

# A DECOMPOSITION OF THE FOURIER-JACOBI COEFFICIENTS OF KLINGEN EISENSTEIN SERIES

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## 1. INTRODUCTION

In analogy to the decomposition of the space of Siegel modular forms of fixed weight and degree into the space of cusp forms and spaces of Eisenstein series of Klingen type associated to cusp forms, Dulinski showed in [3] that the space of Jacobi forms of fixed weight, degree and index admits a natural decomposition into a direct sum of the space of cusp forms and certain spaces of Jacobi Eisenstein series of Klingen type. In [2], Böcherer studied how the Fourier-Jacobi coefficients of square free index of a Klingen Eisenstein series of degree 2 behave under this decomposition, i. e., how one can identify the components in Dulinski's decomposition of these Fourier-Jacobi coefficients. In particular, whereas cusp forms have cuspidal Fourier-Jacobi coefficients and the Siegel Eisenstein series has Siegel-Jacobi Eisenstein series as Fourier-Jacobi coefficients, he showed that the Fourier-Jacobi coefficients of the Klingen Eisenstein series of degree 2 attached to elliptic cusp forms have both a cuspidal and an Eisenstein series part.

We continue this investigation here, using a different method, and obtain an explicit description of the components for arbitrary degree and index. Again, one sees that more than one component appears.

This article and the talk at this RIMS workshop on this topic by the second author on which it is based give an overview of the work of the first author in his doctoral dissertation [7] written at Universität des Saarlandes under the supervision of the second author. Most of the proofs are only sketched, we refer to the dissertation for full details. All results are due to the first author, the second author takes responsibility for the present write-up and all possible mistakes in it.

## 2. PRELIMINARIES

For the basic notions of the theory of Siegel modular forms we refer to [4, 6], for Jacobi forms to [3]. In particular, we consider for  $k > n + 1$  the decomposition  $\mathcal{M}_n^k = \bigoplus_{m=0}^n \mathcal{M}_{n,m}^k$  of the space of Siegel modular forms of weight  $k$  and degree  $n$  for the full modular group  $Sp_n(\mathbb{Z})$  into the spaces  $\mathcal{M}_{n,m}^k$  generated by Eisenstein series  $E_{n,m}^k(f)$  of Klingen type associated to a cusp form  $f \in \mathcal{M}_m^k$ . For  $F \in \mathcal{M}_n^k$  we denote its Fourier coefficient at the symmetric matrix  $T$  by  $A(F, T)$ , here  $T$  runs over the

set  $\widehat{\text{Mat}}_n^{\text{sym}}(\mathbb{Z})$  of positive definite half integral symmetric matrices of size  $n$  with integral diagonal.

For  $n' < n$  and  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{n'}(\mathbb{R}) \subseteq GL_{2n'}(\mathbb{R})$  we write

$$g^{\uparrow n} = \begin{pmatrix} A & 0 & B & 0 \\ 0 & \mathbf{1}_{n-n'} & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n-n'} \end{pmatrix}, g^{\downarrow n} = \begin{pmatrix} \mathbf{1}_{n-n'} & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & \mathbf{1}_{n-n'} & 0 \\ 0 & C & 0 & D \end{pmatrix},$$

for  $U \in GL_n(\mathbb{R})$  write  $L(U) = \begin{pmatrix} {}^tU^{-1} & 0 \\ 0 & U \end{pmatrix} \in Sp_n(\mathbb{R})$ .

We let  $C_{n,r} \subseteq Sp_n(\mathbb{Z})$  denote the intersection with  $Sp_n(\mathbb{Z})$  of the maximal parabolic  $P_{n,r}(\mathbb{Q})$  of  $Sp_n(\mathbb{Q}) \subseteq GL_{2n}(\mathbb{Q})$  characterized as the set of  $g = (g_{ij}) \in Sp_n(\mathbb{Q})$  with  $g_{ij} = 0$  for  $i > n+r, j \leq n+r$  and  $J_{n,r} \subseteq C_{n,r}$  (the Jacobi group of degree  $(n, r)$ ) as the set of elements of  $C_{n,r}$  with an  $(n-r) \times (n-r)$  identity matrix in the lower right hand corner. Notice that, with  $n = r_1 + r_2$ , Dulinski [3] writes  $J^{r_1, r_2} \subseteq C_{r_1+r_2, r_1}$  for this group.

For  $s \leq r$  we divide an  $n \times n$ -matrix into blocks of sizes

$$\begin{pmatrix} s \times s & s \times (r-s) & s \times (n-r) \\ (r-s) \times s & (r-s) \times (r-s) & (r-s) \times (n-r) \\ (n-r) \times s & (n-r) \times (r-s) & (n-r) \times (n-r) \end{pmatrix}$$

and let

$$Q_s^{r, n-r} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z}) \mid C = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, D = \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & \mathbf{1}_{n-r} \end{pmatrix} \right\}.$$

With a block division of type  $\begin{pmatrix} s \times s & s \times (n-r) & s \times (r-s) \\ (n-r) \times s & (n-r) \times (n-r) & (n-r) \times (r-s) \\ (r-s) \times s & (r-s) \times (n-r) & (r-s) \times (r-s) \end{pmatrix}$  we let

$$\tilde{Q}_s^{r, n-r} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z}) \mid C = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, D = \begin{pmatrix} * & * & * \\ 0 & \mathbf{1}_{r-s} & * \\ 0 & 0 & * \end{pmatrix} \right\}.$$

For  $n = r_1 + r_2$  and  $T \in \widehat{\text{Mat}}_{r_2}^{\text{sym}}(\mathbb{Z})$  we denote by  $\mathcal{J}_{r_1, r_2}^k(T)$  the space of Jacobi forms of weight  $k$ , degree  $(r_1, r_2)$  and index  $T$  (which have good transformation behavior under the Jacobi group  $J_{n, r_1}$ ). A Siegel modular form then has a Fourier-Jacobi expansion

$$F(Z) = \sum_{T_4 \in \widehat{M}_{r_2}^{\text{sym}}(\mathbb{Z})} \phi_{T_4}(z_1, z_2) e(T_4 z_4) = \sum_{T_4} \phi^{(T_4)}(Z),$$

with Fourier-Jacobi coefficients  $\phi_{(T_4)} \in \mathcal{J}_{r_1, r_2}^k(T)$  of degree  $(r_1, r_2)$ , index  $T_4$  and weight  $k$ , where  $Z = \begin{pmatrix} z_1 & z_2 \\ {}^t z_2 & z_4 \end{pmatrix}$  is in the Siegel upper half plane  $\mathfrak{H}_n$  of degree  $n$  with  $z_1 \in \mathfrak{H}_{r_1}, z_4 \in \mathfrak{H}_{r_2}, z_2 \in \text{Mat}_{r_1, r_2}(\mathbb{C})$ .

## A DECOMPOSITION OF THE FOURIER-JACOBI COEFFICIENTS

By Theorem 2 of [3] the space  $\mathcal{J}_{r_1, r_2}^k(T)$  has a decomposition

$$\mathcal{J}_{r_1, r_2}^k(T) = \bigoplus_{s=0}^{r_1} \mathcal{J}_{(r_1, r_2), s}^k(T),$$

where the elements of  $\mathcal{J}_{(r_1, r_2), s}^k(T)$  are Jacobi Eisenstein series of Klingen type associated to Jacobi cusp forms of degree  $(s, r_2)$  with varying index  $T'$  for which  $T'[U] = T$  for some integral matrix  $U$ . Dulinski defines these Jacobi Eisenstein series of Klingen type only for index  $T$  of maximal rank. For  $T$  of rank  $t < r_2$  we notice that by [8] the space  $\mathcal{J}_{r_1, r_2}(T)$  is isomorphic to  $\mathcal{J}_{r_1, r_2}(\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix})$  with a  $T_1$  which is positive definite of size  $t$  and that this latter space is isomorphic to  $\mathcal{J}_{r_1, t}(T_1)$ . These isomorphisms allow to transfer Dulinski's definitions to index of arbitrary rank.

Our task is then to identify the components in this decomposition of the Fourier-Jacobi coefficients of an Eisenstein series of Klingen type as explicitly as possible.

## 3. PARTIAL SERIES OF THE KLINGEN EISENSTEIN SERIES

**Lemma 3.1.** *For  $0 \leq m < n, 0 \leq r_1 < n$  and  $0 \leq t \leq \min(n - m, n - r_1)$  let  $M_{n, m, r_1}^t$  denote the set of all  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z})$  for which the lower right  $(n - m) \times (n - r_1)$  block  $C_{22}$  of  $C$  has rank  $t$ .*

*Then the sets  $M_{n, m, r}^t$  are left  $C_{n, m}$  and right  $C_{n, r}$ -invariant, and for fixed  $j, r$  their (disjoint) union over  $0 \leq t \leq \min(n - m, n - r_1)$  is  $Sp_n(\mathbb{Z})$ .*

*Proof.* This is easily checked, see the proof of Proposition 5.2 of [7].  $\square$

**Proposition 3.2.** *Let  $f \in \mathcal{M}_m^k$  be a cusp form.*

- i) *For  $0 \leq m < n, 0 < r_1 < n$  and  $0 \leq t \leq \min(n - m, n - r_1)$  the partial series*

$$H_{n, m, r_1}^t(f; Z) := \sum_{\gamma \in C_{n, m} \setminus M_{n, m, r_1}^t} f(\gamma \langle Z \rangle^*) j(\gamma, Z)^{-k}$$

*of the Eisenstein series  $E_{n, m}^k(f)$  of Klingen type is well defined and invariant under the action  $H \mapsto H|_k g$  of  $g \in J_{n, r_1}$ .*

- ii) *For  $0 \leq m < n$  one has for each  $r_1$  with  $0 \leq r_1 < n$  the decomposition*

$$E_{n, m}^k(f) = \sum_{t=0}^{\min(n-m, n-r_1)} H_{n, m, r_1}^t(f).$$

- iii) *The partial series  $H_{n, m, r_1}^t(f)$  has a Fourier-Jacobi decomposition*

$$H_{n, m, r_1}^t(f; Z) := \sum_T \Psi_{n, m, r_1}^{(T), t}(f; Z) = \sum_T \Psi_{n, m, r_1; T}^t(f; z_1, z_2) e(Tz_4),$$

T. PAUL, R. SCHULZE-PILOT

where the  $\Psi_{n,m,r_1;T}^t(f; z_1, z_2)$  are Jacobi forms of degree  $(r_1, n - r_1)$  and index  $T$ .

*Proof.* Obvious. The last assertion follows since both the existence of an expansion as given and the transformation behavior of the coefficients in it hold for functions on  $\mathfrak{H}_n$  which are  $J_{n,r_1}$ -invariant but not necessarily Siegel modular forms.  $\square$

**Remark 3.3.** Divide a matrix  $M \in \text{Mat}_n(\mathbb{R})$  for  $0 < m, r < n$  into blocks  $M_{11}, M_{12}, M_{21}, M_{22}$  of sizes  $j \times r, j \times (n - r), (n - m) \times r, (n - m) \times (n - r)$  respectively.

For  $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  let  $\gamma'$  be the  $(n + m) \times (n + r)$  matrix obtained from  $\gamma$  by removing the blocks  $A_{21}, A_{22}, B_{22}, B_{12}, D_{12}, D_{22}$  in the second block row and the last block column. Then it can be shown ([7, Satz 5.24]) that the set  $M_{n,m,r}^t$  is the set of all  $\gamma \in \text{Sp}_n(\mathbb{Z})$  for which  $\gamma'$  has rank  $m + r + t$ .

In order to compute the partial series given above one needs explicit coset representatives for  $C_{n,m} \backslash M_{n,m,r}^t$ :

**Theorem 3.4.** Let  $\mathcal{R}_1^s$  for  $s \leq r$  denote a set of representatives of the double cosets in  $L^{-1}(C_{m+r+t-2s,r-s}) \backslash GL_{m+r+t-2s}^{r-s,*}(\mathbb{Z}) / L^{-1}(J_{m+r+t-2s,r-s})$  and  $\mathcal{R}_2^s$  a set of representatives of the cosets in

$$\left( \begin{array}{cc} * & * \\ 0_{n-m-t+s-r,m+t+s} & * \end{array} \right) \in GL_{n-r}(\mathbb{Z}) \backslash GL_{n-r}(\mathbb{Z}),$$

where  $GL_{m+r+t-2s}^{r-s,*}(\mathbb{Z})$  denotes the set of matrices in  $GL_{m+r+t-2s}(\mathbb{Z})$  for which the  $(r - s) \times (r - s)$  block in the lower left corner has full rank  $r - s$ .

For  $u \in GL_{m+r+t-2s}^{r-s,*}(\mathbb{Z})$  we put

$$\hat{u} = \begin{pmatrix} 1_s & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1_{n+s-m-t-r} \end{pmatrix} \in GL_n(\mathbb{Z})$$

and for  $u' \in GL_{n-r}(\mathbb{Z})$  we put  $\tilde{u}' = \begin{pmatrix} 1_r & 0 \\ 0 & u' \end{pmatrix} \in GL_n(\mathbb{Z})$ .

Then a set of representatives of the cosets in  $C_{n,m} \backslash M_{n,m,r}^t$  is given by the matrices

$$\gamma_1^{\uparrow n} L(\hat{u}) \gamma_2 L(\tilde{u}'),$$

where for  $s$  running from  $\max(r + m + t - n, 0)$  to  $\min(j, r)$  one lets  $u$  run through  $\mathcal{R}_1^s$  and  $u'$  through  $\mathcal{R}_2^s$ ,  $\gamma_1$  runs through a set of representatives for  $C_{m+t,m} \backslash M_{m+t,m,s}^t$  and  $\gamma_2$  through a set of representatives of

$$J_{m+t+r-s,r}^{\uparrow n} \cap L(\hat{u}^{-1}) (\tilde{Q}_s^{r,m+t-s})^{\uparrow n} L(\hat{u}) \backslash J_{m+t+r-s,r}^{\uparrow n}.$$

*Proof.* This is Satz 5.21 of [7]. The rather technical proof occupies most of Section 5.  $\square$

## A DECOMPOSITION OF THE FOURIER-JACOBI COEFFICIENTS

## 4. THE FOURIER-JACOBI COEFFICIENTS OF THE PARTIAL SERIES

**Lemma 4.1.** *Let  $f \in M_m^k$  be a cusp form. With the notations of Theorem 3.4 let  $s, u, u'$  be fixed and let  $\gamma_1, \gamma_2$  run through the sets specified there.*

*Then the partial sum*

$$\sum_{\gamma_1} \sum_{\gamma_2} f(\gamma_1^{\uparrow n} L(\hat{u}) \gamma_2 L(\tilde{u}') \langle Z \rangle^*) j(\gamma_1^{\uparrow n} L(\hat{u}) \gamma_2 L(\tilde{u}'), Z)^{-k}$$

*has a Fourier-Jacobi expansion of degree  $(r_1, r_2)$  with coefficients in  $\mathcal{J}_{(r_1, r_2), s}^k(T')$  whose index  $T'$  has rank  $m + t - s$ .*

*In particular, for  $m + t = n$  and  $s = r_1$  the  $T'$  occurring have maximal rank  $r_2$  and the Fourier-Jacobi coefficients are cusp forms.*

*Proof.* The first part of the assertion is formulated on p. 57 of [7] before Lemma 6.3, its proof uses Lemma 6.3, 6.4, 6.6., where Lemma 6.6 is the second part of our assertion.  $\square$

**Theorem 4.2.** i) *The partial series  $H_{n, m, r_1}^t(f)$  has a Fourier-Jacobi expansion whose coefficient  $\Psi(T) := \Psi_{n, m, r_1, T}^t(f)$  at  $T$  is in  $\mathcal{J}_{(r_1, r_2), m+t-\text{rk}(T)}^k(T)$ .*

ii) *Let  $\phi(T)$  denote the Fourier-Jacobi coefficient at  $T \in \widehat{\text{Mat}}_{r_2}^{\text{sym}}(\mathbb{Z})$  of degree  $(r_1, r_2)$  of the Eisenstein series  $E_{n, m}(f)$  and let  $\Psi(T)$  be as in i). Then  $\Psi(T)$  is the component  $\phi_{(r_1, r_2), m+t-\text{rk}(T)}(T)$  of  $\phi(T)$  in the space  $\mathcal{J}_{(r_1, r_2), m+t-\text{rk}(T)}^k(T)$  in Dulinski's decomposition.*

*Proof.* The first assertion is proven in [7] in the calculation following equation (6.2) on page 60 by using the lemma above and carrying out the summation over  $u, u'$  from the set of representatives given in Theorem 3.4. The second assertion follows since the components in Dulinski's decomposition are uniquely determined and  $E_{n, m}(f)$  is the sum of the partial series  $H_{n, m, r_1}^t(f)$ .  $\square$

**Remark 4.3.** *In particular, we see that only the spaces  $\mathcal{J}_{(r_1, r_2), s}^k(T)$  with  $m - \text{rk}(T) \leq s \leq \min(n - \text{rk}(T), m + r_2 - \text{rk}(T))$ . For  $m = n$  the lower bound and  $\text{rk}(T) \leq r_2$  give  $s \geq r_1$ , hence  $s = r_1$ , i.e., the Fourier-Jacobi coefficients of a cusp form are Jacobi cusp forms, which is trivial.*

*For  $m = 0$  we obtain  $s \leq r_2 - \text{rk}(T)$ , so the Fourier-Jacobi coefficients with index of maximal rank of the Siegel Eisenstein series are Jacobi Eisenstein series of Siegel type, which is known from [1]. For  $\text{rk}(T) < r_2$  the Fourier-Jacobi coefficient of degree  $(r_1, r_2)$  with index  $T$  is essentially the Fourier-Jacobi coefficient of degree  $(r_1, \text{rk}(T))$  of the Siegel Eisenstein series of degree  $n - (r_2 - \text{rk}(T))$  at a matrix of maximal rank, so it is again a Jacobi Eisenstein series of Siegel type.*

## 5. PULLBACKS AND FOURIER EXPANSIONS

Having identified the components in Dulinski's decomposition of the Fourier-Jacobi expansion of the Eisenstein series  $E_{n,m}(f)$  in terms of the coefficients of the partial series  $H_{n,m,r_1}^t$  we turn now to the task of computing their Fourier expansion explicitly. For this we adapt and refine ideas from [1] to our situation and divide the series defining the Siegel Eisenstein series of degree  $n + m$  into certain subseries in way similar to what we did in Section 3.

**Lemma 5.1.** *Divide for  $j, r \leq n$  a matrix  $M \in \text{Mat}_{n+m}(\mathbb{R})$  into blocks of types*

$$\begin{pmatrix} j \times r & j \times (n-r) & j \times j \\ (n-m) \times r & (n-m) \times (n-r) & (n-m) \times j \\ j \times r & j \times (n-r) & j \times j \end{pmatrix}$$

and denote these blocks by  $M_{11}, \dots, M_{33}$ . For  $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{n+m}(\mathbb{Z})$  let  $\hat{\gamma} = \begin{pmatrix} C_{11} & C_{12} & D_{11} \\ C_{21} & C_{22} & D_{21} \\ C_{31} & C_{32} & D_{31} \end{pmatrix} \in \text{Mat}_{n+m, n+r}(\mathbb{Z})$  and denote for  $m+r \leq v \leq \min(n+m, n+r)$  the set of all  $\gamma \in Sp_{n+m}(\mathbb{Z})$  with  $\text{rk}(\hat{\gamma}) = v$  by  $X_{n,m,r}^v$ . then  $X_{n,m,r}^v$  is left invariant under  $C_{n+m,0}$  and right invariant under  $Sp_m^{n+m}(\mathbb{Z})$ , and  $Sp_{n+m}(\mathbb{Z})$  is the disjoint union of the  $X_{n,m,r}^v$  for  $m+r \leq v \leq \min(n+m, n+r)$ .

*Proof.* This is Proposition 7.2 of [7]. Since  $\hat{\gamma}$  is obtained from  $\gamma$  by deleting  $n-r$  columns and  $n-m$  rows, its rank  $v$  must be between  $m+r$  and  $\min(n+m, n+r)$ . the assertions about left and right invariance are checked easily.  $\square$

We need an explicit set of representatives of the cosets in  $C_{n+m,0} \backslash X_{n,m,r}^v$ . For this we recall that by [5] a set of representatives for  $C_{n+m,0} \backslash Sp_{n+m}(\mathbb{Z})$  is given by the products

$$g_{j,M}(g'_{j,0})^{\uparrow n+m} g_j^{\uparrow n+m} ((g''_{j,1})^{\uparrow m})^{\downarrow}_{n+m} (g''_j)^{\downarrow}_{n+m},$$

where  $j$  runs from 0 to  $m$ , and for any such  $j$  we let  $g'_{j,0}$  run through  $Sp_j(\mathbb{Z})$ ,  $g'_j$  through a set of representatives for  $C_{n,j} \backslash Sp_n(\mathbb{Z})$  and  $g''$  through a set of representatives for  $C_{m,j} \backslash Sp_m(\mathbb{Z})$ . Moreover, with  $M'$  running through the  $j \times j$  elementary divisor matrices and  $M = \begin{pmatrix} M' & 0 \\ 0 & 0 \end{pmatrix} \in M_{m,n}(\mathbb{Z})$  we let  $g_{j,M} = \begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & 1_m & 0 & 0 \\ 0 & {}^t M & 1_n & 0 \\ M & 0 & 0 & 1_m \end{pmatrix}$  and  $\Gamma_j(M') := Sp_j(\mathbb{Z}) \cap \begin{pmatrix} 0 & M'^{-1} \\ M' & 0 \end{pmatrix} Sp_s(\mathbb{Z}) \begin{pmatrix} 0 & M'^{-1} \\ M' & 0 \end{pmatrix}$  and let  $g''_{j,1}$  run through a set of representatives of  $\Gamma_j(M') \backslash Sp_j(\mathbb{Z})$ .

**Proposition 5.2.** *A set of representatives for  $C_{n+m,0} \backslash X_{n,m,r}^v$  is obtained from the representatives above by restricting  $g'_j$  to a set of representatives of  $C_{n,j} \backslash M_{n,j}^{v-r-j}$ .*

## A DECOMPOSITION OF THE FOURIER-JACOBI COEFFICIENTS

*Proof.* A straightforward computation shows that indeed these are precisely the products which are in  $X_{n,n,r}^v$ , see Satz 7.4 of [7] and the proof given there.  $\square$

**Theorem 5.3.** *For  $0 < s \leq m$  let  $(f_{s,\nu})_\nu$  be an orthonormal basis of Hecke eigenforms for the space of cusp forms of degree  $s$  and weight  $k$ .*

*We set  $A_s^k := \pi^{\frac{s(s-1)}{4}} (4\pi)^{\frac{s(s+1)}{2} - sk} \prod_{i=1}^s \Gamma(k - \frac{s+i}{2})$  and*

$$\beta(s, k) = (-1)^{\frac{sk}{2}} s^{s(k - \frac{s-1}{2})} \prod_{i=0}^{s-1} \frac{\pi^{k - \frac{i}{2}}}{\Gamma(k - \frac{i}{2})} \zeta(k)^{-1} \prod_{i=1}^m \zeta(2k - 2i)^{-1}.$$

*For  $0 \leq m, r < n$  and  $m + r \leq v \leq \min(n + m, n + r)$  we put*

$$G_{n,m,r}^v(Z) := \sum_{\gamma \in C_{n+m,0} \backslash X_{n,m,r}^v} j(\gamma, Z)^{-k}.$$

*Then for  $Z_1 \in \mathfrak{H}_n, Z_2 \in \mathfrak{H}_m$  the pullback  $G_{n,m,r}^v\left(\begin{pmatrix} -\overline{Z_1} & 0 \\ 0 & Z_2 \end{pmatrix}\right)$  of  $G_{n,m,r}^v$  to  $\mathfrak{H}_n \times \mathfrak{H}_m$  can be written as*

$$G_{n,m,r}^v\left(\begin{pmatrix} -\overline{Z_1} & 0 \\ 0 & Z_2 \end{pmatrix}\right) = \sum_{s=0}^m c_s \sum_{\nu} D_{f_{s,\nu}}(k-s) E_{m,s}(f_{s,\nu}; Z_2) \overline{H_{n,s,r}^{v-r-s}(f, Z_1)},$$

*where  $D_{f_{s,\nu}}$  denotes the standard  $L$ -function of the Hecke eigenform  $f_{s,\nu}$  (and this factor doesn't occur for  $s = 0$ ) and where for  $s > 0$  we put  $c_s = 2\beta(s, k)A_s^k$  and set  $c_0 = 1$ .*

*Proof.* This follows from the proof of the theorem in Section 5 of [5] and the explicit evaluation of the constants occurring there in [1].  $\square$

**Corollary 5.4.** *For a Hecke eigenform  $f \in M_m^k$  of Petersson norm 1 one has*

$$H_{n,m,r}^{v-r-m}(f; Z_1) = \lambda(f)^{-1} \left\langle f(\cdot), G_{n,m,r}^v\left(\begin{pmatrix} -\overline{Z_1} & 0 \\ 0 & \cdot \end{pmatrix}\right) \right\rangle$$

*with  $\lambda(f) = 2\beta(m, k)A_m^k D_f(k - m)$  as in the theorem above.*

*Proof.* This follows since taking the Petersson product with  $f$  singles out the summand containing  $H_{n,m,r}^{v-r-m}(f, Z_1)$  from the formula in the theorem.  $\square$

By the corollary we can compute the Fourier expansion of our partial series  $H_{n,m,r}^{v-r-m}(f)$  by computing the Petersson product on the right hand side. We will do this adapting again ideas from [1].

**Lemma 5.5.** i) *Let  $P_{n,m} = \begin{pmatrix} 0 & 1_m \\ 1_n & 0 \end{pmatrix}$ . Then for  $l \leq n$  the set  $M_{n+m,0,n}^l L(P_{n,m}) \cap X_{n,m,r}^v$  is nonempty only if  $l \leq v$  and  $X_{n,m,r}^v$  is contained in the (disjoint) union of the  $M_{n+m,0,n}^l L(P_{n,m})$  for  $0 \leq l \leq v$ .*

ii) *With*

$$G_{n,m,r}^{v,l}(Z) := \sum_{M_{n+m,0,n}^l \cap L(P_{n,m}) \cap X_{n,m,r}^v} j(\gamma, Z)^{-k}$$

one has  $G_{n,m,r}^v(Z) = \sum_{l=0}^v G_{n,m,r}^{v,l}(Z)$ .

iii) *A set of representatives of  $C_{n+m,0} \setminus M_{n+m,0,n}^l \cap L(P_{n,m}) \cap X_{n,m,r}^v$  is given by the  $x^{\uparrow n+m} L(U) y^{\downarrow n+m}$ , where  $x$  runs through a set of representatives of  $C_{l,0} \setminus M_{l,0,0}^l$ ,  $y$  through a set of representatives of  $C_{m,0} \setminus Sp_m(\mathbb{Z})$  and  $U$  through a set of representatives of*

$$\left\{ \left( \begin{array}{ccc} * & * & * \\ 0_{n-l,l} & * & * \\ 0_{m,l} & * & * \end{array} \right) \setminus \left( \begin{array}{ccc} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{array} \right) \in GL_{n+m}(\mathbb{Z}) \mid \text{rk} \begin{pmatrix} u_4 \\ u_7 \end{pmatrix} = v - l, \text{rk} \begin{pmatrix} u_6 \\ u_9 \end{pmatrix} = m \right\},$$

where  $U$  has a block division of type  $\begin{pmatrix} l \times r & l \times (n-r) & l \times m \\ (n-l) \times r & (n-l) \times (n-r) & (n-l) \times m \\ m \times r & m \times (n-r) & m \times m \end{pmatrix}$ .

*Proof.* This is Satz 8.1 of [7]. For the proof one checks which of the representatives of  $C_{n+m,0} \setminus M_{n+m,0,n}^l \cap L(P_{n,m})$  obtained from Theorem 3.4 are in  $X_{n,m,r}^v$ , see [7] for details.  $\square$

**Lemma 5.6.** *Let  $U$  run through the set of representatives from the previous lemma and write a matrix in  $\text{Mat}_{n+m,l}(\mathbb{Z})$  as  $\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ , where  $w_1, w_2, w_3$  have  $r, n-r, m$  rows respectively. Then the matrix formed by the first  $l$  columns of  $U^{-1}$  runs through a set of representatives of*

$$\left\{ \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \text{ primitive} \mid \text{rk} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = l, \text{rk} \begin{pmatrix} w_2 \\ w_3 \end{pmatrix} = v - r \right\} / GL_l(\mathbb{Z}).$$

*Proof.* This is Lemma 8.4 a) of [7]. The proof uses computations from Lemma 5.7 and Remark 5.8 of [7].  $\square$

**Lemma 5.7.** *We denote by  $a_l(T)$  the Fourier coefficient at  $T$  of the Siegel Eisenstein series of degree  $l$  and weight  $k$  and write  $\mathcal{A}_l^+$  for the set of positive definite matrices in  $\widehat{\text{Mat}}_l^{\text{sym}}(\mathbb{Z})$ . Then*

$$G_{n,m,r}^{v,l}(Z) = \sum_{T \in \mathcal{A}_l^+} \sum_{w_1, w_2} \sum_{w_3} \sum_y a_l(T) \times e\left(T \left( (y^{\downarrow n+m} \langle Z \rangle)^* \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right) \right) j(y, z_4)^{-k},$$



## A DECOMPOSITION OF THE FOURIER-JACOBI COEFFICIENTS

where  $y$  runs through a set of representatives of  $C_{m,0} \backslash Sp_m(\mathbb{Z})$ ,  $w_1, w_2, w_3$  are as in the previous lemma, and  $z_4 \in \mathfrak{H}_m$  is the lower left  $m \times m$  corner of  $Z$ .

*Proof.* We carry out the summation over the coset representatives given in Lemma 5.5, expanding the automorphy factor  $j$  using its cocycle relation and  $j(L(U), \cdot) = 1$ . The summation over  $x$  gives then by [1, Lemma 3]

$$\sum_{T \in \mathcal{A}_l^+} \sum_U \sum_y a_l(T) e(T(L(U)y^{\downarrow n+m} \langle Z \rangle)^*) j(Y, z_4)^{-k},$$

Using  $L(U)y^{\downarrow n+m} \langle Z \rangle = y^{\downarrow n+m} \langle Z \rangle [U^{-1}]$  and writing the upper left block of  $U^{-1}$  in terms of  $w_1, w_2, w_3$  as in the previous lemma, we obtain the assertion.  $\square$

**Lemma 5.8.** Write  $\mathbb{Z}_s^{m \times l} = \{w \in \text{Mat}_{m,l}(\mathbb{Z}) \mid \text{rk}(w) = s\}$ ,  $\mathbb{Z}_{s,0}^{m \times l} = \{(0_{m-s,l}^* \in \mathbb{Z}_s^{m \times l})\}$ .

Let  $GL_m(\mathbb{Z})_s = \{(0_{m-s,s}^* \in GL_m(\mathbb{Z}))\}$  and  $GL_m(\mathbb{Z})_s^1 = \{(0_{m-s,s}^1 \in GL_m(\mathbb{Z}))\}$ .

Let  $w'_3$  run through a set of representatives of  $GL_m(\mathbb{Z})_s^1 \backslash \mathbb{Z}_{s,0}^{m \times l}$  and  $w''_3$  through a set of representatives of  $GL_m(\mathbb{Z})/GL_m(\mathbb{Z})_s^1$ . Then every element of  $\mathbb{Z}_s^{m \times l}$  has a unique expression as a product  $w''_3 w'_3$ , and all these products are in  $\mathbb{Z}_s^{m \times l}$ .

For  $w_1, w_2$  fixed, the matrix  $\begin{pmatrix} w_1 \\ w_2 \\ w'_3 \end{pmatrix}$  is primitive if and only if  $\begin{pmatrix} w_1 \\ w_2 \\ w''_3 w'_3 \end{pmatrix}$  is primitive, and one has  $\text{rk}\begin{pmatrix} w_2 \\ w'_3 \end{pmatrix} = \text{rk}\begin{pmatrix} w_2 \\ w''_3 w'_3 \end{pmatrix}$ .

*Proof.* This is Lemma 8.4 b) of [7]. It is clear that any  $u \in \mathbb{Z}_s^{m \times l}$  can be written as  $uw'_3$  with  $w \in GL_m(\mathbb{Z})$  and  $w'_3 \in \mathbb{Z}_{s,0}^{m \times l}$ , where  $w'_3$  is unique up to multiplication with an element of  $GL_m(\mathbb{Z})_s$  from the left. Moreover, if  $w'_3$  is fixed,  $w$  is unique up to right multiplication by an element of  $GL_m(\mathbb{Z})_s^1$ . The second assertion is obvious.  $\square$

**Lemma 5.9.** i) With notations as in Lemma 5.7 the sum

$$\sum_{w_3} e\left(T \left( (y^{\downarrow n+m} \langle Z \rangle)^* \left[ \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right] \right) j(y, z_4)^{-k}$$

for  $T, y, w_1, w_2$  fixed is equal to

$$\sum_s \sum_{w'_3} \sum_{w''_3} e\left(T \left( (L(w''_3)^{-1})^{\downarrow n+m} y^{\downarrow n+m} \langle Z \rangle)^* \left[ \begin{pmatrix} w_1 \\ w_2 \\ w'_3 \end{pmatrix} \right] \right) j(y, z_4)^{-k},$$

where  $s$  runs from 0 to  $\min(l, m)$ ,  $w'_3$  runs over the set of matrices in  $\mathbb{Z}_s^{m \times l}$  for which  $\begin{pmatrix} w_2 \\ w'_3 \end{pmatrix}$  has rank  $v - r$ , and  $w''_3$  runs over a set of representatives of  $GL_m(\mathbb{Z})/GL_m(\mathbb{Z})_s^1$ .

T. PAUL, R. SCHULZE-PILLOT

ii) For a block diagonal matrix  $Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$  with  $Z_1 \in \mathfrak{H}_n, Z_2 \in \mathfrak{H}_m$  one has

$$G_{n,m,r}^{v,l}(Z) = \sum_{T \in \mathcal{A}_l^+} a(T) \sum_{w_1, w_2} e\left(T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) Z_1 \\ \times \sum_{s=0}^{\min(l,m)} \epsilon(s) \sum_{w'_3} g_{m,s}^k(Z_2, T \begin{bmatrix} w'_3 \end{bmatrix}),$$

with  $\epsilon(0) = 1$  and  $\epsilon(s) = 2$  otherwise, where the summations over  $w_1, w_2, w'_3$  are as before and where the Poincaré series  $g_{m,s}^k(Z_2, T \begin{bmatrix} w'_3 \end{bmatrix})$  is given by

$$g_{m,s}^k(Z_2, T_1^\uparrow) = \sum_{\gamma \in \mathfrak{U}_{m,s} \backslash Sp_m(\mathbb{Z})} e(T_1^\uparrow(\gamma \langle Z \rangle)) j(\gamma, Z)^{-k},$$

where  $\mathfrak{U}_{m,s} \subseteq C_{m,0}$  is the group of matrices  $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in C_{m,0} \subseteq Sp_m(\mathbb{Z})$ , with  $A = \begin{pmatrix} \pm 1 & * \\ 0 & * \end{pmatrix}$ .

*Proof.* For a) we use the decomposition  $w_3 = w''_3 w'_3$  from the previous lemma and order the sum over  $w'_3$  by the rank  $s$  of  $w'_3$ . For b), with  $\mathfrak{U}_{m,s}^+ = \{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathfrak{U}_{m,s} \mid A = \begin{pmatrix} 1_s & * \\ 0 & * \end{pmatrix} \}$  we see that  $L(w''_3{}^{-1})$  runs through a set of representatives of  $\mathfrak{U}_{m,s}^+ \backslash C_{m,0}$ , so that  $L(w''_3{}^{-1})y$  runs through a set of representatives  $\tilde{y}$  of  $\mathfrak{U}_{m,s}^+ \backslash Sp_m(\mathbb{Z})$  which satisfy  $j(\tilde{y}, Z_2) = j(y, Z_2)$  for  $\tilde{y} = L(w''_3{}^{-1})y$  and  $Z_2 \in \mathfrak{H}_m$ . For  $s = 0$  one has  $\mathfrak{U}_{m,s} = \mathfrak{U}_{m,s}^+$ , for  $s > 0$  each coset modulo  $\mathfrak{U}_{m,s}$  is the union of two cosets modulo  $\mathfrak{U}_{m,s}^+$ , which explains the factor  $\epsilon(s)$ . The expression obtained in a) then transforms (with  $z_4 = Z_2$ ) to

$$a_l(T) e\left(T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) Z_1 \sum_s \epsilon(s) \sum_{w'_3} \sum_{\tilde{y} \in \mathfrak{U}_{m,s} \backslash Sp_m(\mathbb{Z})} e(T \begin{bmatrix} w'_3 \end{bmatrix} \tilde{y} \langle Z_2 \rangle) j(\tilde{y}, Z_2)^{-k},$$

and the sum over  $\tilde{y}$  equals the Poincaré series  $g_{m,s}^k(Z_2, T \begin{bmatrix} w'_3 \end{bmatrix})$  (notice that  $T \begin{bmatrix} w'_3 \end{bmatrix}$  has the block diagonal shape required).  $\square$

**Theorem 5.10.** Let  $f(Z) = \sum_{S \in \widehat{\text{Mat}}_m^{\text{sym}}(\mathbb{Z})} b(S) e(SZ) \in M_m^k$  be a cusp form with Fourier coefficients  $b(S)$ .

Then the Fourier coefficient of  $H_{n,m,r}^t(f)$  at  $R \in \widehat{\text{Mat}}_n^{\text{sym}}(\mathbb{Z})$  with  $\text{rk}(R) = l$  is

$$\beta(m, k)^{-1} D_f(k - m)^{-1} \sum_{T \in \mathcal{A}_l^+} \sum_{w_1, w_2} \sum_{w'_3} b(T \begin{bmatrix} w'_3 \end{bmatrix}) \det(T \begin{bmatrix} w'_3 \end{bmatrix})^{\frac{m+1}{2} - k}$$

with  $\beta(m, k), D_f(k - m)$  as in Theorem 5.3.

## A DECOMPOSITION OF THE FOURIER-JACOBI COEFFICIENTS

In the sum,  $\begin{pmatrix} w_1 \\ w_2 \\ w'_3 \end{pmatrix} \in \text{Mat}_{n+m,l}(\mathbb{Z})$  with  $w_1 \in \text{Mat}_{r,l}(\mathbb{Z})$ ,  $w_2 \in \text{Mat}_{n-r,l}(\mathbb{Z})$ ,  $w'_3 \in \text{Mat}_{m,l}(\mathbb{Z})$  runs through those primitive elements of a set of representatives of  $\text{Mat}_{n+m,l}(\mathbb{Z})/GL_l(\mathbb{Z})$  which satisfy

$$R = T \left[ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right], \quad \text{rk}(w'_3) = m, \quad \text{rk} \begin{pmatrix} w_2 \\ w'_3 \end{pmatrix} = t + m.$$

*Proof.* By our previous results only the Petersson product

$$\langle f(\cdot), G_{n,m,r}^{t+m+r,l} \begin{pmatrix} -\bar{z}_1 & 0 \\ 0 & \cdot \end{pmatrix} \rangle$$

contributes to the Fourier coefficient of  $H_{n,m,r}^t(f)$  at a matrix  $R$  of rank  $l$ , and we have reduced the computation of this Petersson product to the product with the Poincaré series  $g_{m,s}^k(Z_2, T[\begin{smallmatrix} t \\ w'_3 \end{smallmatrix}])$ . For  $s < m$ , these are known to be orthogonal to cusp forms (being Eisenstein series of Klingen type), for  $s = m$  the Petersson product has been computed in [6, p.90,94]. Plugging in that result gives the assertion.  $\square$

**Remark 5.11.** *It should be noticed that the sum in the formula of the theorem is a finite sum.*

**Corollary 5.12.** *As in Theorem 4.2 denote by  $\phi_{m+t-\text{rk}(R_4)}^{(R_4)}$  the component in  $\mathcal{J}_{(r_1,r_2),m+t-\text{rk}(T)}^k(T)$  of the Fourier-Jacobi coefficient at the  $r_2 \times r_2$ -symmetric matrix  $R_4$  of the Klingen Eisenstein series  $E_{n,m}^k(f)$ . Then the Fourier coefficient at  $(R_1, R_2)$  of  $\phi_{m+t-\text{rk}(R_4)}^{(R_4)}$  is given by the formula in the previous theorem for the Fourier coefficient of  $H_{n,m,r_1}^t(f)$  at  $R = \begin{pmatrix} R_1 & R_2 \\ tR_2 & R_4 \end{pmatrix}$ .*

*Proof.* This follows directly from the previous theorem and Theorem 4.2.  $\square$

6. THE CASE  $n = 2$ 

We consider here  $r = r_1 = r_2 = m = 1$ , i.e., we study the Klingen Eisenstein series attached to an elliptic cusp form  $f(z) = \sum_{n=1}^{\infty} b(n)e(nz)$ , which we assume to be a Hecke eigenform.

One obtains here  $\beta(m, k)^{-1} D_f(k-1)^{-1} = \frac{1}{2} \zeta(1-k) \zeta(2k-2) L_2(f, 2k-2)^{-1}$ , where  $L_2(f, s) = \zeta(2s - sk + 2) \sum_{n=1}^{\infty} b(n^2) n^{-s}$  is the symmetric square  $L$ -function of  $f$ . We have to consider the  $H_{2,1,1}^t$  for  $t = 0, t = 1$ . For  $t = 1$  our computation in the previous paragraph shows that  $H_{2,1,1}^1$  has nonzero Fourier coefficients only at matrices  $R = \begin{pmatrix} r_1 & \frac{r_2}{2} \\ \frac{r_2}{2} & r_4 \end{pmatrix}$  of rank 2. The Fourier coefficient at such an  $R$  is then computed as

$$\begin{aligned} & \frac{1}{2} \zeta(1-k) \zeta(2k-2) L_2(f, 2k-2)^{-1} \sum_{a,b,d} a_2(T) \\ & \times \sum_{u,v} b(u^2 t_1 + uvt_2 + v^2 t_4) (u^2 t_1 + uvt_2 + v^2 t_4)^{1-k}, \end{aligned}$$

where the summation over  $a, b, d$  runs over  $a, d > 0$  and  $0 \leq b < a$  such that

$$T = \begin{pmatrix} t_1 & \frac{t_2}{2} \\ \frac{t_2}{2} & t_4 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} R \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} \in \widehat{\text{Mat}}_2^{\text{sym}}(\mathbb{Z})$$

and the summation over  $u, v$  runs over  $u, v \in \mathbb{Z}$  satisfying  $u \neq 0$ ,  $\gcd(u, a) = \gcd(av - ub, d) = 1$ . If  $-\det(2R)$  is a fundamental discriminant only  $a = d = 1$  occurs, and one checks that this agrees with the result in [2]. One can proceed from here to obtain asymptotic formulas as in [2]. For details see [7, Section 9].

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