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A tensor product of certain two simple modules for finite Chevalley groups

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1 Introduction

Let k be an algebraically closed field of prime characteristic $p > 0$, and let G be a simply connected and simple algebraic k-group which is defined and split over the finite field $\mathbb{F}_p$ (for example, $G = \mathrm{SL}_{l+1}(k)$ for type A_l and $G = \mathrm{Sp}_{2l}(k)$ for type C_l).

The representation theory of G plays an important role to study that of the corresponding finite Chevalley groups $G(p^r)$. Indeed, a simple $kG(p^r)$-module can be obtained by restricting a simple rational G-module, and a projective indecomposable $kG(p^r)$-module also can be obtained by restricting a certain rational G-module.

In this article, we give some formulas on a direct sum decomposition of a tensor product of the r-th Steinberg module and a 'small' simple $kG(p^r)$-module, using the representation theory of G.

2 Preliminaries

We shall use the following standard notation:

(1) T : maximal split torus of G
(2) $X = X(T) := \mathrm{Hom}(T, k^\times)$: character group
(3) Φ : root system relative to the pair (G, T)
(4) $\Delta := \{\alpha_1, \cdots, \alpha_l\}$: set of simple roots for the ordering as in [2]
(5) Φ^+ : set of positive roots containing Δ
(6) α_0 : highest short root in Φ^+
(7) s_α : reflection for $\alpha \in \Phi^+$ in $\mathbb{E} := X \otimes_{\mathbb{Z}} \mathbb{R}$
(8) $W := N_G(T)/T = \langle s_\alpha | \alpha \in \Delta \rangle$: Weyl group
(9) w_0 : longest element of W
(10) $\pi : G \to G$: graph automorphism (which induces $\pi : X \to X$)
(11) $F : G \to G$: standard Frobenius map relative to $\mathbb{F}_p$ (if $G = \mathrm{SL}_{l+1}(k)$, then $F : (a_{ij}) \mapsto (a_{ij}^p)$)
(12) $G(p^r) := \{g \in G \mid F^r(\pi(g)) = g\}$: finite Chevalley group (for example, if $G = \mathrm{SL}_{l+1}(k)$, then $G(p^r) = \mathrm{SL}_{l+1}(\mathbb{F}_{p^r})$ if $\pi = id$ and $G(p^r) = \mathrm{SU}_{l+1}(\mathbb{F}_{p^{2r}})$ if $\pi \neq id$)
(13) $\langle \cdot, \cdot \rangle$: W-invariant inner product on $\mathbb{E} = X \otimes_{\mathbb{Z}} \mathbb{R}$
(14) $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$: coroot of $\alpha \in \Phi$
(15) ω_i : fundamental weight with $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$ (then $X = \sum_{i=1}^{l} \mathbb{Z}\omega_i$)

(16) $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha = \sum_{i=1}^{l}\omega_i$
(17) $X^+ := \sum_{i=1}^{l}\mathbb{Z}_{\geq 0}\omega_i$: set of dominant weights

In this article, all modules are assumed to be finite-dimensional and even rational for an algebraic group. For a T-module V, set $V_\lambda := \{v \in V | tv = \lambda(t)v, \forall t \in T\}$. If $V_\lambda \neq 0$, this space is called the *weight space* in V of *weight* $\lambda \in X$. Then there is a direct sum decomposition $V = \bigoplus_{\lambda\in X} V_\lambda$. Let $V^{[i]}$ be the i-th Frobenius twist for a G-module V. Let $L(\lambda)$ be the simple G-module with highest weight $\lambda \in X^+$. Then $\{L(\lambda)|\lambda \in X^+\}$ is a set of all non-isomorphic simple G-modules, where $L(0) \cong k$ is trivial and $L((p^n-1)\rho) = \mathrm{St}_n$ is called the *n-th Steinberg module.* For $n \in \mathbb{Z}_{>0}$, set $X_n := \{\sum_{i=1}^{l} c_i\omega_i \in X^+ | c_i < p^n, \forall i\}$, whose elements are called *p^n-restricted weights.* The symbol $\otimes$ denotes a tensor product over k.

Theorem 1 (Steinberg). *Consider $\lambda \in X^+$ and its p-adic expansion $\lambda = \sum_{i=0}^{n-1} p^i\lambda_i$ ($\lambda_i \in X_1$). Then*

$$L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes \cdots \otimes L(\lambda_{n-1})^{[n-1]}$$

as G-modules.

It is well-known that $\{L(\lambda)|\lambda \in X_r\}$ is a set of all non-isomorphic simple $kG(p^r)$-modules, where $L((p^r-1)\rho) = \mathrm{St}_r$ is the unique simple projective $kG(p^r)$-module. Let $U_r(\lambda)$ be the projective cover of the simple $kG(p^r)$-module $L(\lambda)$ ($\lambda \in X_r$).

3 Minuscule weights and (p, a, r)-minuscule weights

Definition 1. $\lambda \in X^+$ is *minuscule* if $\langle \lambda, \alpha_0^\vee \rangle = 1$.

Remark. A minuscule weight is one of the following:

- $\omega_1, \ldots, \omega_l$ in type A_l ($l \geq 1$),
- ω_l in type B_l ($l \geq 2$),
- ω_1 in type C_l ($l \geq 3$),
- $\omega_1, \omega_{l-1}, \omega_l$ in type D_l ($l \geq 4$),
- ω_1, ω_6 in type E_6,
- ω_7 in type E_7.

Definition 2. Consider $\lambda \in X_r$ and its p-adic expansion $\lambda = \sum_{i=0}^{r-1} p^i\lambda_i$ ($\lambda_i \in X_1$). Suppose that $a \in \mathbb{Z}_{\geq 0}$ satisfies $a \leq p$. Then
(i) λ is *(p, a, r)-minuscule* if $\langle \lambda_i, \alpha_0^\vee \rangle \leq a$ for each i.

(ii) λ is (p,r)-*minuscule* if it is (p,p,r)-minuscule.

Remark. If λ is (p,a,r)-minuscule, then clearly it is (p,r)-minuscule.

Example 1. Let $G = \mathrm{SL}_{l+1}(k)$. Then we have $\alpha_0 = \alpha_1 + \cdots + \alpha_l$ and $\langle \omega_i, \alpha_0^\vee \rangle = 1$ $(1 \le i \le l)$, and the $(p,a,1)$-minuscule weights $(a \le p)$ are

$$\{\sum_{i=1}^{l} c_i\omega_i | 0 \le c_i \le p-1,\ \sum_{i=1}^{l} c_i \le a\}.$$

4 Formal characters

Let $\mathbb{Z}X$ be the $\mathbb{Z}$-group algebra of X with basis $\{e(\lambda) \mid \lambda \in X\}$, whose multiplication is defined by $e(\lambda)e(\mu) = e(\lambda+\mu)$ for $\lambda, \mu \in X$.

Definition 3. For a T-module V, define the *(formal) character* of V as

$$\mathrm{ch}(V) := \sum_{\lambda \in X} (\dim_k V_\lambda) e(\lambda) \in \mathbb{Z}X.$$

Proposition 1. *Let V_1 and V_2 be T-modules. Then the following holds.*
(i) $\mathrm{ch}(V_1 \oplus V_2) = \mathrm{ch}(V_1) + \mathrm{ch}(V_2)$.
(ii) $\mathrm{ch}(V_1 \otimes V_2) = \mathrm{ch}(V_1) \cdot \mathrm{ch}(V_2)$.
(iii) *If V_1 and V_2 are G-modules, then* $\mathrm{ch}(V_1) = \mathrm{ch}(V_2)$ *if and only if V_1 and V_2 have the same composition factors with multiplicity.*

For $\lambda \in X^+$, set $s(\lambda) := \sum_{\mu \in W\lambda} e(\mu) \in \mathbb{Z}X$.

Proposition 2. *If $\lambda \in X^+$ is minuscule, then* $\mathrm{ch}(L(\lambda)) = s(\lambda)$.

For $\lambda \in X^+$ and its p-adic expansion $\lambda = \sum_{i=0}^{n-1} p^i \lambda_i$, set

$$s_n(\lambda) := s(\lambda_0) s(\lambda_1)^{[1]} \cdots s(\lambda_{n-1})^{[n-1]},$$

where $s(\mu)^{[i]} = s(p^i\mu)$.

Lemma 1. *For a (p,r)-minuscule weight $\mu \in X_r$, the character* $\mathrm{ch}(L(\mu))$ *can be written uniquely as*

$$\mathrm{ch}(L(\mu)) = \sum_{\kappa \in X_r} b_\kappa s_r(\kappa) \quad (b_\kappa \in \mathbb{Z}_{\ge 0}).$$

Example 2. Let $G = \mathrm{SL}_3(k)$, $p = 3$, and

$$X = \{c_1\omega_1 + c_2\omega_2 | c_1, c_2 \in \mathbb{Z}\} = \{(c_1, c_2) | c_1, c_2 \in \mathbb{Z}\}.$$

Take $(6,1)=(0,1)+3(2,0)\in X_2$. Then

$$\mathrm{ch}(L(0,1)) = e(0,1)+e(1,-1)+e(-1,0)=s(0,1),$$

$$\begin{aligned}\mathrm{ch}(L(2,0)) &= e(2,0)+e(-2,2)+e(0,-2)\\ &\quad +e(0,1)+e(1,-1)+e(-1,0)\\ &= s(2,0)+s(0,1),\end{aligned}$$

$$\begin{aligned}\mathrm{ch}(L(6,1)) &= \mathrm{ch}(L(0,1)\otimes L(2,0)^{[1]})\\ &= \mathrm{ch}(L(0,1))\cdot \mathrm{ch}(L(2,0)^{[1]})\\ &= s(0,1)\cdot (s(2,0)^{[1]}+s(0,1)^{[1]})\\ &= s_2(6,1)+s_2(0,4).\end{aligned}$$

5 Main results

The following is a result by Anwar in 2011, which gives a direct sum decomposition of a tensor product of the r-th Steinberg module St_r and a simple G-module $L(\lambda)$ for a (p,r)-minuscule weight λ.

Theorem 2 ([1, Theorem 2]). *Suppose that $\lambda\in X_r$ is (p,r)-minuscule, and let* $\mathrm{ch}(L(\lambda))=\sum_{\kappa\in X_r} b_\kappa s_r(\kappa)$ *with $b_\kappa\in\mathbb{Z}_{\geq 0}$. Then*

$$\mathrm{St}_r\otimes L(\lambda)\cong \bigoplus_{\kappa\in X_r} b_\kappa T((p^r-1)\rho+\kappa)$$

as G-modules, where $T(\mu)$ is the indecomposable tilting module with highest weight $\mu\in X^+$.

Now we describe the main results, which are analogous to Theorem 2 for $kG(p^r)$. These are also generalizations of Tsushima's results in [3].

Theorem 3 ([4, Theorem 3.4]). *Suppose that $\lambda\in X_r$ is $(p,p-1,r)$-minuscule, and let $\mathrm{ch}(L(\lambda))=\sum_{\kappa\in X_r} b_\kappa s_r(\kappa)$ with $b_\kappa\in\mathbb{Z}_{\geq 0}$. Then*

$$\mathrm{St}_r\otimes L(\lambda)\cong \left(\bigoplus_{\kappa\in X_r} b_\kappa U_r((p^r-1)\rho+w_0\kappa)\right)\oplus \varepsilon\mathrm{St}_r$$

as $kG(p^r)$-modules, where

$$\varepsilon=\begin{cases} 1 & \text{if } \lambda=(p^r-1)\omega \text{ for some minuscule weight } \omega,\\ 0 & \text{otherwise.}\end{cases}$$

Theorem 4 ([4, Theorem 3.3]). *Suppose that $\lambda \in X_r$ is (p,r)-minuscule, and let $\mathrm{ch}(L(\lambda)) = \sum_{\kappa \in X_r} b_\kappa s_r(\kappa)$ with $b_\kappa \in \mathbb{Z}_{\geq 0}$. Moreover, for a p-adic expansion $\lambda = \sum_{i=0}^{r-1} p^i \lambda_i$, suppose that there exists an integer $j \in \{0, 1, \cdots, r-1\}$ such that $\langle \lambda_j, \alpha_0^\vee \rangle \leq p-1$ and $\lambda_j \neq (p-1)\omega$ for any minuscule weight ω. Then*

$$\mathrm{St}_r \otimes L(\lambda) \cong \bigoplus_{\kappa \in X_r} b_\kappa U_r((p^r-1)\rho + w_0\kappa)$$

as $kG(p^r)$-modules.

Example 3. Consider $G(p^r) = \mathrm{SL}_3(\mathbb{F}_{5^2})$ $(l=2, p=5, r=2)$. Then $\Delta = \{\alpha_1, \alpha_2\}$,

$$X = \{c_1\omega_1 + c_2\omega_2 | c_1, c_2 \in \mathbb{Z}\} = \{(c_1, c_2) | c_1, c_2 \in \mathbb{Z}\},$$

$$X^+ = \{(c_1, c_2) | c_1, c_2 \in \mathbb{Z}_{\geq 0}\}$$

and

$$X_2 = \{(c_1, c_2) | c_1, c_2 \in \{0, 1, \cdots, 24\}\}.$$

Since

$$\begin{aligned}
&\mathrm{ch}(L(3,11)) \\
&= \mathrm{ch}(L(3,1) \otimes L(0,2)^{[1]}) \\
&= (s(3,1) + s(1,2) + s(2,0) + s(0,1)) \cdot (s(0,2)^{[1]} + s(1,0)^{[1]}) \\
&= s_2(3,11) + s_2(1,12) + s_2(2,10) + s_2(0,11) \\
&\quad + s_2(8,1) + s_2(6,2) + s_2(7,0) + s_2(5,1),
\end{aligned}$$

$$\begin{aligned}
&\mathrm{ch}(L(15,11)) \\
&= \mathrm{ch}(L(0,1) \otimes L(3,2)^{[1]}) \\
&= s(0,1) \cdot (s(3,2)^{[1]} + s(1,3)^{[1]} + s(4,0)^{[1]} \\
&\quad + 2s(2,1)^{[1]} + 2s(0,2)^{[1]} + 2s(1,0)^{[1]}) \\
&= s_2(15,11) + s_2(5,16) + s_2(20,1) + 2s_2(10,6) + 2s_2(0,11) + 2s_2(5,1),
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{ch}(L(24,0)) \\
&= \mathrm{ch}(L(4,0) \otimes L(4,0)^{[1]}) \\
&= (s(4,0) + s(2,1) + s(0,2) + s(1,0)) \\
&\quad \cdot (s(4,0)^{[1]} + s(2,1)^{[1]} + s(0,2)^{[1]} + s(1,0)^{[1]}) \\
&= s_2(24,0) + s_2(22,1) + s_2(20,2) + s_2(21,0) + s_2(14,5) + s_2(12,6) \\
&\quad + s_2(10,7) + s_2(11,5) + s_2(4,10) + s_2(2,11) + s_2(0,12) + s_2(1,10) \\
&\quad + s_2(9,0) + s_2(7,1) + s_2(5,2) + s_2(6,0),
\end{aligned}$$

we obtain

$$\begin{aligned}\mathrm{St}_2 \otimes L(3,11) &\cong U_2(13,21) \oplus U_2(12,23) \oplus U_2(14,22) \oplus U_2(13,24)\\ &\quad \oplus U_2(23,16) \oplus U_2(22,18) \oplus U_2(24,17) \oplus U_2(23,19)\end{aligned}$$

and

$$\begin{aligned}&\mathrm{St}_2 \otimes L(24,0)\\ &\cong U_2(24,0) \oplus U_2(23,2) \oplus U_2(22,4) \oplus U_2(24,3) \oplus U_2(19,10) \oplus U_2(18,12)\\ &\quad \oplus U_2(17,14) \oplus U_2(19,13) \oplus U_2(14,20) \oplus U_2(13,22) \oplus U_2(12,24)\\ &\quad \oplus U_2(14,23) \oplus U_2(24,15) \oplus U_2(23,17) \oplus U_2(22,19) \oplus U_2(24,18) \oplus \mathrm{St}_2\end{aligned}$$

by Theorem 3, and

$$\begin{aligned}\mathrm{St}_2 \otimes L(15,11) &\cong U_2(13,9) \oplus U_2(8,19) \oplus U_2(23,4) \oplus 2U_2(18,14)\\ &\quad \oplus 2U_2(13,24) \oplus 2U_2(23,19)\end{aligned}$$

by Theorem 4.

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