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NEW GLOBAL INTEGRALS FOR TENSOR PRODUCT L-FUNCTIONS

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ABSTRACT. This is a report on my joint work [CFGK17] with Solomon Friedberg, David Ginzburg, and Eyal Kaplan. We present a family of global integrals representing tensor product L-functions of classical groups with general linear groups. Our construction is uniform over all classical groups and their non-linear coverings, and is applicable to all cuspidal representations.

1. INTRODUCTION

In the 1980s, Piatetski-Shapiro-Rallis [GPSR87] introduced a Rankin-Selberg integral for the standard L-function for an irreducible cuspidal automorphic representation on classical groups. This construction grew out of Rallis' work on the inner products of theta lifts – the Rallis inner product formula. This is known as the doubling method.

The purpose of this note is to describe a generalization of the doubling method. We start with some notations.

Let F be a number field and $\mathbb{A}$ be its adele ring. Let m be a positive integer. Let G_n be a split classical group of rank n $(GL_n, Sp_{2n}, SO_{2n}, SO_{2n+1}, \cdots)$. Let $G_n^{(m)}(\mathbb{A})$ be an m-fold metaplectic cover of $G_n(\mathbb{A})$ and let $GL_k^{(m)}(\mathbb{A})$ be an m-fold metaplectic cover of $GL_k(\mathbb{A})$. Let π and τ be irreducible genuine cuspidal representations on $G_n^{(m)}(\mathbb{A})$ and $GL_k^{(m)}(\mathbb{A})$, respectively. We give a global integral, representing the tensor product L-function $L(s, \pi \times \tau)$.

Remark 1.1. If $m = 1$, we take $G_n^{(m)}(\mathbb{A}) = G_n(\mathbb{A})$ and $GL_k^{(m)}(\mathbb{A}) = GL_k(\mathbb{A})$. When $k = 1$, our construction recovers the original doubling construction.

Remark 1.2. Our construction gives L-function for π twisted by τ for arbitrary k. It is a "twisted doubling" construction.

Remark 1.3. If $G_n = Sp_{2n}$ and m is even, we need to replace $GL_k^{(m)}(\mathbb{A})$ by $GL_k^{(m/2)}(\mathbb{A})$ to make the covering consistent.

Remark 1.4. The global construction relies on the following unique models: matrix coefficients on $G_n^{(m)}$ and a degenerate type unique models related to τ. Thus, this construction works for arbitrary cuspidal representations on $G_n^{(m)}(\mathbb{A})$, not necessarily generic representations. It is also crucial for covering groups since the dimension of Whittaker models is in general greater than 1.

Remark 1.5. Our construction also works for all non-linear coverings of G_n and GL_k. However, for $m > 1$ (covering group case) we must assume a certain non-vanishing result on residues of metaplectic Eisenstein series. (Currently, this assumption is only known to be valid for a few cases).

2. The first new example

In this section, we give the global integral in the first new case.

Let π be an irreducible cuspidal automorphic representation on $Sp_2(\mathbb{A})$ and τ be an irreducible cuspidal automorphic representation on $GL_2(\mathbb{A})$.

The global integral is

$$\int_{(Sp_2(F)\backslash Sp_2(\mathbb{A}))^2} \varphi_1(g_1)\overline{\varphi_2(g_2)} \int_{U_0(F)\backslash U_0(\mathbb{A})} E_{\sigma(\tau)}(u_0\iota(g_1,g_2);f,s)\psi_{U_0}(u)\, du_0\, dg_1\, dg_2. \tag{1}$$

Here

- φ_1, φ_2 are in the space of π,
- U_0 is a unipotent subgroup of Sp_8, defined by
$$U_0 = \left\{ u_0 = \begin{pmatrix} I_2 & X & Y & Z \\ & I_2 & 0 & Y' \\ & & I_2 & X' \\ & & & I_2 \end{pmatrix} \in Sp_8 : X, Y \in \mathrm{Mat}_{2\times 2} \right\}.$$
The matrices X', Y' and Z are chosen so that the matrix u_0 is in Sp_8.
- $\psi_{U_0}(u_0) = \psi(X_{11} + Y_{22})$ where ψ is a nontrivial additive character on $F\backslash\mathbb{A}$. Here $X_{i,j}$ is the (i,j)-coordinate of X.
- $\iota : Sp_2 \times Sp_2 \to Sp_8$ ("doubling map") is given by
$$(g_1, g_2) \mapsto \begin{pmatrix} g_1 & & & & \\ & g_{11} & & g_{12} & \\ & & g_2 & & \\ & g_{13} & & g_{14} & \\ & & & & g_1^* \end{pmatrix} \text{ where } g_1 = \begin{pmatrix} g_{11} & g_{12} \\ g_{13} & g_{14} \end{pmatrix}.$$
Here g_1^* is chosen so that $\iota(g_1, g_2)$ is in Sp_8.
- The representation $\sigma(\tau)$ is a generalized Speh representation. It is defined as residues of Eisenstein series. The representation $\sigma(\tau)$ can also be understood as the unique irreducible subrepresentation of
$$\mathrm{Ind}_{P(GL_2\times GL_2)(\mathbb{A})}^{GL_4(\mathbb{A})}(\tau|\cdot|^{-1/2} \otimes \tau|\cdot|^{1/2})\delta_{P(GL_2\times GL_2)}^{1/2}.$$
- $E_{\sigma(\tau)}(g; f, s)$ is the Siegel Eisenstein series attached to $\sigma(\tau)$ and f is a flat section in
$$\mathrm{Ind}_{P(GL_4)(\mathbb{A})}^{Sp_8(\mathbb{A})} \sigma(\tau)\delta_{P(GL_4)}^s.$$
Here $P = \left\{ \begin{pmatrix} a & u \\ & a^* \end{pmatrix} \in Sp_8 : a \in GL_4 \right\}$, and
$$E_{\sigma(\tau)}(g; f, s) = \sum_{\gamma \in P(GL_4)(F)\backslash Sp_8(F)} f(\gamma g; s).$$

By a standard unfolding argument, when $\mathrm{Re}(s) \gg 0$, Eq. (1) equals

$$\int_{Sp_2(\mathbb{A})} \langle \pi(g)\varphi_1, \overline{\varphi_2}\rangle \int_{U_1(\mathbb{A})} f_{W_{\sigma(\tau)}}(\delta u_1 \iota(1, g), s)\psi_{U_0}(u_1)\, du_1\, dg. \tag{2}$$

Here

- the pairing $\langle \cdot, \cdot \rangle$ is the matrix coefficient:

$$\langle \varphi_1, \varphi_2 \rangle = \int\limits_{Sp_2(F)\backslash Sp_2(\mathbb{A})} \varphi_1(h)\varphi_2(h)\, dh.$$

- $\delta = \begin{pmatrix} & I_4 \\ -I_4 & \end{pmatrix} \begin{pmatrix} I_2 & & & \\ & I_2 & I_2 & \\ & & I_2 & \\ & & & I_2 \end{pmatrix}$.
- $U_1 = \left\{ \begin{pmatrix} I_2 & 0 & Y & Z \\ & I_2 & 0 & Y' \\ & & I_2 & 0 \\ & & & I_2 \end{pmatrix} \in U_0 : Y \in \mathrm{Mat}_{2\times 2} \right\}$.
- $W_{\sigma(\tau)}(\xi)$ is the global Shalika integral, given by

$$W_{\sigma(\tau)}(\xi) = \int\limits_{(F\backslash\mathbb{A})^4} \xi \begin{pmatrix} I_2 & V \\ & I_2 \end{pmatrix} \psi(\mathrm{tr} V)\, dV$$

  for $\xi \in \sigma(\tau)$.

Both the matrix coefficient $\langle \cdot, \cdot \rangle$ and $W_{\sigma(\tau)}$ are factorizable. Thus Eq. (2) is Eulerian, that is, it decomposes as an infinite product over all places of F. At an unramified place, Eq. (2) equals

$$\frac{L(5s-2, \pi \times \tau)}{L(5s-1, \tau)L(10s-3, \wedge^2, \tau)L(10s-4, \vee^2, \tau)}.$$

3. Comparison with previous works

In this section, we compare our construction with Langlands's calculation of constant terms of Eisenstein series and Shahidi's theory of local coefficients.

- Langlands [Lan71] computed constant terms of Eisenstein series and used it to deduce meromorphic continuation of certain L-functions. This is extended to Brylinski-Deligne covering groups in [Gaoar].
- Shahidi developed the theory of local coefficients to deduce functional equations for certain L-functions (see [Sha10]). His method relies on uniqueness of Whittaker models. Thus, it works for generic representations. It is not clear if one can extend the theory of local coefficients to general covering groups.
- Twisted doubling (for $G_n^{(m)} \times GL_k^{(m)}$): when $m = 1$, it works for non-generic representations as well. When $m > 1$, this construction relies on an assumption on metaplectic Eisenstein series. However, it might provide a definition for local factors for covering groups.

4. The general global integral

We now give some more details of the global integral.

The general global integral is of the form

$$\int_{(G(F)\backslash G(\mathbb{A}))^2} \varphi_1(g_1)\overline{\varphi_2(g_2)}E_{\sigma(\tau)}^{U_0,\psi_{U_0}}(\iota(g_1,g_2);f,s)\,dg_1\,dg_2.$$

Here

- $E_{\sigma(\tau)}^{U_0,\psi_{U_0}}$: a certain (U_0,ψ_{U_0})-Fourier coefficient of $E_{\sigma(\tau)}$.
- $E_{\sigma(\tau)}$: Siegel Eisenstein series on $H^{(m)}$ with special inducing data $\sigma(\tau)$.
- H: a "large" classical group. For example, if $G_n = Sp_{2n}$ and m is odd, then $H = Sp_{4nkm}$.

Key ideas in our construction:

- $\iota(G_n^{(m)} \times G_n^{(m)})$ is contained in the stabilizer of ψ_{U_0} in $H^{(m)}$.
- The representation $\sigma(\tau)$ has a suitable Gelfand-Kirillov dimension so that the dimension equation in [Gin14] is satisfied. In particular, $\sigma(\tau)$ affords a certain nonzero Fourier coefficient. Moreover, the corresponding local model is unique. (Together with uniqueness of the spherical model, this implies that the integral is Eulerian.)
- The representation $\sigma(\tau)$ carries the same representation-theoretic information as τ (see Theorem 4.4).

4.1. **Construction of $\sigma(\tau)$.** We now describe the construction of $\sigma(\tau)$. Let n be a positive integer. We define a representation $\sigma(\tau)$ on $GL_{mkn}^{(m)}(\mathbb{A})$ by residues of Eisenstein series.

Let P be the standard parabolic subgroup of GL_{mkn} whose Levi subgroup is $GL_k \times \cdots \times GL_k$, where GL_k appears mn times. Consider the Eisenstein series $E(g;f,\underline{s})$ attached to the induced representation

$$\mathrm{Ind}_{P^{(m)}(\mathbb{A})}^{GL_{kmn}^{(m)}(\mathbb{A})}(\tau|\cdot|^{s_1}\otimes\cdots\otimes\tau|\cdot|^{s_{mn}})\delta_P^{1/2}.$$

Here $\underline{s} = (s_1,\cdots,s_{mn}) \in \mathbb{C}^{mn}$ and $P^{(m)}(\mathbb{A})$ is the preimage of $P(\mathbb{A})$ in $GL_{kmn}^{(m)}(\mathbb{A})$. We remark that, when $m > 1$, the tensor product process here is complicated. (See [Tak16, Takar].)

Conjecture 4.1. *The Eisenstein series $E(g;f,\underline{s})$ has a pole at*

$$m(s_i - s_{i+1}) = 1, \qquad s_1 + \cdots + s_{mn} = 0.$$

When $m = 1$, this is a theorem of Jacquet and Shalika. When $k = 2$, this conjecture is known since the Shimura correspondence is known (thanks to the work of Flicker [Fli80]). For the double covers, this can be proved by the adapting the method in [JR92]. It is not known in general, but might be accessible via other Rankin-Selberg integrals or by establishing a generalized Shimura lift.

Now assume the Conjecture is true. Then we define $\sigma(\tau)$ as the representation generated by the residues. When $k = 1$, this is the theta representation studied in [KP84]. When $n = 1$, the construction is considered in [Suz98].

4.2. **Properties of $\sigma(\tau)$.** We now explain properties of $\sigma(\tau)$ in detail.

Let V be the unipotent subgroup of GL_{mkn} consisting elements of the form

$$v = \begin{pmatrix} I_n & X_1 & * & \cdots & * & * \\ & I_n & X_2 & \cdots & * & * \\ & & & \cdots & \cdots & \cdots \\ & & & & \cdots & * \\ & & & & I_n & X_{mk-1} \\ & & & & & I_n \end{pmatrix}.$$

Define a character ψ_V on V by

$$\psi_V(v) = \psi(\mathrm{tr}(X_1 + \cdots + X_{mk-1})).$$

Notice that when $n = 1$, this is the character used to define Whittaker coefficient.

Theorem 4.2. *The unipotent orbit attached to $\sigma(\tau)$ is $((mk)^n)$, in the sense of* [Gin06]. *This means the following:*

(1) The orbit $((mk)^n)$ supports a nonzero Fourier coefficient. That is

$$\int_{V(F)\backslash V(\mathbb{A})} \varphi(vg)\psi_V(v)\, dv \neq 0$$

for some $\varphi \in \sigma(\tau)$.

(2) Any orbit larger than or not comparable with $((mk)^n)$ does not support any Fourier coefficient.

Theorem 4.3. *Let ν be an unramified place of F and $\sigma(\tau)_\nu$ be the local component of $\sigma(\tau)$ at ν. Then the unipotent orbit attached to $\sigma(\tau)$ is $((mk)^n)$ and*

$$\dim \mathrm{Hom}_{V(F_\nu)}(\sigma(\tau)_\nu, \psi_V) = 1.$$

When $n = 1$, this is proved in [Suz98]. (That is, uniqueness of Whittaker models holds for $\sigma(\tau)$.) When $k = 1$, this is proved in [Cai16].

Finally, we state a Casselman-Shalika type formula for $\sigma(\tau)$. Let ν be an unramified place of F and let $n = 1$. Let $Sh(\tau_\nu)$ be the local Shimura lift of τ_ν to $GL_k(F_\nu)$. Notice that in this case uniqueness of Whittaker models holds for both $\sigma(\tau)_\nu$ and $Sh(\tau_\nu)$.

Let $W_{Sh(\tau_\nu)}$ be the normalized unramified Whittaker function for $Sh(\tau_\nu)$. Let $W_{\sigma(\tau)_\nu}$ be the normalized unramified Whittaker function for $\sigma(\tau)_\nu$.

Theorem 4.4. *We have the following identity between two Casselman-Shalika type formulas:*

$$W_{\sigma(\tau)_\nu}\begin{pmatrix} \varpi^{ml} & \\ & I_{mk-1} \end{pmatrix} = q^* W_{Sh(\tau_\nu)}\begin{pmatrix} \varpi^{l} & \\ & I_{k-1} \end{pmatrix} \text{ for } l \geq 0.$$

Here ϖ is a uniformizer for F_ν, $|\varpi| = q^{-1}$, and $*$ is some explicit constant.

This result is proved in [Suz98] when $k = 1, 2$. We manage to prove it for arbitrary k.

5. Applications in progress

These doubling integrals are expected to have far-reaching consequences. We mention a few applications here. This is work in progress.

The first application is on the functoriality from classical groups to general linear groups, which should follow from Arthur's work on the trace formula. By combining these new integrals and the converse theorem of Cogdell and Piatetski-Shapiro, we might give a new proof of this case of functoriality.

One can also develop a local theory of the doubling construction. This would give a definition of local L-factor and ϵ-factor at every local place. Moreover, using these integrals, one can potentially locate possible poles of the global L-functions.

Finally, we remark that, it is also possible to extend the above projects to covering groups.

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