] = S/I_H$ its semigroup ring over K.

We are interested in the minimal graded free S-resolution $(\mathcal{F}, d)$ of $K[H]$. For each i, we have $F_i = \bigoplus_j S(-\beta_{ij})$, where the β_{ij} are the graded Betti numbers of $K[H]$. Moreover, $\beta_i = \sum_j \beta_{ij} = \mathrm{rank}(F_i)$ is the ith Betti number of $K[H]$. Note that $\mathrm{proj\,dim}_S K[H] = e - 1$ and that $F_{e-1} \cong \bigoplus_{f \in \mathrm{PF}(H)} S(-f - N)$, where we put $N = \sum_{i=1}^{e} n_i$

Recall from Section 1 that R is almost symmetric if the cokernel of a natural morphism

$$R \to \omega_R(-\mathrm{F}(H))$$

[2] In our previous article [HW], Lemma 4.11 was not true. Hence our proof needs some more lemma.

is annihilated by the graded maximal ideal of $K[H]$. In other words, there is an exact sequence of graded S-modules

$$0 \to R \to \omega_R(-\mathrm{F}(H)) \to \bigoplus_{f \in \mathrm{PF}'(H)} K(-f) \to 0.$$

Note that, we used the symmetry of $\mathrm{PF}(H)$ given in Lemma 1.1 when H is almost symmetric.

Since $\omega_S \cong S(-N)$, the minimal free resolution of ω_R is given by the S-dual $\mathcal{F}^\vee$ of $\mathcal{F}$ with respect to $S(-N)$). Now, the injection $R \to \omega_R(-\mathrm{F}(H))$ lifts to a morphism $\varphi : \mathcal{F} \to \mathcal{F}^\vee(-\mathrm{F}(H))$, and the resolution of the cokernel of $R \to K_R(-\mathrm{F}(H))$ is given by the mapping cone $\mathrm{MC}(\varphi)$ of φ.

On the other hand, the free resolution of the residue field K is given by the Koszul complex $\mathbb{K} = \mathbb{K}(x_1, \ldots, x_e; K)$. Hence we get

Lemma 3.1. The mapping cone $\mathrm{MC}(\varphi)$ gives a (non-minimal) free S-resolution of $\bigoplus_{f \in \mathrm{PF}'(H)} K(-f)$. Hence, the minimal free resolution obtained from $\mathrm{MC}(\varphi)$ is isomorphic to $\bigoplus_{f \in \mathrm{PF}'(H)} K(-f)$

Let us discuss the case $e = 4$ in more details. For $K[H]$ with $t = \mathrm{type}(K[H])$ we have the graded minimal free resolution

$$0 \to \bigoplus_{f \in \mathrm{PF}(H)} S(-f-N) \to \bigoplus_{i=1}^{m+t-1} S(-b_i) \to \bigoplus_{i=1}^{m} S(-a_i) \to S \to K[H] \to 0$$

of $K[H]$. The dual with respect to $\omega_S = S(-N)$ shifted by $-F(H)$ gives the exact sequence

$$\begin{aligned} 0 \to S(-\mathrm{F}(H) - N) \quad &\to \bigoplus_{i=1}^{m} S(a_i - F(H) - N) \to \bigoplus_{i=1}^{m+t-1} S(b_i - F(H) - N) \\ &\to \bigoplus_{f \in \mathrm{PF}(H)} S(f - \mathrm{F}(H)) \to \omega_{K[H]}(-\mathrm{F}(H)) \to 0. \end{aligned}$$

Considering the fact that for the map $\varphi : \mathcal{F} \to \mathcal{F}^\vee$ the component

$$\varphi_0 : S \to \bigoplus_{f \in \mathrm{PF}(H)} S(f - F(H))$$

maps S isomorphically to $S(\mathrm{F}(H) - \mathrm{F}(H)) = S$, these two terms can be canceled against each others in the mapping cone. Similarly, via

$$\varphi_4 : \bigoplus_{f \in \mathrm{PF}(H)} S(-f-N) \to S(-\mathrm{F}(H) - N)$$

the summands $S(-\mathrm{F}(H)-N)$ can be canceled. Observing then that $\mathrm{PF}'(H) = \{\mathrm{F}(H)-f : f \in \mathrm{PF}'(H)\}$, we obtain the reduced mapping cone

$$\begin{aligned} 0 \to & \bigoplus_{f\in \mathrm{PF}'(H)} S(-f-N) \to \bigoplus_{i=1}^{m+t-1} S(-b_i) \to \bigoplus_{i=1}^{m} S(-a_i) \oplus \bigoplus_{i=1}^{m} S(a_i - F(H) - N) \\ \to & \bigoplus_{i=1}^{m+t-1} S(b_i - F(H) - N) \to \bigoplus_{f\in \mathrm{PF}'(H)} S(-f) \to \bigoplus_{f\in \mathrm{PF}'(H)} K(-f) \to 0, \end{aligned}$$

which provides a graded free resolution of $\bigoplus_{f\in \mathrm{PF}'(H)} K(-f)$. Comparing this resolution with the minimal graded free resolution of $\bigoplus_{f\in \mathrm{PF}'(H)} K(-f)$, which is

$$\begin{aligned} 0 \to & \bigoplus_{f\in \mathrm{PF}'(H)} S(-f-N) \to \bigoplus_{\substack{f\in \mathrm{PF}'(H)\\ 1\le i\le 4}} S(-f-N+n_i) \to \bigoplus_{\substack{f\in \mathrm{PF}'(H)\\ 1\le i<j\le 4}} S(-f-n_i-n_j) \\ \to & \bigoplus_{\substack{f\in \mathrm{PF}'(H)\\ 1\le i\le 4}} S(-f-n_i) \to \bigoplus_{f\in \mathrm{PF}'(H)} S(-f) \to \bigoplus_{f\in \mathrm{PF}'(H)} K(-f) \to 0, \end{aligned}$$

we notice that $m \geq 3(t-1)$. If $m = 3(t-1)$, then the reduced mapping cone provides a graded minimal free resolution of $\bigoplus_{f\in \mathrm{PF}'(H)} K(-f)$.

A comparison of the mapping cone with the graded minimal free resolution of $\bigoplus_{f\in \mathrm{PF}'(H)} K(-f)$ yields the following numerical result.

Proposition 3.2. Let H be a 4-generated almost symmetric numerical semigroup of type t for which I_H is generated by $m = 3(t-1)$ elements. Then, with the notation introduced, one has the following equalities of multisets:

$$\begin{aligned} &\{a_1, \ldots, a_m\} \cup \{\mathrm{F}(H) + N - a_1, \cdots, \mathrm{F}(H) + N - a_m\} \\ &= \{f + n_i + n_j \;\; f \in \mathrm{PF}'(H), 1 \le i < j \le 4\}, \end{aligned}$$

and

$$\{b_1, \ldots, b_{m+t-1}\} = \{f + N - n_i \;\; f \in \mathrm{PF}'(H), 1 \le i \le 4\}.$$

Theorem 3.3. Let H be a 4-generated almost symmetric numerical semigroup of type t for which I_H is generated by $m = 3(t-1)$ elements. Then I_H is generated by RF-relations.

Conjecture 3.4. Assume that H is AS with $\langle n_1, n_2, n_3, n_4\rangle$ and $\mathrm{type}(H) = 3$ with $\mathrm{PF}(H) = \{f, f', \mathrm{F}(H)\}$ with $f + f' = \mathrm{F}(H)$. Then I_H is minimally generated by 6 or 7 elements and 6 of the minimal generators are obtained with no cancellation from $\mathrm{RF}(f)$ or $\mathrm{RF}(f')$ as in Example 2.3. If $\mu(I_H) = 7$, then $X_1X_4 - X_2X_3 \in I_H$.

4. When is $H+m$ almost symmetric for infinitely many m?

Definition 4.1. For $H = \langle n_1, \dots, n_e \rangle$, we put $H+m = \langle n_1+m, \dots, n_e+m \rangle$. When we write $H+m$, we assume that $H+m$ is a numerical semigroup, that is, $\mathrm{GCD}(n_1+m, \dots, n_e+m) = 1$. In this section, we always assume that $n_1 < n_2 < \dots < n_e$. We put $s = n_e - n_1$ and $d = \mathrm{GCD}(n_2 - n_1, \dots, n_e - n_1)$.

First, we will give a lower bound of Frobenius number of $H+m$.

Proposition 4.2. For $m \gg 1$, $\mathrm{F}(H+m) \geq m^2/s$.

The following fact is trivial but very important in our argument.

Lemma 4.3. If $\phi = \prod_{i=1}^{e} X_i^{a_i} - \prod_{i=1}^{e} X_i^{b_i} \in I_H$ is homogeneous, namely, if $\sum_{i=1}^{e} a_i = \sum_{i=1}^{e} b_i$, then $\phi \in I_{H+m}$ for every m.

We define $\alpha_i(m)$ to be the minimal positive integer such that

$$\alpha_i(m)(n_i+m) = \sum_{j=1, j\neq i}^{e} \alpha_{ij}(m)(n_j+m),$$

as in Definition 4.5.

Lemma 4.4. Let $H+m$ be as in Definition 4.1. Then, if m is sufficiently big compared with $n_1, \dots, n_e$, then $\alpha_2(m), \dots, \alpha_{e-1}(m)$ is constant, $\alpha_1(m) \geq (m+n_1)/s$ and $\alpha_4(m) \geq (m+n_1)/s - 1$. Moreover, if we put $d = \mathrm{GCD}\{n_e - n_j \mid j = 1, \dots, e-1\}$ and $s' = s/d$, there is a constant C depending only on H such that $\alpha_1(m) - (m+n_1)/s' \leq C$ and $\alpha_4(m) - (m+n_1)/s' \leq C$.

Remark 4.5. By Lemma 4.4 $\alpha_i(m)$ does not depend on m for $m \gg 0$. Therefore we simply write $\alpha_i = \alpha_i(m)$ for $m \gg 0$ and $i = 1, \dots, e$.

Question 4.6. If we assume $H = \langle n_1, n_2, n_3, n_4 \rangle$ is almost symmetric of type 3, we have some examples of $d > 1$ and odd, like $H = \langle 20, 23, 44, 47 \rangle$ with $d = 3$ or $H = \langle 19, 24, 49, 54 \rangle$ with $d = 5$. But in all examples we know, at least one of the minimal generators is even. What does it mean is even? Is this true in general? Note that we have examples of 4 generated symmetric semigroup all of whose minimal generators are odd. What does it mean is odd?

$H+m$ is almost symmetric of type 2 for only finitely many m. We show

Theorem 4.7. Assume $H+m = \langle n_1+m, \dots, n_4+m \rangle$. Then for large enough m, $H+m$ is not almost symmetric of type 2.

We then give the classification of the numerical semigroups H such that $H+m$ is almost symmetric of type 3 for infinitely many m. Unlike the case of type 2, there are infinite series of $H+m$, which are almost symmetric of type 3 for infinitely many m. The first one of the following examples was given by T. Numata and the most basic one.

Example 4.8. For the following H, $H+m$ is almost symmetric with type 3 if

(1) $H = \langle 10, 11, 13, 14\rangle$, m is a multiple of 4.
(2) $H = \langle 10, 13, 15, 18\rangle$, m is a multiple of 8.
(3) $H = \langle 14, 19, 21, 26\rangle$, m is a multiple of 12.
(4) $H = \langle 18, 25, 27, 34\rangle$, m is a multiple of 16.

From now on, we assume that $H+m$ is almost symmetric of type 3 and assume that m is sufficiently bigger than n_1, n_2, n_3, n_4. We say some invariant $\sigma(m)$ (e.g. $\mathrm{F}(H+m), f(m), f'(m)$) of $H+m$ is $O(m^2)$ (resp. $O(m)$) if there is some positive constants $c < c'$ such that $cm^2 < \sigma(m) \le c'm^2$ (resp. $cm \le \sigma(m) \le c'm$) for all m.

Lemma 4.9. The invariants $\mathrm{F}(H+m)$ and $f'(m)$ are $O(m^2)$ and $f(m)$ is $O(m)$.

We assume that $H+m$ is almost symmetric of type 3 for infinitely many m and we will write $\mathrm{PF}(H+m) = \{f(m), f'(m), \mathrm{F}(H+m)\}$ with $f(m) < f'(m)$ and $f(m) + f'(m) = \mathrm{F}(H+m)$.

If $H+m$ is AS of type 3 for infinitely many m, we get the following Proposition. We also assume that H is AS of type 3, too.

Proposition 4.10. Assume $H+m$ is almost symmetric of type 3 for infinitely many m. We use notation as above and we put $d = \mathrm{GCD}(n_2 - n_1, n_3 - n_2, n_4 - n_3)$. If $H+m$ is almost symmetric of type 3 for sufficiently big m, then the following statements hold:

(1) We have $\alpha_2 = \alpha_3$ and $\alpha_1(m) = \alpha_4(m) + 1$. If we put $a = \alpha_2 = \alpha_3$, then

$$\mathrm{RF}(f(m)) = \begin{pmatrix} -1 & a-1 & 0 & 0 \\ 1 & -1 & a-2 & 0 \\ 0 & a-2 & -1 & 1 \\ 0 & 0 & a-1 & -1 \end{pmatrix},$$

(2) If we put $b = \alpha_1(m)$, then $\alpha_4(m) = b - d$ and

$$\mathrm{RF}(f'(m)) = \begin{pmatrix} -1 & 0 & 1 & b-d-2 \\ 0 & -1 & 0 & b-d-1 \\ b-1 & 0 & -1 & 0 \\ b-2 & 1 & 0 & -1 \end{pmatrix}.$$

(3) The integer $a = \alpha_2 = \alpha_3$ is odd and we have $n_2 = n_1 + (a-2)d, n_3 = n_1 + ad, n_4 = n_1 + (2a-2)d$.

Theorem 4.11. Assume that $H = \langle n_1, n_2, n_3, n_4\rangle$ with $n_1 < n_2 < n_3 < n_4$ and we assume that H and $H+m$ are almost symmetric of type 3 for infinitely many m. Then putting $d = \mathrm{GCD}(n_2 - n_1, n_3 - n_2, n_4 - n_3)$, $a = \alpha_2, b = \alpha_1$ and $\mathrm{PF}(H) = \{f, f', \mathrm{F}(H)\}$, H has the following characterization.

(1) a and d are odd, $\mathrm{GCD}(a, d) = 1$ and $b \geq d + 2$.

(2) $\mathrm{RF}(f)$ and $\mathrm{RF}(f')$ have the following form.

$$\mathrm{RF}(f) = \begin{pmatrix} -1 & a-1 & 0 & 0 \\ 1 & -1 & a-2 & 0 \\ 0 & a-2 & -1 & 1 \\ 0 & 0 & a-1 & -1 \end{pmatrix}, \mathrm{RF}(f') = \begin{pmatrix} -1 & 0 & 1 & b-d-2 \\ 0 & -1 & 0 & b-d-1 \\ b-1 & 0 & -1 & 0 \\ b-2 & 1 & 0 & -1 \end{pmatrix}.$$

(1) $n_1 = 2a+(b-d-2)(2a-2), n_2 = n_1+(a-2)d, n_3 = n_1+ad, n_4 = 2a+(b-2)(2a-2)$.

(3) If we put $H(a, b; d) = \langle n_1, n_2, n_3, n_4\rangle$, then $H(a, b+1; d) = H(a, b; d) + (2a - 2)$. Since $H(a, b; d)$ is almost symmetric of type 3 for every a, d odd, $\mathrm{GCD}(a, d) = 1$ and $b \geq d + 2$, it follows that $H(a, b; d) + m$ is almost symmetric of type 3 for infinitely many m.

(4) $I_H = (xw - yz, y^a - x^2z^{a-2}, z^a - y^{a-2}w^2, xz^{a-1} - y^{a-1}w, x^b - z^2w^{b-d-2}, w^{b-d} - x^{b-2}y^2, x^{b-1}y - zw^{b-d-1}))$.

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