164/5 and 236/7

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Abstract

We show that there does not exist a C_2-*cofinite and rational* vertex operator algebra of CFT type with central charge either 164/5 or 236/7 satisfying conditions (a) the space of characters of simple modules is contained in the space of solutions of a modular linear differential equation of order 3, (b) the graded subspace with weight 1 is trivial.

The central charges 164/5 and 236/7 first appeared in the paper by Tuite and Van in the classification of central charges of *exceptional* vertex operator algebras with lowest primary weight 2, which satisfy the conditions (a) and (b). Later we rediscovered these central charges in the characterization problem of the minimal model. By theorems in this paper our characterization problem is completely proved.

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In the theory of vertex operator algebras (simply VOAs), the notions of C_2-cofiniteness (we use terminology "finite type" in this paper) and rationality play an important role. Many significant results (e.g., the modular invariance property of characters of simple modules([13]) and the Verlinde formula [8], and so on) are proved under these conditions together with some mild conditions. It is then very natural to study the classification of rational VOAs of finite type. However, it seems to be impossible to complete this problem without additional conditions. One of typical conditions, which we take in this paper (and previous papers) is that we first fix an order of a modular linear differential equation (simply MLDE) whose space of solutions contains the vector space which is linearly generated by characters of simple modules of a VOA (cf. [2], [3], [7] and [9]). Therefore, our classification problem will be studied under the condition that the space of characters of simple modules is *contained* in the space of solutions of an MLDE of order n, but is vector subspace of the spaces of solutions of MLDEs of order less than the n. We denote this condition by $(\mathrm{MD})_n$.

An MLDE (of weight 0) is a linear differential equation $\mathfrak{d}^n(f)+\sum_{i=0}^{n-2} P_i\mathfrak{d}^i(f)=0$ defined by the Serre derivation $\mathfrak{d}: M_*(\Gamma_1)\to M_{*+2}(\Gamma_1)$, where $\mathfrak{d}(f)=qd/dq(f)-(k/12)E_2(f)$ for any $f\in M_k(\Gamma_1)$. The function E_2 is the normalized Eisenstein series of weight 2 and P_i is a holomorphic modular form of weight $2(n-i)$. The MLDEs naturally arise in the theory of VOAs as explained below. As an important process to prove modular invariance property of characters, Y. Zhu ([13]) showed that there exists a linear differential equation with a regular singularity whose space of solutions contains the vector space linearly generated by characters of simple modules (see also [5], [13]), which was used for showing convergence of the characters.

The classification of rational VOAs of finite type satisfying the condition $(\mathrm{MD})_2$ is studied in [9]. It was found in [9] that there are 10 rational numbers which can be central charges in the sense that any solutions of the corresponding MLDEs have non-negative integral coefficients. It is also shown that all but one of these rational numbers have the corresponding VOAs and that these VOAs are one of the Virasoro minimal model with central charge $-22/5$ and the affine VOAs with level 1 associated with the Deligne exceptional series

$$A_1\subset A_2\subset G_2\subset D_4\subset F_4\subset E_6\subset E_7\subset E_8.$$

The remaining central charge (which is 38/5) ha no corresponding rational VOAs of finite type because the "Verlinde formula" gives a negative fusion coefficient [10].

We concern the classification of (central charges of) rational VOAs of finite type whose associated MLDEs satisfy $(\mathrm{MD})_3$. Unlike the situation of $n=2$, there are *infinitely many* such VOAs. In fact, the affine VOA $L_{B_\ell,1}$ for each $\ell\geq 2$ satisfies $(\mathrm{MD})_3$ (see [1]). Therefore, extra conditions will be needed to a obtain finite number of central charges.

Tuite introduced the notion of the *exceptional* VOAs V with *lowest primary weight* ℓ (see [11], [12] for the definition), where ℓ is the smallest integer such that $(V^\omega)_n=V_n$ for $n<\ell$ and $(V^\omega)_\ell\subsetneq V_\ell$ (where V^ω is the vertex operator subalgebra of V generated by the Virasoro element ω of V). Let V be an exceptional VOA which is not necessary of finite type. Then it is shown in [12] that any characters of simple modules of V with lowest *primary weight* ℓ is a solution of an MLDE of order $\ell+1$. By studying these MLDEs, they listed rational numbers which can be central charges of exceptional VOAs with lowest primary weight $\ell<9$ ([12]) and found corresponding VOAs for almost all these rational numbers (except 164/5 and 236/7). It is also shown in [12] that the exceptional VOAs with lowest primary weight 2 satisfy the condition $(\mathrm{MD})_3$ (No. 3–No. 11 in Table 1). The Virasoro minimal models are not in the list of [12] since they do not contain primary vectors.

In the classification of central charges of exceptional VOAs done by Tuite and Van, it is assumed $V_n=(V^\omega)_n$ for $n<\ell-1$ to obtain finite possible central charges of simple, rational VOAs of finite and CFT type of CFT type satisfying $(\mathrm{MD})_{\ell-1=k}$. However, [2] and [7] have obtained the list under the single condition $V_1=(V^\omega)_1=0$ and shown that these rational numbers can be central charges of simple, rational VOAs V of CFT and finite type whose spaces of characters coincide with the spaces of solutions of MLDEs of order 3 . Moreover, the corresponding VOAs whose have the central charge in the list (except 164/5

No.	Central charge	VOA
1	$-68/7$	$L(-68/7,0)$
2	$1/2$	$L(1/2,0)$
3	$-44/5$	$L(-22/5,0)\otimes L(-22/5,0)$
4	8	$V^+_{\sqrt{2}E_8}$
5	16	$V^+_{BW_{16}}$
6	$47/2$	$VB^\natural_{\mathbb{Z}}$
7	24	$V^\natural$
8	32	$V_L^+\oplus(V_L)_T^+$ where L is extremal.
9	$164/5$	unknown
10	$236/7$	unknown
11	40	$V_L^+\oplus(V_L)_T^+$ where L is extremal.

Table 1: Central charges and corresponding VOAs

and 236/7)are completely determined (No. 1–No. 6, No. 9 and No. 10 in Table 1). According to the results of [2], [7], [11] and [12], the central charges of simple, rational VOAs of CFT and finite type satisfying the condition $(\mathrm{MD})_3$ are given by Table 1 except two rational numbers. It is also shown in [2],[7] and [12] that there exists a rational VOA of finite type for each central charge in Table 1 except 164/5 and 236/7. This list contains the Virasoro minimal models (No. 1 and No. 2), the 2-fold tensor product of the Virasoro minimal model with the central charge $-22/5$ (No. 3), $\mathbb{Z}_2$-orbifold models of lattice VOAs (No. 4 and No. 5), the baby monster VOA $VB^\natural_{\mathbb{Z}}$ (No. 6, see [6]), the moonshine VOA $V^\natural$ (No. 7) and holomorphic VOAs constructed by $\mathbb{Z}_2$-orbifold construction from lattice VOAs associated with the extremal unimodular lattices (No. 8 and No. 11).

Our purpose of this paper is to give a sketch only that there does not exist any simple, rational VOAs of CFT and finite type whose central charge is either 164/5 (No. 9) or 236/7 (No. 10) in Table 1.

We prove our main result by estimating the *global dimensions* of VOAs, which is defined in [4] as the square sum of quantum dimensions of simple V-modules (see [4] for more details). Let V be a simple, rational VOA of finite and CFT type. It is shown in [4] that if any conformal weight of a simple V-module except V is positive, then the quantum dimension of each simple V-module is not smaller than 1 and the global dimension of V coincides with $1/S_{00}^2$, where S_{00} is the $(0,0)$-entry of the S-matrix of V.

Let V be a simple VOA of CFT type. Suppose that V satisfies $(\mathrm{MD})_3$ and that V is of finite type and rational. If a central charge V is either 164/5 or 236/7, then the exact form of the corresponding MLDE is already known ([2] [7] and [12]). The solutions of each MLDE which are found in [2], which are homogeneous polynomials of characters of simple modules of the Virasoro minimal models $L(-22/5,0)$ and $L(-68/7,0)$ are found. We can show that there are at least 3 simple V-modules (except V) whose conformal weights are positive. Since the quantum dimensions of simple V-modules are not smaller than one by [4], we see that $\mathrm{glob}(V)$ is not smaller than 2 by the definition of global dimensions.

The global dimension of V described above is determined explicitly. By using these expressions, we can obtain the S-transformation (all components of the S-matrix) of solutions with the help of a computer. It follows from the formula $\mathrm{glob}(V) = 1/S_{00}^2$ that the global dimensions of V with central charges $164/5$ and $236/7$ are $100/(5+\sqrt{5})^2 \approx 1.90983$ and $7/\cos^2(3\pi/14) \approx 2.86294$, respectively. This gives our main theorems.

Theorem 1. *Let V be a simple vertex operator algebra of CFT type satisfying $V_1 = 0$ whose central charge is $164/5$. Suppose that the space of characters is contained in the space of solutions of a modular linear differential equation of order 3. Then*

(a) *Any character of a simple V-module is a solution of the modular differential equation*

$$f''' - \frac{1}{2}E_2 f'' + \left(\frac{1}{2}E_2' - \frac{169}{100}E_4\right) f' + \frac{1271}{1080}E_6 f = 0\,, \quad ' = q\frac{d}{dq} = \frac{1}{2\pi\sqrt{-1}}\frac{d}{d\tau}\,. \tag{1}$$

(b) *The vertex operator algebra V is not of finite type and rational.*

Theorem 2. *Let V be a simple vertex operator algebra of CFT type satisfying $V_1 = 0$ whose central charge is central charge $236/7$. Suppose that the space of characters is contained in the space of solutions of a modular linear differential equation of order 3. Then*

(a) *Any character of a simple V-module is a solution of the modular differential equation*

$$f''' - \frac{1}{2}E_2 f'' + \left(\frac{1}{2}E_2' - \frac{149}{84}E_4\right) f' + \frac{93869}{74088}E_6 f = 0\,. \tag{2}$$

(b) *The vertex operator algebra V is not of finite type and rational.*

Let f_1, f_2 and f_3 be solutions of the MLDE (1) whose exponents are $-41/30$, $5/6$ and $31/30$, respectively. Then we have

$$\begin{aligned} f_1 &= q^{-41/30}\Big(1 + 90118q^2 + 53459408q^3 + \cdots\Big)\,, \\ f_2 &= 11271q^{5/6}\Big(8 + 2915q + 266160q^2 + \cdots\Big)\,, \\ f_3 &= 5084q^{31/30}\Big(121 + 30008q + 2304726q^2 + \cdots\Big)\Big)\,. \end{aligned} \tag{3}$$

Let g_1, g_2 and g_3 be solutions of the MLDE (2) whose exponents are $-59/42$, $37/42$ and $43/42$, respectively.

$$\begin{aligned} g_1 &= q^{-59/42}\left(1 + 63366q^2 + 46421200q^3 + \cdots\right)\,, \\ g_2 &= 31093q^{37/42}\left(23 + 8288q + 774410q^2 + \cdots\right)\,, \\ g_3 &= 3422q^{43/42}\left(248 + 67983q + 5611328q^2 + \cdots\right)\,. \end{aligned}$$

Let ϕ and ψ be the characters of the Virasoro minimal model $L(c_{2,5}, 0)$ whose central charge is $c_{2,5} = -22/5$ and its simple module whose conformal weight is $-1/5$, respectively. Let x, y and z be the characters of the simple modules of the Virasoro minimal model $L(c_{2,7}, 0)$ ($c_{2,7} = -68/7$) with conformal weights 0, $-2/7$ and $-3/7$, respectively.

It was observed in [7, (3.8) and Table 2] that the j-invariant

$$j = 1728E_4^3/(E_4^3 - E_6^2) = q^{-1} + 744 + 196884q + O(q^2)$$

is expressed as polynomials in $\{f_i, \psi, \phi\}$ and $\{g_i, \mathrm{x}, y, z\}$ in two ways, respectively. The expected appearance is given by

$$j - 744 = \psi^2 f_1 - 50\phi\psi f_2 + \phi^2 f_3, \quad (4)$$

$$j - 744 = xg_1 - yg_2 + zg_3. \quad (5)$$

Theorem 3. *We have $j - 744 = \psi^2 f_1 - 50\phi\psi f_2 + \phi^2 f_3$ and $j - 744 = xg_1 - yg_2 + zg_3$. The j-invariant $j - 744$ is a homogeneous polynomial in ϕ and ψ of degree* 84 *and a homogeneous polynomial in x, y and z of degree* 60.

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