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Fundamental Domains of Arithmetic Quotients of the General Linear Group and Humbert Forms

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1 Introduction

This is a résumé that covers the main results in the article *Fundamental domains of arithmetic quotients of reductive groups over number fields (with appendix by Takao Watanabe)* [7]. The paper mainly focuses on the determination and construction of fundamental domains associated to certain arithmetic quotients of reductive algebraic groups over an algebraic number field $\mathbf{k}$.

Definition. Let T be a locally compact Hausdorff space and Γ a discrete group with a properly discontinuous action on T. A subset Ω of T satisfying

(i) $T=\Gamma\Omega^{-}$,

(ii) $\Omega^{\circ}\cap\gamma\Omega^{-}=\emptyset$ for all $\gamma\in\Gamma\setminus\{e\}$

is called a **fundamental domain of T with respect to Γ** or just a **fundamental domain of $\Gamma\backslash T$** (T/Γ in the case of a right action). Here $\Omega^{\circ},\Omega^{-}$ denote the interior and closure of Ω in T respectively.

In particular we study arithmetic quotients of GL_n, and the results of which are used to construct fundamental domains for P_n, the cone of positive definite Humbert forms over $\mathbf{k}$, with respect to arithmetic subgroups of $GL_n(\mathbf{k})$.

For the first part of the paper we consider a general connected reductive isotropic algebraic group G over $\mathbf{k}$ and investigate fundamental domains of the quotients $G(\mathbf{k})\backslash G(\mathbf{A})^1$ and $\Gamma_i\backslash G(\mathbf{k}_\infty)^1$ with arithmetic subgroups $\Gamma_1,\ldots,\Gamma_{n_G}$ of $G(\mathbf{k})$ (n_G: the class number of G).

The results here are an extension of Watanabe's results in [9]. A maximal $\mathbf{k}$-parabolic subgroup of G, Q, is taken and we define the Ryshkov domain of G associated to Q, R_Q. This was introduced in [9] for the purpose of constructing a fundamental domain for $G(\mathbf{k})\backslash G(\mathbf{A})^1$ well-matched with the Hermite function of Q, m_Q. Watanabe also considered the case when G is of class number 1, and obtained a fundamental domain for $G(\mathbf{k}_\infty)$ with respect to $G_{\mathcal{O}}=G(\mathbf{k})\cap G_{\mathbf{A},\infty}$ (this coincides with Γ_1). Here however, we will consider algebraic groups of any general class number n_G.

The second topic of interest in this paper is the special case when G is the general linear group GL_n defined over $\mathbf{k}$. It is well known that the class number of G in this case is equal to h, the class number of $\mathbf{k}$. The Γ_i in this case are the subgroups of $GL_n(\mathbf{k})$ stabilizing certain $\mathcal{O}$-lattices in $\mathbf{k}^n$.

In the final section we proceed onto P_n, the space of positive definite Humbert forms over $\mathbf{k}_\infty$, with the usual identification $P_n=\prod_\sigma P_n(\mathbf{k}_\sigma)$ where $P_n(\mathbf{k}_\sigma)$ denotes the set of n by n positive-definite real symmetric/complex Hermitian matrices depending on whether σ is a real/imaginary, the product taken over all infinite places σ of $\mathbf{k}$.

When $\mathbf{k}=\mathbb{Q}$, P_n is just the cone of positive-definite real symmetric matrices, and fundamental domains for $P_n/GL_n(\mathbb{Z})$ in this case have been historically constructed by Korkin and Zolotarev [6], Minkowski [8] and later on Grenier [4]. For P_n over a general number field, in [5] Humbert has previously provided a fundamental domain constructed with respect to the particular group $GL_n(\mathcal{O})$. As $GL_n(\mathcal{O})$ coincides with one of the Γ_i we study in this paper, the question can be raised about fundamental domains for P_n with respect to each of the groups Γ_i when $n_G>1$.

As such we proceed in the final sections to provide a general way of constructing fundamental domains for P_n/Γ_i given any number field. The method of construction follows and generalizes the example given by Watanabe in [9] for the specific case $\mathbf{k}=\mathbb{Q}$. As already noted in [9], when $\mathbf{k}=\mathbb{Q}$ the fundamental domain for $P_n/GL_n(\mathbb{Z})$ resulting from this method coincides with Grenier's ([4]). It was observed by Dutour Sikirić and Schürmann that this fundamental domain is in fact equivalent to the one previously developed by Korkin and Zolotarev. Regarding $P_n/GL_n(\mathcal{O})$ for general number fields however, we note that the fundamental domain produced by the method here differs from Humbert's construction which utilizes the matrix trace, whereas the domain here is defined using the adele norm of matrix determinants.

Notation

We fix $\mathbf{k}$, an algebraic number field of finite degree over $\mathbb{Q}$, and denote its ring of integers by $\mathcal{O}$ and the adele ring by $\mathbf{A}$. $\mathbf{p}_\infty$ and $\mathbf{p}_f$ denote the sets of infinite and finite places of $\mathbf{k}$ respectively and we let $\mathbf{p} = \mathbf{p}_\infty \cup \mathbf{p}_f$. $\mathbf{k}_\infty$ denotes the usual étale $\mathbf{R}$-algebra $\mathbf{k} \otimes_{\mathbf{Q}} \mathbf{R}$ which we identify with $\prod_{\sigma \in \mathbf{p}_\infty} \mathbf{k}_\sigma$.

2 Fundamental domains of $G(\mathbf{k})\backslash G(\mathbf{A})^1$ and $\Gamma_i\backslash G(\mathbf{k}_\infty)^1$

2.1 The Ryshkov domain of G associated to Q

Let G be a connected reductive isotropic affine algebraic group defined over $\mathbf{k}$. Fix a minimal $\mathbf{k}$-parabolic subgroup of G and let Q be a proper maximal $\mathbf{k}$-parabolic subgroup of G containing it.

Definition ([9, §4]). The **Ryshkov domain of G associated to Q** is defined by

$$R_Q := \{g \in G(\mathbf{A})^1 \mid \mathsf{m}_Q(g) = H_Q(g)\}$$

where $H_Q : G(\mathbf{A}) \to \mathbb{R}_{>0}$ and $\mathsf{m}_Q : G(\mathbf{A})^1 \to \mathbb{R}_{>0}$ are respectively the **height function** and **Hermite function** associated to Q given by

$$\begin{aligned} H_Q(umh) &:= |\chi_Q(m)|_{\mathbf{A}}^{-1} \quad (u \in U(\mathbf{A}), m \in M(\mathbf{A}), h \in K), \\ \mathsf{m}_Q(g) &:= \min_{x \in Q(\mathbf{k})\backslash G(\mathbf{k})} H_Q(xg). \end{aligned}$$

Here,

- U and M: the unipotent radical and Levi subgroup of Q,
- K: maximal compact subgroup of $G(\mathbf{A})$,
- χ_Q the $\mathbf{k}$-rational character of $M/$(the maximal central $\mathbf{k}$-split torus of G) spanning the (rank 1) $\mathbb{Z}$-module of all such characters.
- $G(\mathbf{A})^1 := \{g \in G(\mathbf{A}) \mid |\chi(g)|_{\mathbf{A}} = 1$ for all $\mathbf{k}$-rational characters χ of $G\}$

The Ryshkov domain is useful to us because of the following theorem from [9].

Theorem 1. Let Ω be an open fundamental domain of $(R_Q^\circ)^-$ (closure of interior of R_Q in $G(\mathbf{A})^1$) with respect to $Q(\mathbf{k})$. Then Ω° is an open fundamental domain of $G(\mathbf{k})\backslash G(\mathbf{A})^1$.

Thus by starting with the Ryshkov domain, we can proceed to construct a fundamental domain for $G(\mathbf{k})\backslash G(\mathbf{A})^1$. The following subsection details this.

2.2 Constructing R_Q and Ω

Notation

- $K_f = \prod_{\sigma \in \mathbf{p}_f} K_\sigma$ (the finite part of K),
- $G_{\mathbf{A},\infty} = G(\mathbf{k}_\infty) \times K_f$, $\quad G^1_{\mathbf{A},\infty} = G_{\mathbf{A},\infty} \cap G(\mathbf{A})^1$,
- $G(\mathbf{k}_\infty)^1 = G(\mathbf{k}_\infty) \cap G(\mathbf{A})^1$.

Also we will denote the **class number of** G, that is the finite number $|G(\mathbf{k})\backslash G(\mathbf{A})/G_{\mathbf{A},\infty}|$, by n_G. We note here that $|G(\mathbf{k})\backslash G(\mathbf{A})^1/G^1_{\mathbf{A},\infty}|$ is also equal to n_G.

First, take a complete set of representatives $\{\eta_i\}_{i=1}^{n_G}$ for $G(\mathbf{k})\backslash G(\mathbf{A})^1/G^1_{\mathbf{A},\infty}$. We then define the **arithmetic subgroups** $\Gamma_1, \ldots, \Gamma_{n_G}$ by

$$\Gamma_i = \eta_i G^1_{\mathbf{A},\infty} \eta_i^{-1} \cap G(\mathbf{k}).$$

Also for each $i = 1, \ldots, n_G$ take a complete set of representatives $\{\xi_{ij}\}_{j=1}^{h_i}$ for $Q(\mathbf{k})\backslash G(\mathbf{k})/\Gamma_i$ (where the number of double cosets h_i is finite, see [2, §7]) and define groups

$$Q_{i,j} = Q \cap \xi_{ij}\Gamma_i\xi_{ij}^{-1} = Q(\mathbf{k}) \cap \xi_{ij} G_i \xi_{ij}^{-1}$$

and the sets

$$R_{i,j,\infty} = \{g \in G(\mathbf{k}_\infty)^1 : \mathsf{m}_Q(g\xi_{ij}\eta_i) = H_Q(g\xi_{ij}\eta_i)\}$$

for $j = 1, \ldots, h_i$.

We can immediately verify that

- $G(\mathbf{A})^1 = \coprod_{i=1}^{n_G} \coprod_{j=1}^{h_i} Q(\mathbf{k})G(\mathbf{k}_\infty)^1 \xi_{ij}\eta_i K_f,$
- $R_Q = \coprod_{i=1}^{n_G} \coprod_{j=1}^{h_i} Q(\mathbf{k}) R_{i,j,\infty}\xi_{ij}\eta_i K_f$

Also, by taking a complete set of representatives $\{\theta_{ijk}\}_k$ for $Q(\mathbf{k})/Q_{i,j}$, we obtain

$$\begin{aligned} R_Q &= \coprod_{i=1}^{n_G} \coprod_{j=1}^{h_i} Q(\mathbf{k}) R_{i,j,\infty}\xi_{ij}\eta_i K_f = \coprod_{i=1}^{n_G} \coprod_{j=1}^{h_i} \Big(\coprod_k \theta_{ijk} Q_{i,j}\Big) R_{i,j,\infty}\xi_{ij}\eta_i K_f \\ &= \coprod_{i=1}^{n_G} \coprod_{j=1}^{h_i} \coprod_k \theta_{ijk} R_{i,j,\infty}\xi_{ij}\eta_i K_f \end{aligned} \tag{1}$$

Denote $(R^\circ_{i,j,\infty})^-$ by $R^*_{i,j,\infty}$ where the interior and closure is taken in $G(\mathbf{k}_\infty)^1$. Similarly write R^*_Q for $(R^\circ_Q)^-$ in $G(\mathbf{A})^1$. From (1) we have

$$R^*_Q = \coprod_{i=1}^{n_G} \coprod_{j=1}^{h_i} \coprod_k \theta_{ijk} R^*_{i,j,\infty}\xi_{ij}\eta_i K_f. \tag{2}$$

We have the following main result.

Theorem 2. For each $i = 1, \ldots, n_G$ and $j = 1, \ldots, h_i$, take open fundamental domains $\Omega_{i,j,\infty}$ of $R^*_{i,j,\infty}$ with respect to $Q_{i,j}$. Then the set

$$\Omega = \coprod_{i=1}^{n_G} \coprod_{j=1}^{h_i} \Omega_{i,j,\infty}\xi_{ij}\eta_i K_f$$

is an open fundamental domain of R^*_Q with respect to $Q(\mathbf{k})$.

Corollary 3. Ω°(interior of Ω in $G(\mathbf{A})^1$) is an open fundamental domain of $G(\mathbf{A})^1$ with respect to $G(\mathbf{k})$.

Proof. From (2) we have

$$\begin{aligned} R^*_Q &= \coprod_{i=1}^{n_G} \coprod_{j=1}^{h_i} \coprod_k \theta_{ijk} R^*_{i,j,\infty}\xi_{ij}\eta_i K_f = \coprod_{i=1}^{n_G} \coprod_{j=1}^{h_i} \coprod_k \theta_{ijk} (Q_{i,j}\Omega^-_{i,j,\infty})\xi_{ij}\eta_i K_f \\ &= \coprod_{i=1}^{n_G} \coprod_{j=1}^{h_i} Q(\mathbf{k})\Omega^-_{i,j,\infty}\eta_i K_f = Q(\mathbf{k})\Omega^-. \end{aligned}$$

Now suppose $\Omega \cap q\Omega^- \neq \emptyset$ for $q \in Q(\mathbf{k})$. So for some i, i', j, j' we must have $q(\Omega_{i,j,\infty}\xi_{ij}\eta_i K_f) \cap (\Omega^-_{i',j',\infty}\xi_{i'j'}\eta_{i'}K_f) \neq \emptyset$. Writing $q = \theta_{ijk}q'$ with $q' \in Q_{i,j}$ and some k, we have

$$\theta_{ijk}(q')_\infty \Omega_{i,j,\infty}\xi_{ij}\eta_i K_f \cap \Omega^-_{i',j',\infty}\xi_{i'j'}\eta_{i'}K_f \neq \emptyset$$

since $(q')_f\xi_{ij}\eta_i K_f \subset \xi_{ij}\eta_i K_f$. Then (2) implies $i = i'$, $j = j'$, and $\theta_{ijk} = e$. Thus $\Omega_{i,j,\infty} \cap (q')_\infty \Omega^-_{i,j,\infty} = \Omega_{i,j,\infty} \cap q'\Omega^-_{i,j,\infty}$ must be non-empty, which means $q' = e$ and hence $q = e$. This proves the theorem, and the corollary follows from Theorem 1. □

Additionally, for any fixed $1 \leq i \leq n_G$, we have the following theorem.

Theorem 4. The set $\Omega_{i,\infty} = \bigcup_{j=1}^{h_i} \xi_{ij}^{-1}\Omega_{i,j,\infty}\xi_{ij}$ is a fundamental domain of $G(\mathbf{k}_\infty)^1$ with respect to Γ_i.

Proof. To show that $G(\mathbf{k}_\infty)^1 = \Gamma_i\Omega^-_{i,\infty}$, consider an arbitrary $g \in G(\mathbf{k}_\infty)^1$. From corollary 3

$$\begin{aligned} G(\mathbf{A})^1 = G(\mathbf{k})\Omega^- &= G(\mathbf{k}) \bigsqcup_{i=1}^{n_G} \bigsqcup_{j=1}^{h_i} \Omega^-_{i,j,\infty}\xi_{ij}\eta_i K_f \\ &= G(\mathbf{k}) \bigsqcup_{i=1}^{n_G} \bigsqcup_{j=1}^{h_i} \xi_{ij}(\xi_{ij}^{-1}\Omega^-_{i,j,\infty}\xi_{ij})\eta_i K_f \\ &\subset G(\mathbf{k}) \bigcup_{i=1}^{n_G} \Omega^-_{i,\infty}\eta_i K_f \end{aligned}$$

so we may write $g\eta_i = g'\omega\eta_i h$ with $g' \in G(\mathbf{k})$, $\omega \in \Omega^-_{i,\infty}$ and $h \in K_f$. Rearranging we get $g' = (g\omega^{-1})(\eta_i h^{-1}\eta_i^{-1})$ which belongs to $G(\mathbf{k}_\infty)^1\eta_i K_f \eta_i^{-1} = G_i$. Hence $g' \in \Gamma_i$. Since $g = (g'\omega)(\eta_i h \eta_i^{-1})$ and $g \in G(\mathbf{k}_\infty)^1$, $\eta_i h \eta_i^{-1}$ must necessarily be trivial. Thus $g \in \Gamma_i\Omega^-_{i,\infty}$.

Now suppose that $\Omega^\circ_{i,\infty} \cap g\Omega^-_{i,\infty}$ is non-empty for a $g \in \Gamma_i$. Then we must have $\xi_{ij}^{-1}\Omega^\circ_{i,j,\infty}\xi_{ij} \cap g\xi_{ij'}^{-1}\Omega^-_{i,,j',\infty}\xi_{ij'} \neq \emptyset$ for some j, j'. Since $g_f\eta_i K_f = \eta_i K_f$,

$$\begin{aligned} &\xi_{ij}^{-1}\Omega^\circ_{i,j,\infty}\xi_{ij} \cap g\xi_{ij'}^{-1}\Omega^-_{i,,j',\infty}\xi_{ij'} \neq \emptyset \\ &\Rightarrow (\Omega_{i,j,\infty}\xi_{ij}\eta_i K_f)^\circ \cap \xi_{ij}g\xi_{ij'}^{-1}(\Omega_{i,j',\infty}\xi_{ij'}\eta_i K_f)^- \neq \emptyset \\ &\Rightarrow \Omega^\circ \cap (\xi_{ij}g\xi_{ij'}^{-1})\Omega^- \neq \emptyset \end{aligned}$$

and thus $\xi_{ij}g\xi_{ij'}^{-1} = e$ by Corollary 3. Hence $Q(\mathbf{k})\xi_{ij}\Gamma_i = Q(\mathbf{k})\xi_{ij'}\Gamma_i$, which implies $j = j'$ whereby $g = \xi_{ij}^{-1}\xi_{ij'} = e$. □

3 The case $G = GL_n$

In this section we will consider the case where G is a general linear group GL_n defined over $\mathbf{k}$. Fixing an integer $1 \leq m < n$, we consider the maximal standard $\mathbf{k}$-parabolic subgroup Q defined by

$$Q(\mathbf{k}) = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a \in GL_m(\mathbf{k}), b \in M_{m,n-m}(\mathbf{k}), d \in GL_{n-m}(\mathbf{k}) \right\}.$$

For the maximal compact subgroup K of $G(\mathbf{A})$ let $K = K_\infty \times K_f$ where

$$K_\infty = \{g \in GL_n(\mathbf{k}_\infty) : {}^t\bar{g}g = I_n\}, \quad K_f = \prod_{\sigma \in \mathbf{p}_f} GL_n(\mathcal{O}_\sigma).$$

Here we identify $GL_n(\mathbf{k}_\infty)$ with $\prod_{\sigma \in \mathbf{p}_\infty} GL_n(\mathbf{k}_\sigma)$, and for $g = (g_\sigma)_{\sigma \in \mathbf{p}_\infty} \in GL_n(\mathbf{k}_\infty)$ we write ${}^t\bar{g}$ for the element $({}^t\bar{g}_\sigma)_{\sigma \in \mathbf{p}_\infty}$ of $GL_n(\mathbf{k}_\infty)$.

We shall see that in this case the number of double cosets of $Q(\mathbf{k})\backslash GL_n(\mathbf{k})/\Gamma_i$ for each i is invariant and equal to $|GL_n(\mathbf{k})\backslash GL_n(\mathbf{A})^1/G^1_{\mathbf{A},\infty}|$, the class number of GL_n.

Denote the set of all $\mathcal{O}$-lattices in $\mathbf{k}^r$ ($r \geq 1$) by $\mathfrak{L}_r$, and the standard unit vectors of $\mathbf{k}^r$ by $\mathbf{e}_1^{(r)}, \dots, \mathbf{e}_r^{(r)}$. For this section we simply write $\mathfrak{L}$ for $\mathfrak{L}_n$ and $\mathbf{e}_k$ for $\mathbf{e}_k^{(n)}$ ($1 \leq k \leq n$).

For $L \in \mathfrak{L}_r$ and $g = (g_\sigma)_{\sigma \in \mathbf{p}} \in GL_r(\mathbf{A})$ put

$$gL = \Big((\mathbf{k}_\infty)^r \times \prod_{\sigma \in \mathbf{p}_f} g_\sigma L_\sigma \Big) \cap \mathbf{k}^r \in \mathfrak{L}_r. \tag{3}$$

This defines a transitive left action of $GL_r(\mathbf{A})^1$ on $\mathfrak{L}_r$. Note that if $g \in GL_r(\mathbf{k})$ then gL as defined above coincides with the usual image of L under the linear transformation $v \mapsto gv$ of $\mathbf{k}^r$. The subset of $\mathfrak{L}$ consisting of all $\mathcal{O}$-lattices of the form gL with $g \in GL_n(\mathbf{k})$ will be referred to as the **$\mathcal{O}$-lattice class of** L or just the lattice class of L in $\mathfrak{L}$. Since every $\mathcal{O}$-lattice in a lattice class has the same Steinitz class, we refer to the Steinitz class of any lattice representing the class as the Steinitz class for that lattice class.

From the previous section, we will require a complete set representing $GL_n(\mathbf{k})\backslash GL_n(\mathbf{A})^1/G^1_{\mathbf{A},\infty}$. Take $\{\eta_1, \dots, \eta_h\}$ to be such a set of matrices. Then for each $i = 1, \dots, h$ put $L_i = \eta_i(\mathcal{O}\mathbf{e}_1 + \dots + \mathcal{O}\mathbf{e}_n) \in \mathfrak{L}$. We then have a one-to-one correspondence between $GL_n(\mathbf{k})\backslash GL_n(\mathbf{A})^1/G^1_{\mathbf{A},\infty}$ and the set of $\mathcal{O}$-lattices classes in $\mathfrak{L}$ by mapping each η_i to the lattice class of L_i. That this is a bijection follows from $G^1_{\mathbf{A},\infty}$ being the stabilizer group of the $\mathcal{O}$-lattice $\mathcal{O}\mathbf{e}_1 + \dots + \mathcal{O}\mathbf{e}_n$ under the action of $GL_n(\mathbf{A})^1$ on $\mathfrak{L}$.

Continuing the map to $St(L_i)$, the Steinitz class of L_i gives us a bijection from $GL_n(\mathbf{k})\backslash GL_n(\mathbf{A})^1/G^1_{\mathbf{A},\infty}$ to $Cl(\mathbf{k})$. As a result the class number of GL_n is equal to the class number of $\mathbf{k}$, which we write as h.

We can proceed on to our next main results, that $h_i = |Q(\mathbf{k})\backslash GL_n(\mathbf{k})/\Gamma_i|$ is also equal to h for every $i = 1 \dots, h$.

Identify $Q(\mathbf{k})\backslash GL_n(\mathbf{k})$ with the set of all m-dimensional linear subspaces of $\mathbf{k}^n$ denoted by Gr_m (the Grassmanian) via the bijection

$$Q(\mathbf{k})\backslash GL_n(\mathbf{k}) \ni Q(\mathbf{k})g \longmapsto g^{-1}\big(\sum_{k=1}^{m} \mathbf{k}\mathbf{e}_k\big) \in Gr_m. \tag{4}$$

Fix $i \in \{1, ..., h\}$. Considering the left action of $\Gamma_i \subset GL_n(\mathbf{k})$ on Gr_m, the map (4) gives rise to the bijection

$$Q(\mathbf{k})\backslash GL_n(\mathbf{k})/\Gamma_i \ni Q(\mathbf{k})g\Gamma_i \longmapsto \Gamma_i g^{-1}\big(\sum_{k=1}^{m} \mathbf{k}\mathbf{e}_k\big) \in \Gamma_i\backslash Gr_m \tag{5}$$

which lets us identify $Q(\mathbf{k})\backslash GL_n(\mathbf{k})/\Gamma_i$ with $\Gamma_i\backslash Gr_m$.

Lemma 5.

$$\Gamma_i = \{g \in GL_n(\mathbf{k}) : gL_i = L_i\}$$

i.e. Γ_i is the stabilizer of L_i in $GL_n(\mathbf{k})$, under the action of $GL_n(\mathbf{A})^1$ on $\mathfrak{L}$.

Theorem 6. The map

$$\lambda_i : \Gamma_i\backslash Gr_m \longrightarrow Cl(\mathbf{k}), \quad \lambda_i(\Gamma_i V) = St(L_i \cap V) \quad (V \in Gr_m) \tag{6}$$

is a well-defined bijection, and thus $h_i = h$.

The above bijections give us an explicit way to find candidates for $\{\eta_i\}_{i=1}^h$ and $\{\xi_{ij}\}_{j=1}^h$ as follows. Let $\{\mathfrak{a}_1, \dots, \mathfrak{a}_h\}$ be a complete set of fractional ideals representing the ideal class of $\mathbf{k}$. For each $i = 1, \dots, h$, we shall require an element $\eta_i \in GL_n(\mathbf{A})^1$ such that the Steinitz class of the resulting lattice $L_i = \eta_i(\sum_{k=1}^n \mathcal{O}\mathbf{e}_k)$ is the ideal class represented by $\mathfrak{a}_i$.

Let $D_n(x)$ ($x \in \mathbf{A}$) denote the unit matrix of size n with bottom-most diagonal entry replaced by x. For each $1 \leq i \leq h$ we can choose $\alpha_i \in \mathbf{A}^\times$ such that $\alpha_{i\sigma}$ generates the principal ideal $\mathfrak{a}_i\mathcal{O}_\sigma$ for every finite σ and $|\alpha_i|_\infty = N(\mathfrak{a}_i)$, the ideal norm of $\mathfrak{a}_i$. Then $D_n(\alpha_i) \in GL_n(\mathbf{A})^1$ since $|\det D_n(\alpha_i)|_{\mathbf{A}} = |\alpha_i|_{\mathbf{A}} = 1$, and

$$D_n(\alpha_i)\big(\sum_{k=1}^{n} \mathcal{O}\mathbf{e}_k\big) = \sum_{1 \leq k < n} \mathcal{O}\mathbf{e}_k + \mathfrak{a}_i\mathbf{e}_n.$$

Hence putting $\eta_i = D_n(\alpha_i)$ ($1 \leq i \leq h$) gives us our required set of representatives for $GL_n(\mathbf{k})\backslash GL_n(\mathbf{A})^1/G^1_{\mathbf{A},\infty}$. The corresponding $\mathcal{O}$-lattice L_i and its stabilizer group Γ_i will be denoted by $L_n(\mathfrak{a}_i)$ and $\Gamma_n(\mathfrak{a}_i)$ respectively whenever we want to call to attention the fractional ideal $\mathfrak{a}_i$ or the dimension n.

We can also proceed similarly to find for a fixed i a suitable set of representatives for $Q(\mathbf{k})\backslash GL_n(\mathbf{k})/\Gamma_i$. We do this using the bijection

$$Q(\mathbf{k})\backslash GL_n(\mathbf{k})/\Gamma_i \ni Q(\mathbf{k})g\Gamma_i \mapsto St(L_i \cap g^{-1}V_m) \in Cl(\mathbf{k})$$

formed by composing λ_i with the bijection (5), where $V_m = \sum_{k=1}^{m} \mathbf{k}\mathbf{e}_k$.

For each $j \in \{1, \ldots, h\}$ the ideal $\mathfrak{a}_i\mathfrak{a}_j^{-1}$ shares the same ideal class as a unique $\mathfrak{a}_{j'}$ ($j' \in \{1, \ldots, h\}$), that is $[\mathfrak{a}_j][\mathfrak{a}_{j'}] = [\mathfrak{a}_i]$. Putting $\tau_i(j) := j'$ defines a permutation τ_i on $\{1, \ldots, h\}$.

Call a set of matrices $\{\xi_1, \ldots, \xi_h\} \subset GL_n(\mathbf{k})$ an **(n, m)-splitting set for $L_n(\mathfrak{a}_i)$** if for each $j = 1, \ldots, h$

$$\begin{aligned} \xi_j L_n(\mathfrak{a}_i) &= \Big(\sum_{1 \le k < m} \mathcal{O}\mathbf{e}_k + \mathfrak{a}_j \mathbf{e}_m \Big) + \Big(\sum_{m < k < n} \mathcal{O}\mathbf{e}_k + \mathfrak{a}_{\tau_i(j)}\mathbf{e}_n \Big) \\ &\simeq L_m(\mathfrak{a}_j) \oplus L_{n-m}(\mathfrak{a}_{\tau_i(j)}). \end{aligned} \tag{7}$$

Since $St(L_i \cap \xi_j^{-1}V_m) = St(\xi_j L_i \cap V_m) = [\mathfrak{a}_j]$ ($i \le j \le h$), such a set of matrices completely represents $Q(\mathbf{k})\backslash GL_n(\mathbf{k})/\Gamma_i$.

One such set is given as follows. For each $j = 1, \ldots, h$, first take $\kappa_{ij} \in \mathbf{k}$ such that $\mathfrak{a}_j\mathfrak{a}_{\tau_i(j)} = \kappa_{ij}\mathfrak{a}_i$. Then choose elements $\alpha_{ij} \in \mathfrak{a}_j$, $\alpha'_{ij} \in \mathfrak{a}_{\tau_i(j)}$, $\beta_{ij} \in \mathfrak{a}_j^{-1}$ and $\beta'_{ij} \in \mathfrak{a}_{\tau_i(j)}^{-1}$ satisfying

$$\alpha_{ij}\beta_{ij} - \alpha'_{ij}\beta'_{ij} = 1$$

(see [3, §1, Prop. 1.3.12 or Algorithm 1.3.16]) and define the matrix

$$\xi_{ij} := \begin{bmatrix} I_{m-1} & & & \\ & \alpha_{ij} & & \kappa_{ij}\beta'_{ij} \\ & & I_{n-m+1} & \\ & \alpha'_{ij} & & \kappa_{ij}\beta_{ij} \end{bmatrix} \in GL_n(\mathbf{k}).$$

By direct calculation it is easily verified that $\{\xi_{ij}\}_{j=1}^{h}$ is indeed an (n, m)-splitting set for $L_n(\mathfrak{a}_i)$ and thus fully represents $Q^{n,m}(\mathbf{k})\backslash GL_n(\mathbf{k})/\Gamma_n(\mathfrak{a}_i)$.

4 Fundamental domains of $GL_n(\mathbf{k})\backslash GL_n(\mathbf{A})^1$ and P_n/Γ_i

We will apply the general results of section 2 to GL_n, before proceeding to P_n. The matrices $\{\eta_i\}_{i=1}^{h}$ and $\{\xi_{ij}\}_{j=1}^{h}$ used from here on are the same ones chosen in the end of the previous section.

4.1 The height function

The height function associated to the parabolic subgroup Q used in the previous section is given by

$$H_Q\left(u \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} h\right) = |\det a|_{\mathbf{A}}^{-(n-m)/l} |\det d|_{\mathbf{A}}^{m/l} \quad \left(u \in U(\mathbf{A}), \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in M(\mathbf{A}), h \in K\right)$$

where l is the greatest common divisor of $n - m$ and m.

Definition. For each $\sigma \in \mathbf{p}$ define $H_\sigma : \bigwedge^m \mathbf{k}_\sigma^n \to \mathbb{R}_{>0}$ by

$$H_\sigma\Big(\sum_I a_I(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_m})\Big) = \begin{cases} \Big(\sum_I |a_I|_\sigma^2\Big)^{[\mathbf{k}_\sigma:\mathbb{R}]/2} & (\sigma \in \mathbf{p}_\infty), \\ \sup_I |a_I|_\sigma & (\sigma \in \mathbf{p}_f), \end{cases}$$

the sum and the supremum taken over all $I = \{i_1 < \cdots < i_m\} \subset \{1, \ldots, n\}$. We call this the **local height function** at σ. H_σ can be extended to a function of $GL_n(\mathbf{k}_\sigma)$ by defining

$$H_\sigma(\gamma) = H_\sigma(\gamma\mathbf{e}_1 \wedge \cdots \wedge \gamma\mathbf{e}_m), \quad \gamma \in GL_n(\mathbf{k}_\sigma).$$

The following lemma allows us to express the height function H_Q (restricted to $GL_n(\mathbf{A})^1$) in terms of these local heights.

Lemma 7.

$$H_Q(g) = \prod_{\sigma \in \mathbf{p}} H_\sigma(g_\sigma^{-1})^{n/l}$$

for $g = (g_\sigma)_{\sigma \in \mathbf{p}} \in GL_n(\mathbf{A})^1$.

We proceed to describe the sets $R_{i,j,\infty}$ using the matrices η_i and ξ_{ij} chosen at the end of the previous section. For the rest of this paper, for a square matrix A with entries in $\mathbf{A}$ or $\mathbf{k}_\infty$, we will write $|A|_{\mathbf{A}}$ and $|A|_\infty$ to denote $|\det A|_{\mathbf{A}}$ and $|\det A|_\infty$ respectively. When the size of A is at least m, we write $A^{[m]}$ for the top-left m by m submatrix of A, and use $|A|_\infty^{[m]}$ to denote $|A^{[m]}|_\infty$.

Theorem 8. Let X_{ij} denote the n by m matrix formed by the first m columns of ξ_{ij}^{-1}. Then

$$H_Q(\xi_{ij}\gamma g\eta_i) = N(\mathfrak{a}_j)^{n/l} |{}^t\overline{X}_{ij}{}^t\bar{\gamma}^{-1}{}^t\bar{g}^{-1}(\eta_i)_\infty^{-2} g^{-1}\gamma^{-1}X_{ij}|_\infty^{n/2l} \tag{8}$$

for any $1 \le i, j \le h$, $\gamma \in \Gamma_i$ and $g \in GL_n(\mathbf{k}_\infty)^1$.

Proof. This can be proved by verifying that

- $\prod_{\sigma \in \mathbf{p}_f} H_\sigma((\xi_{ij}\gamma g\eta_i)_\sigma^{-1}) = N(\mathfrak{a}_j)$ from our choices of η_i and ξ_{ij},
- $\prod_{\sigma \in \mathbf{p}_\infty} H_\sigma((\xi_{ij}\gamma g\eta_i)_\sigma^{-1}) = |{}^t\overline{X}_{ij}{}^t\bar{\gamma}^{-1}{}^t\bar{g}^{-1}(\eta_i)_\infty^{-2} g^{-1}\gamma^{-1}X_{ij}|_\infty$ which can be shown using the Cauchy-Binet formula.

The result follows from the previous lemma. □

Now fix $1 \le i, j \le h$ and first consider the set $\xi_{ij}^{-1}R_{i,j,\infty}\xi_{ij}$. It is easy to directly verify that

$$\xi_{ij}^{-1}R_{i,j,\infty}\xi_{ij} = \{g \in G(\mathbf{k}_\infty)^1 : H_Q(\xi_{ij}g\eta_i) = \mathsf{m}_Q(g\eta_i)\}$$

Hence for $g \in \xi_{ij}^{-1}R_{i,j,\infty}\xi_{ij}$ we have

$$H_Q(\xi_{ij}g\eta_i) = \mathsf{m}_Q(g\eta_i) = \min_{x \in Q(\mathbf{k})\backslash GL_n(\mathbf{k})} H_Q(xg\eta_i) = \min_{\substack{1 \le k \le h \\ \gamma \in \Gamma_i}} H_Q(\xi_{ik}\gamma g\eta_i)$$

which in this case can be written using (8) as

$$|{}^t\overline{X}_{ij}{}^t\bar{g}^{-1}(\eta_i)_\infty^{-2} g^{-1}X_{ij}|_\infty \le \left(\frac{N(\mathfrak{a}_k)}{N(\mathfrak{a}_j)}\right)^2 |{}^t\overline{X}_{ik}{}^t\bar{\gamma}{}^t\bar{g}^{-1}(\eta_i)_\infty^{-2} g^{-1}\gamma X_{ik}|_\infty$$

for all $k = 1, \ldots, h$ and $\gamma \in \Gamma_i$.

Now ${}^t\overline{X}_{ik}{}^t\bar{\gamma}{}^t\bar{g}^{-1}(\eta_i)_\infty^{-2} g^{-1}\gamma X_{ik} = ({}^t\bar{\xi}_{ik}^{-1}{}^t\bar{\gamma}{}^t\bar{g}^{-1}(\eta_i)_\infty^{-2} g^{-1}\gamma\xi_{ik}^{-1})^{[m]}$, which by letting $g_{[ij]} = \xi_{ij}g\xi_{ij}^{-1}$ can be rewritten as

$$\left({}^t(\overline{\xi_{ij}\gamma\xi_{ik}^{-1}}){}^t\bar{g}_{[ij]}^{-1}({}^t\bar{\xi}_{ij}^{-1}(\eta_i)_\infty^{-2}\xi_{ij}^{-1})g_{[ij]}^{-1}(\xi_{ij}\gamma\xi_{ik}^{-1})\right)^{[m]}.$$

This lets us express the set $R_{i,j,\infty}$ as follows. For $g \in GL_n(\mathbf{k}_\infty)$ let $\pi_{ij}(g)$ denote ${}^t\bar{g}^{-1}({}^t\bar{\xi}_{ij}^{-1}(\eta_i)_\infty^{-2}\xi_{ij}^{-1})g^{-1}$. Then $g \in R_{i,j,\infty}$ if and only if

$$|\pi_{ij}(g)|_\infty^{[m]} \le \left(\frac{N(\mathfrak{a}_k)}{N(\mathfrak{a}_j)}\right)^2 |{}^t(\overline{\xi_{ij}\gamma\xi_{ik}^{-1}})\pi_{ij}(g)(\xi_{ij}\gamma\xi_{ik}^{-1})|_\infty^{[m]} \tag{9}$$

for all $k = 1, \ldots, h$ and $\gamma \in \Gamma_i$.

4.2 Fundamental domains of P_n/Γ_i

For each infinite place σ of $\mathbf{k}$ let $P_n(\mathbf{k}_\sigma)$ denote the subset of $GL_n(\mathbf{k}_\sigma)$ consisting of all positive definite real symmetric matrices when σ is real and positive definite Hermitian matrices when σ is imaginary. We consider the subset of $GL_n(\mathbf{k}_\infty)$ defined by $P_n = \prod_{\sigma\in\mathbf{p}_\infty} P_n(\mathbf{k}_\sigma)$. This is the **space of positive definite Humbert forms over $\mathbf{k}_\infty$.**

We have the following right action of $GL_n(\mathbf{k}_\infty)$ on P_n

$$A\cdot g = {}^t\bar{g}Ag \quad (g\in GL_n(\mathbf{k}_\infty),\ A\in P_n). \tag{10}$$

To determine fundamental domains in P_n with respect to subgroups of $GL_n(\mathbf{k})$, we consider instead the induced action $A\cdot gZ = {}^t\bar{g}Ag$ of $GL_n(\mathbf{k})/Z$ on P_n, where $Z=\{z\in\mathbf{k} : \bar{z}z=1\}$, the set of roots of unity in $\mathbf{k}$. Here $\{zI_n : z\in Z\}$ is naturally seen to be the intersection of K_∞ and the center of $GL_n(\mathbf{k})$.

Now for each $1\le i,j\le h$, put

$$K_{i,j,\infty} = (\xi_{ij}\eta_i)_\infty K_\infty (\xi_{ij}\eta_i)_\infty^{-1}, \quad P_n^{ij} = \{A\in P_n : |A|_\infty = N(\kappa_{ij}\mathfrak{a}_i)^{-2}\},$$

and define the map $\pi_{ij} : G(\mathbf{k}_\infty)\ni g\mapsto {}^t\bar{g}^{-1}({}^t\bar{\xi}_{ij}^{-1}(\eta_i)_\infty^{-2}\xi_{ij}^{-1})g^{-1}\in P_n$. Note that $K_{i,j,\infty}$ is the stabilizer of ${}^t\bar{\xi}_{ij}^{-1}(\eta_i)_\infty^{-2}\xi_{ij}^{-1}\in P_n$ under the action of $GL_n(\mathbf{k}_\infty)$ on P_n and that π_{ij} preserves this action. Thus the surjective map π_{ij} gives us the isomorphisms

$$GL_n(\mathbf{k}_\infty)/K_{i,j,\infty}\simeq P_n \quad\text{and}\quad GL_n(\mathbf{k}_\infty)^1/K_{i,\infty}\simeq \pi_{ij}(GL_n(\mathbf{k}_\infty)^1) = P_n^{ij}$$

since $|{}^t\bar{\xi}_{ij}^{-1}(\eta_i)_\infty^{-2}\xi_{ij}^{-1}|_\infty = N(\kappa_{ij}\mathfrak{a}_i)^{-2}$.

Lastly let $F_{i,j}$ denote the following closed subset of P_n:

$$\{A\in P_n :\ |A|_\infty^{[m]} \le \left(\frac{N(\mathfrak{a}_k)}{N(\mathfrak{a}_j)}\right)^2 |{}^t(\overline{\xi_{ij}\gamma\xi_{ik}^{-1}})A(\xi_{ij}\gamma\xi_{ik}^{-1})|_\infty^{[m]},\ 1\le k\le h,\ \gamma\in\Gamma_i\}.$$

From (9), π_{ij} maps $R_{i,j,\infty}$ onto $F_{i,j}\cap P_n^{ij}$. We also note that $F_{i,j}$ is right $Q_{i,j}$-invariant under the action (10).

Thus the subgroup $Q_{i,j}$ of $GL_n(\mathbf{k}_\infty)$ acts on $R_{i,j,\infty}$ from the left and on $F_{i,j}$ from the right, and π_{ij} preserves this. Hence by constructing a fundamental domain for $F_{i,j}/Q_{i,j}$, we can find one for $Q_{i,j}\backslash R_{i,j,\infty}$ by taking the inverse image under π_{ij}.

We start by observing that $\xi_{ij}\Gamma_i\xi_{ij}^{-1}$ is the stabilizer in $GL_n(\mathbf{k})$ of the $\mathcal{O}$-lattice $\xi_{ij}L_i$ described in (7). This gives us the following expression for $Q_{i,j} = Q(\mathbf{k})\cap\xi_{ij}\Gamma_i\xi_{ij}^{-1}$:

$$\left\{\begin{bmatrix} a & b\\ 0 & d\end{bmatrix} :\quad a\in\Gamma_m(\mathfrak{a}_j),\quad d\in\Gamma_{n-m}(\mathfrak{a}_{\tau_i(j)}),\quad bL_{n-m}(\mathfrak{a}_{\tau_i(j)})\subset L_m(\mathfrak{a}_j)\right\}.$$

Any $A\in P_n$ can be written uniquely in the form

$$A = \begin{bmatrix} I_m & 0\\ {}^t\overline{u_{A,m}} & I_{n-m}\end{bmatrix}\begin{bmatrix} A^{[m]} & 0\\ 0 & A_{[n-m]}\end{bmatrix}\begin{bmatrix} I_m & u_{A,m}\\ 0 & I_{n-m}\end{bmatrix} \tag{11}$$

with $A^{[m]}\in P_m$, $A_{[n-m]}\in P_{n-m}$ and $u_{A.m}\in M_{m,n-m}(\mathbf{k}_\infty)$ (The symbol $A^{[m]}$ here coincides with its prior use to denote the top left m by m submatrix of A). It is easy to verify that the action of $q=\left[\begin{smallmatrix} a & b\\ 0 & d\end{smallmatrix}\right]\in Q_{i,j}$ on A result in

$$({}^t\overline{q}Aq)^{[m]} = {}^t\overline{a}A^{[m]}a, \quad ({}^t\overline{q}Aq)_{[n-m]} = {}^t\overline{d}A_{[n-m]}d,$$

$$u_{{}^t\overline{q}Aq,m} = a^{-1}(u_{A,m}d+b).$$

These equations will determine the necessary form of our fundamental domain.

For each $k=1,\dots,h$ choose sets $\mathfrak{d}_k$, $\mathfrak{d}'_k$ and $\mathfrak{d}_{ik}$ that are fundamental domains for $\mathbf{k}_\infty$ with respect to addition by $\mathfrak{a}_k$, $\mathfrak{a}_k^{-1}$ and $\mathfrak{a}_k\mathfrak{a}_{\tau_i(k)}^{-1}$ respectively. We require each of these sets are closed under multiplication by Z. Then choose also a subset $\tilde{\mathfrak{d}}_{ik}$ of $\mathfrak{d}_{ik}$ that is a fundamental domain for $\mathfrak{d}_{ik}$ with respect to multiplication by Z. Also if necessary (which will be the case when $m>1$ and $n-m>1$) take a fundamental domain $\mathfrak{d}_\mathcal{O}$ of $\mathbf{k}_\infty$ with respect to addition by $\mathcal{O}$.

Using these, we define for $1 < i, j < h$ the sets

$$\mathfrak{D}_{i,j} = \left\{ \begin{bmatrix} d_{11} & \cdots & d_{1,n-m} \\ \vdots & \ddots & \vdots \\ d_{m1} & \cdots & d_{m,n-m} \end{bmatrix} : \; d_{m,n-m} \in \tilde{\mathfrak{d}}_{ij}, \;\; d_{rs} \in \begin{cases} \mathfrak{d}_{\mathcal{O}} & r < m, \; s < n-m \\ \mathfrak{d}'_{\tau_i(j)} & r < m, \; s = n-m \\ \mathfrak{d}_j & r = m, \; s < n-m \end{cases} \right\}.$$

By observing the action of $Q_{i,j}$ on $F_{i,j}$, we establish the following result.

Theorem 9. Let $\mathfrak{B}$ and $\mathfrak{C}$ be fundamental domains for $P_m/\Gamma_m(\mathfrak{a}_j)$ and $P_{n-m}/\Gamma_{n-m}(\mathfrak{a}_{\tau_i(j)})$ respectively. Then $F_{i,j}(\mathfrak{B}, \mathfrak{C}) = \{A \in F_{i,j} : \; A^{[m]} \in \mathfrak{B}, \;\; A_{[n-m]} \in \mathfrak{C}, \;\; u_{A,m} \in \mathfrak{D}_{i,j}\}$ is a fundamental domain of $F_{i,j}/Q_{i,j}$.

As a result, the inverse image of $F_{i,j}(\mathfrak{B}_j, \mathfrak{C}_{\tau_i(j)}) \cap P_n^{ij}$ under π_{ij} is a fundamental domain of $Q_{i,j} \backslash R_{i,j,\infty}$. Also, if we have fundamental domains $\mathfrak{B}_k$ for $P_m/\Gamma_m(\mathfrak{a}_k)$, as well as fundamental domains $\mathfrak{C}_k$ of $P_{n-m}/\Gamma_{n-m}(\mathfrak{a}_k)$ for each $k = 1, \dots, h$, we can then construct the sets $F_{i,j}(\mathfrak{B}_j, \mathfrak{C}_{\sigma_i(j)})$ $(1 \le i, j \le h)$. Then by Corollary 3 a fundamental domain for $GL_n(\mathbf{k}) \backslash GL_n(\mathbf{A})^1$ is given by the set

$$\bigsqcup_{1 \le i,j \le h} \pi_{ij}^{-1}(F_{i,j}(\mathfrak{B}_j, \mathfrak{C}_{\tau_i(j)}) \cap P_n^{ij}) \xi_{ij} \eta_i K_f.$$

Also Theorem 4 shows us that $\bigcup_{j=1}^{h} \xi_{ij}^{-1} \pi_{ij}^{-1}(F_{i,j}(\mathfrak{B}_j, \mathfrak{C}_{\tau_i(j)}) \cap P_n^{ij}) \xi_{ij}$ is a fundamental domain for $GL_n(\mathbf{k}_\infty)^1$ with respect to Γ_i. Now let

$$\Omega_i(\mathfrak{B}_1, \dots, \mathfrak{B}_h, \mathfrak{C}_1, \dots, \mathfrak{C}_h) = \bigcup_{j=1}^{h} {}^t\bar{\xi}_{ij} F_{i,j}(\mathfrak{B}_j, \mathfrak{C}_{\tau_i(j)}) \xi_{ij}.$$

Theorem 10. $\Omega_i(\mathfrak{B}_1, \dots, \mathfrak{B}_h, \mathfrak{C}_1, \dots, \mathfrak{C}_h) \cap P_n^{ij}$ is a fundamental domain of P_n^{ij} with respect to Γ_i. In addition, if we assume that each of the $\mathfrak{B}_k$ and $\mathfrak{C}_k$ are closed under positive multiplication (viewing $\mathbb{R}_{>0}$ as a subset of $\mathbf{k}_\infty$ via the usual diagonal embedding), then

$$\mathbb{R}_{>0}\mathfrak{B}_k = \mathfrak{B}_k, \quad \mathbb{R}_{>0}\mathfrak{C}_k = \mathfrak{C}_k,$$

then $\Omega_i(\mathfrak{B}_1, \dots, \mathfrak{B}_h, \mathfrak{C}_1, \dots, \mathfrak{C}_h)$ is a fundamental domain of P_n/Γ_i.

Using the theorem, we can construct fundamental domains for P_n with respect to Γ_i for each i and $n \ge 1$. Since $\Gamma_i = \mathcal{O}^\times$ for any i when $n = 1$, we can start by choosing a fixed fundamental domain, Ω^1, for P_1 with respect to $\mathcal{O}^\times/Z$ that is closed under multiplication by $\mathbb{R}_{>0}$ (The existence of such a set can be shown using Voronoï reduction, as demonstrated in the appendix of [7]). Then for each $i = 1, \dots, h$, let $\Omega_i^1 = \Omega^1$ and define

$$\Omega_i^n = \Omega_i^{n,n-1}(\Omega_1^{n-1} \dots, \Omega_h^{n-1}, \Omega^1, \cdots, \Omega^1)$$

inductively for $n \ge 2$. By construction $\mathbb{R}_{>0}\Omega_i^n = \Omega_i^n$ so for each $1 \le i \le h$ and $n \ge 1$, Ω_i^n gives us a fundamental domain for P_n/Γ_i.

4.3 An example ($\mathbf{k} = \mathbb{Q}(\sqrt{-5})$)

When $\mathbf{k}$ is an imaginary quadratic field, we have $\mathbf{k}_\infty = \mathbb{C}$. For $n = 1$ we have $P_1 = \mathbb{R}_{>0}(\subset \mathbb{C})$ and $\Gamma_i = \mathcal{O}^\times = Z$ acts trivially on P_1, hence P_1 itself is a fundamental domain for $P_1/\Gamma_1(\mathfrak{a}_i)$.

Consider in particular $\mathbf{k} = \mathbb{Q}(\sqrt{-5})$ of class number $h = 2$. We can choose representatives $\mathfrak{a}_1, \mathfrak{a}_2$ for $Cl(\mathbf{k})$ by putting $\mathfrak{a}_1 = \mathcal{O}$ and $\mathfrak{a}_2 = \langle 2, 1 + \sqrt{-5} \rangle$. Then following the procedure at the end of section 4, we see that

$$\begin{aligned} &\mathfrak{a}_1^2 = \mathfrak{a}_1, \quad \mathfrak{a}_2^2 = 2\mathfrak{a}_1 \quad (\tau_1 = (\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}) \quad \kappa_{11} = 1, \kappa_{12} = 2), \\ &\mathfrak{a}_1\mathfrak{a}_2 = \mathfrak{a}_2, \quad \mathfrak{a}_2\mathfrak{a}_1 = \mathfrak{a}_2 \quad (\tau_2 = (\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}) \quad \kappa_{21} = \kappa_{22} = 1). \end{aligned}$$

$(2,1)-$splitting sets for $L_2(\mathfrak{a}_i)$ are given by

$$\left\{ \xi_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; \xi_{12} = \begin{bmatrix} 2 & 2+\sqrt{-5} \\ 2 & 3+\sqrt{-5} \end{bmatrix} \right\} \quad (i = 1),$$

$$\left\{ \xi_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; \xi_{22} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\} \quad (i = 2).$$

For $1 \leq i,j,k \leq 2$ denote by $\Xi_{i,j,k}$ the set of the first columns of the matrices $\xi_{ij}\gamma\xi_{ik}^{-1}$ as γ ranges over $\Gamma(\mathfrak{a}_i)$. Then for $A \in P_2$

$$\min_{\gamma\in\Gamma_i} |{}^t(\overline{\xi_{ij}\gamma\xi_{ik}^{-1}})A(\xi_{ij}\gamma\xi_{ik}^{-1})|_\infty^{[1]} = \min_{\mathbf{x}\in\Xi_{i,j,k}} |{}^t\overline{\mathbf{x}}A\mathbf{x}| = \min_{\left[\begin{smallmatrix} e \\ f \end{smallmatrix}\right]\in\Xi_{i,j,k}} A^{[1]}|e+u_{A,1}f|^2 + A_{[1]}|f|^2$$

and so $F_{i,j}^{2,1}$ can be expressed as

$$F_{i,j}^{2,1} = \left\{ \begin{bmatrix} 1 & 0 \\ \bar{d} & 1 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} : \begin{array}{l} b,c\in\mathbb{R}_{>0},\ d\in\mathbb{C} \\ |e+df|^2 + \frac{c}{b}|f|^2 \geq 1, \\ \left[\begin{smallmatrix} e \\ f \end{smallmatrix}\right] \in \frac{1}{N(\mathfrak{a}_j)}\Xi_{i,j,1} \cup \frac{2}{N(\mathfrak{a}_j)}\Xi_{i,j,2} \end{array} \right\}.$$

Now for $\alpha,\beta\in\mathbf{k}$ let $\mathfrak{d}(\alpha,\beta) = \{x\alpha + y\beta : -1/2 < x,y \leq 1/2\}$. When α and β generate a fractional ideal $\mathfrak{a}$, $\mathfrak{d}(\alpha,\beta)$ is a fundamental domain for $\mathbb{C}$ with respect to addition by $\mathfrak{a}$. Also if we let $\tilde{\mathfrak{d}}(\alpha,\beta)$ denote the subset of $\mathfrak{d}(\alpha,\beta)$ where the range of y is restricted to $0 \leq y \leq 1/2$, this gives us a fundamental domain for $\mathfrak{d}(\alpha,\beta)$ with respect to multiplication by $Z = \{\pm1\}$

In particular $\mathfrak{d}(1,\sqrt{-5}), \mathfrak{d}(2, 1+\sqrt{-5}), \mathfrak{d}(1, \frac{1-\sqrt{-5}}{2})$ are fundamental domains for $\mathbb{C}$ with respect to addition by $\mathcal{O}, \mathfrak{a}_2$ and $\mathfrak{a}_2^{-1}$ respectively, and we can put $\tilde{\mathfrak{d}}_{11} = \tilde{\mathfrak{d}}_{12} = \tilde{\mathfrak{d}}(1,\sqrt{-5})$, $\tilde{\mathfrak{d}}_{21} = \tilde{\mathfrak{d}}(1,\frac{1-\sqrt{-5}}{2})$ and $\tilde{\mathfrak{d}}_{22} = \tilde{\mathfrak{d}}(2,\sqrt{-5})$. Then

$$F_{i,j}^{2,1}(P_1,P_1) = \left\{ \begin{bmatrix} 1 & 0 \\ \bar{d} & 1 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} : \begin{array}{l} b,c\in\mathbb{R}_{>0},\ d\in\tilde{\mathfrak{d}}_{ij} \\ |e+df|^2 + \frac{c}{b}|f|^2 \geq 1, \\ \left[\begin{smallmatrix} e \\ f \end{smallmatrix}\right] \in \frac{1}{N(\mathfrak{a}_j)}\Xi_{i,j,1} \cup \frac{2}{N(\mathfrak{a}_j)}\Xi_{i,j,2} \end{array} \right\}.$$

Writing $F_{i,j}^{2,1}(P_1,P_1)$ as $F_{i,j}$ for short, we obtain the fundamental domains $\Omega_1^2 = F_{1,1} \cup {}^t\bar{\xi}_{12}F_{1,2}\xi_{12}$ for $P_2/\Gamma_2(\mathfrak{a}_1)$ and $\Omega_2^2 = F_{1,1} \cup {}^t\bar{\xi}_{22}F_{2,2}\xi_{22}$ for $P_2/\Gamma_2(\mathfrak{a}_2)$.

References

[1] A. Borel, *Ensembles fondamentaux pour les groupes arithmetiques*, Colloque sur la Théorie des Groupes Algébriques, Centre Belge de Recherches Mathématiques (1962), 23-40

[2] A. Borel, *Some finiteness properties of adele groups over number fields*, I.H.É.S. Publications Mathmatiques **16** (1963), 121-122.

[3] H. Cohen, Advanced Topics in Computational Number Theory, Graduate Texts in Mathematics **193** (2000), Springer-Verlag New York

[4] D. Grenier, *Fundamental domains for the general linear group*, Pacific Journal of Mathematics, **132** (1988), 293-317.

[5] P. Humbert, *Théorie de la réduction des formes quadratiques définies positives dans un corps algébrique K fini*, Comment. Math. Helv., **23** (1939), 263-306.

[6] A. Korkin and G. Zolotarev, *Sur les formes quadratiques*, Math. Annalen **6** (1873), 366-389.

[7] T.W. Lee, *Fundamental domains of arithmetic quotients of reductive groups over number fields (with appendix by Takao Watanabe)*, Pacific Journal of Mathematics, in press.

[8] H. Minkowski, *Gesammelte Abhandlungen*, Chelsea, New York, 1967.

[9] T. Watanabe, *Ryshkov domains of reductive algebraic groups*, Pacific Journal of Mathematics, **270** (2014), 237-255.

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