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NEW FORMS IN THE KOHNEN PLUS SPACE

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1. Introduction

Let $k \geq 2$ be an odd integer, χ a Dirichlet character $\mod 4N$ where N is a natural number. By $S_{k+1/2}(4N, \chi)$ and $S_{2k}(2N, \chi^2)$ we denote the spaces of cusp forms of weight $k+1/2$ and $2k$ with respect to the congruence subgroups $\Gamma_0(4N)$ and $\Gamma_0(2N)$, respectively. For any $f \in S_{k+1/2}(4N, \chi)$ and square-free integer t, Shimura showed that there exists $Sh_t(f) \in S_{2k}(2N, \chi^2)$ which can be described exactly by the Fourier coefficients. If f is an eigenform, then so does $Sh_t(f)$ and they share the same eigenvalues for all Hecke operators T_p and T_{p^2}, respectively, where p is an odd prime number. Note that the above is also true for $k = 1$ if f is in the complement of the subspace of $S_{3/2}(4N, \chi)$ spanned by all single variable theta functions (otherwise $Sh_t(f)$ may not be a cusp form). By taking linear combinations of such correspondences for square-free t's, one gets various liftings from $S_{k+1/2}(4N, \chi)$ to $S_{2k}(2N, \chi^2)$. However, in general, one cannot get a bijective lifting in such a way. A natural problem is to identify the image of such liftings, or the subspace of $S_{k+1/2}(4N, \chi)$ by restricting some lifting to which one can get a injective lifting. A partial answer to this question comes from the Kohnen plus space.

Definition 1.1. *For N odd and square-free and χ quadratic, the plus space $S^+_{k+1/2}(4N, \chi)$ is the subspace of $S_{k+1/2}(4N, \chi)$ consisting of those forms whose n-th Fourier coefficients vanish for all natural number n such that $(-1)^k \chi(-1) n \equiv 2$ or $3 \mod 4$.*

Kohnen initially introduced the plus space in 1980 [3] for the classical case and generalized it to the version as the definition above in 1982 [4]. He showed that there exists a one-to-one correspondence, which is a lifting introduced above, between $S^+_{k+1/2}(4N, \chi)$ and $S_{2k}(2N, \chi^2)$. From now we want to consider the case for general totally real number field, that is, the Hilbert case.

2. Definitions

Let F be a totally real number field with degree n over $\mathbf{Q}$. As usual, $\mathfrak{o}$ and $\mathfrak{d}$ denote its ring of integers and different over $\mathbf{Q}$, respectively. We fix a odd square-free ideal $\mathfrak{I}$ of $\mathfrak{o}$ and a primitive quadratic character χ of (o) with conductor $(\mathfrak{f})$, a principal ideal generated by some $\mathfrak{f} \in \mathfrak{o}$. Thus explicitly, we can write χ in the form

$$\chi(d) = \prod_{v \mid 2} (\mathfrak{f}, d)_v \prod_{v \mid \mathfrak{f}} (\mathfrak{f}, d)$$

where v runs over places of F and $(\cdot,\cdot)_v$ is the Hilbert symbol of the local field F_v corresponding to v.

For ideals $\mathfrak{b}$ and $\mathfrak{c}$ of F such that $\mathfrak{bc} \subset \mathfrak{o}$, we put

$$\Gamma[\mathfrak{b},\mathfrak{c}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) \,\middle|\, a,d \in \mathfrak{o}, b \in \mathfrak{b}, c \in \mathfrak{c} \right\}$$

and

$$\Gamma_0(\mathfrak{a}) = \Gamma[\mathfrak{d}^{-1}, \mathfrak{a}\mathfrak{d}]$$

for ideal $\mathfrak{a}$ of $\mathfrak{o}$.

For simplicity, we let $k \in \mathbf{N}^n$ be parallel and $\mathfrak{f}$ be with the sign $(-1)^k$, that is, the norm of $\mathfrak{f}$ over $\mathfrak{Q}$ has the same sign with $(-1)^k$.

We define the theta function θ on $\mathfrak{h}^n$, where $\mathfrak{h}$ is the upper-half part of the complex plane, by

$$\theta(z) = \sum_{\xi \in \mathfrak{o}} \exp\left(2\pi\sqrt{-1}\mathrm{tr}(\xi^2 z)\right).$$

Applying θ, we can define the factor of automorphy of weight $1/2$ by

$$j(\gamma, z) = \theta(\gamma z)/\theta(z)$$

where $\gamma \in \Gamma_0(4)$ and γz denotes the image of z under the Möbius transformation by γ.

Putting $S_{k+1/2}(4\mathfrak{I}, \chi)$ to be the space consisting of Hilbert cusp forms with respect to the factor of automorphy given by $j(\gamma, z)^{2k+1}\chi(\gamma)$ where

$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \chi(d) \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4\mathfrak{I}),$$

we give the definition of the plus space.

Definition 2.1. *With the notations stated above, the Kohnen plus space $S^+_{k+1/2}(4\mathfrak{I}, \chi)$ of weight $k + 1/2$, level $4\mathfrak{I}$ and character χ is defined to be the subspace of $S_{k+1/2}(4\mathfrak{I}, \chi)$ such that $h \in S^+_{k+1/2}(4\mathfrak{I}, \chi)$ if and only if the ξ-th Fourier coefficient of h vanishes unless there exists $\lambda \in \mathfrak{o}$ such that $\xi - \mathfrak{f}\lambda^2 \in 4\mathfrak{o}$.*

Let $\mathbb{A}$ be the adele ring of F with finite part $\mathbb{A}_{\mathrm{f}}$ and $\mathrm{Mp}_2(\mathbb{A}_{\mathrm{f}})$ be the metaplectic double covering of $\mathrm{SL}_2(\mathbb{A}_{\mathrm{f}})$.

An eigenform $h \in S^{+}_{k+1/2}(4\mathfrak{I}, \chi)$ generates an irreducible representation $\pi_{\mathrm{f}} = \prod_{v \in \infty} \pi_v$ of $\mathrm{Mp}_2(\mathbb{A}_{\mathrm{f}})$ where π_v is an irreducible representation of $\mathrm{Mp}_2(F_v)$. An eigenform h is called a Hecke new form if for any finite place v dividing $\mathfrak{I}$, π_v is quivalent to a Steinberg representation, which is a certain ramified subrepresentation of some principal series representation. We let $S^{+,\mathrm{NEW}}_{k+1/2}(4\mathfrak{I}, \chi)$ be the $\mathbf{C}$-space spanned by Hecke new forms given above. Any form in $S^{+,\mathrm{NEW}}_{k+1/2}(4\mathfrak{I}, \chi)$ is called a new form. Note that the definition of new forms coincides with the one given by Kohnen.

3. An if-and-only-if condition for the Hecke new forms

In this section, for simplicity, we set $\chi = 1$.

For $v \mid \mathfrak{I}$, we let

$$\Gamma_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F_v) \,\middle|\, a, d \in \mathfrak{o}_v, b \in \mathfrak{d}_v^{-1}, c \in \varpi_v \mathfrak{d}_v \right\}$$

where $\varpi_v \in \mathfrak{o}_v$ is the uniformizer corresponding to the place v. We denote the inverse image of Γ_v in $\mathrm{Mp}_2(F_v)$ by $\widetilde{\Gamma_v}$.

Let $\widetilde{\mathcal{H}_v} = \widetilde{\mathcal{H}_v}(\widetilde{\Gamma_v} \backslash \mathrm{Mp}_2(F_v)/\widetilde{\Gamma_v}, \varepsilon_v)$ be the Hecke algebra with respect to the genuine character ε_v of $\widetilde{\Gamma_v}$ which comes from some Weil representation of $\mathrm{Mp}_2(F_v)$.

Definition 3.1. *Let $\widetilde{\mathcal{T}_v}$ and $\widetilde{\mathcal{U}_v}$ be the Hecke operators in $\widetilde{\mathcal{H}_v}$ which are supported on $\widetilde{\Gamma_v} \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v^{-1} \end{pmatrix} \widetilde{\Gamma_v}$ and $\widetilde{\Gamma_v} \begin{pmatrix} 0 & -\boldsymbol{\delta}_v^{-1}\varpi_v^{-1} \\ \boldsymbol{\delta}_v \varpi_v & 0 \end{pmatrix} \widetilde{\Gamma_v}$, respectively, such that*

$$\widetilde{\mathcal{T}_v}\left(\begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v^{-1} \end{pmatrix} \right) = q_v^{-1/2} \frac{\alpha_v(\varpi_v)}{\alpha_v(1)}$$

and

$$\widetilde{\mathcal{U}_v}\left(\begin{pmatrix} 0 & -\boldsymbol{\delta}_v^{-1}\varpi_v^{-1} \\ \boldsymbol{\delta}_v \varpi_v & 0 \end{pmatrix} \right) = \alpha_v(\boldsymbol{\delta}_v \varpi_v).$$

Here $\boldsymbol{\delta}_v \in \mathfrak{o}_v$ is one which generates the local principal ideal $\mathfrak{d}_v$, α_v denotes the Weil constant and q_v is the index of the local residue field with respect to v.

In the definition above, $\widetilde{\mathcal{T}_v}$ is the usual Hecke operator and $\widetilde{\mathcal{U}_v}$ is the Atkin-Lehner operator.

Theorem 3.1. *An eigenform $h \in S^+_{k+1/2}(4\mathfrak{I},1)$ is a Hecke new form if and only if*

$$\widetilde{T_v}\widetilde{\mathcal{U}_v}h = -h = \widetilde{\mathcal{U}_v}\widetilde{T_v}h$$

for all finite $v|\mathfrak{I}$.

This theorem is motivated by a result from [1]. They treated the case for integral weight, $F = \mathbf{Q}$, $\chi = 1$ and general level.

4. Application of Waldspurger's theory

Theorem 4.1. *The plus space $S^+_{k+1/2}(4\mathfrak{I},\chi)$ is the E^K-fixed subspace of $S^+_{k+1/2}(4\mathfrak{I},\chi)$ for some Hecke operator $E^K = \otimes_{v<\infty}E^K_v \in \otimes_{v<\infty}\widetilde{\mathcal{H}_v}$ where for each $\widetilde{\mathcal{H}_v} = (\widetilde{\Gamma_v}\backslash \mathrm{Mp}_2(F_v)/\widetilde{\Gamma_v}, \varepsilon_v)$ we set*

$$\Gamma_v = \begin{cases} \Gamma_0(1)_v & \text{if } v \nmid 2\mathfrak{I}, \\ \Gamma_0(4)_v & \text{if } v \mid 2 \\ \Gamma_0(\varpi_v)_v & \text{if } v \mid \mathfrak{I}. \end{cases}$$

The Hecke operator E^K is an idempotent and can be written down explicitly, but we omit its definition here. The following proposition was given by Hiraga and Ikeda [2].

Proposition 4.1. *Let v be a finite place of F not dividing $\mathfrak{I}$ and $\mathcal{B}$ be the Borel subgroup of $\mathrm{SL}_2(F_v)$ consisting of upper-triangular matrices. For $s \in \mathbf{C}$, if the principal series $Ind_{\widetilde{\mathcal{B}}}^{\mathrm{Mp}_2(F_v)}(\left(\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right) \mapsto \frac{\alpha_v(1)}{\alpha_v(a)}|a|_v^{s+1})$ is irreducible, then its E^K_v-fixed subspace is of one dimension.*

Proposition 4.2. *The E^K_v-fixed subspace of a Steinberg representation is of one dimension for $v \mid \mathfrak{I}$.*

Now let $k \geq 2$. By Waldspurger's results, each irreducible representation π of $\mathrm{Mp}_2(\mathbb{A})$ from an eigenform $h \in S^+_{k+1/2}(4\mathfrak{I},\chi)$ corresponds to an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$, which gives a non-zero unique-up-to-non-zero-scalar-multiplications eigenform in the space of cuspidal automorphic forms

$$\mathcal{A}^{\mathrm{CUSP}}_{2k}(\mathfrak{I}) = \mathcal{A}^{\mathrm{CUSP}}_{2k}(\mathrm{PGL}_2(F)\backslash \mathrm{PGL}_2(\mathbb{A})/\prod_{v<\infty}\Gamma'_v(\mathfrak{I}))$$

where $\Gamma'_v(\mathfrak{I})$ is a congruence subgroup which is maximal compact if $v \nmid \mathfrak{I}$ and Iwahori if $v \mid \mathfrak{I}$. We put $\mathcal{A}^{\mathrm{CUSP,NEW}}_{2k}(\mathfrak{I})$ to be the subspace of $\mathcal{A}^{\mathrm{CUSP}}_{2k}(\mathfrak{I})$ spanned by g such that its corresponding representation of $\mathrm{PGL}_2(\mathbf{A})$ is locally a Steinberg representation at any finite $v \mid \mathfrak{I}$.

Theorem 4.2. *The plusspace* $S_{k+1/2}^{+,\mathrm{NEW}}(4\mathfrak{I},\chi)$ *is Hecke isomorphic to* $\mathcal{A}_{2k}^{\mathrm{CUSP},\mathrm{NEW}}(\mathfrak{I})$.

Note that for the case $\mathfrak{I}=1$ the theorem was treated by Hiraga and Ikeda in [2].

Using Theorem 3.1 and Theorem 4.2 we can get an analogue of the result from Baruch and Purkait in [1] for the Hilbert modular forms of integral weight.

References

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