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On Inequalities about Instantaneous Amplitudes

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Abstract

For a real signal (a real-valued function) $f(t)$, we consider its analytic signal $(\mathcal{A}f)(t) = f(t) + i(\mathcal{H}f)(t)$, where $(\mathcal{H}f)(t)$ is the Hilbert transform of $f(t)$. Its absolute value $A(t) = |(\mathcal{A}f)(t)|$, which is called *instantaneous amplitude*, often represents a coarse variation of $f(t)$, and the graph of $A(t)$ looks like an "envelope" of the graph of $|f(t)|$. However, for some signals, $A(t)$ changes rather rapidly, and it doesn't look like an "envelope" of the graph of $|f(t)|$. We give mathematically rigorous inequalities about $\widehat{A^2}(\xi)$ $(A^2(t) = \{A(t)\}^2)$ which can be considered to explain this difference. We also consider the best possibility of the constants of the inequalities.

Keywords : analytic signal, Hilbert transform, instantaneous amplitude, envelope, frequency band

1 Introduction

A real signal can be considered mathematically as a real-valued function. For a real-valued function $f(t)$, its analytic signal $(\mathcal{A}f)(t) = f(t) + i(\mathcal{H}f)(t)$, where $(\mathcal{H}f)(t)$ is the Hilbert transform of $f(t)$, is well-known to be useful. It has only positive frequency components, while it becomes complex-valued. Its absolute value $A(t) = |(\mathcal{A}f)(t)| = \sqrt{|f(t)|^2 + |(\mathcal{H}f)(t)|^2}$ is called the *instantaneous amplitude* of $f(t)$, and it is known that it often represents a coarse variation of $f(t)$, and the graph of $A(t)$ looks

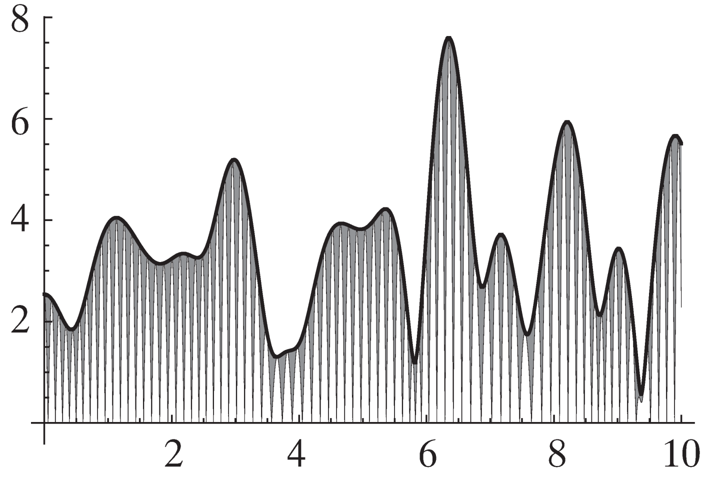

Figure 1: An example of $|f(t)|$(thin) and $A(t) = |(\mathcal{A}f)(t)|$(thick).

like an "envelope" of the graph of $|f(t)|$. An example of $|f(t)|$ and $A(t) = |(\mathcal{A}f)(t)|$ is given in Figure 1. Note that $|f(t)| \leq A(t)$, and the shaded area is between $|f(t)|$ and $A(t)$.

This property of analytic signals is often used to extract a coarse variation of a real signal. However, for some signals, $A(t)$ changes rather rapidly, and its graph does not look like an "envelope" of the graph of $|f(t)|$, as is shown in Figure 2.

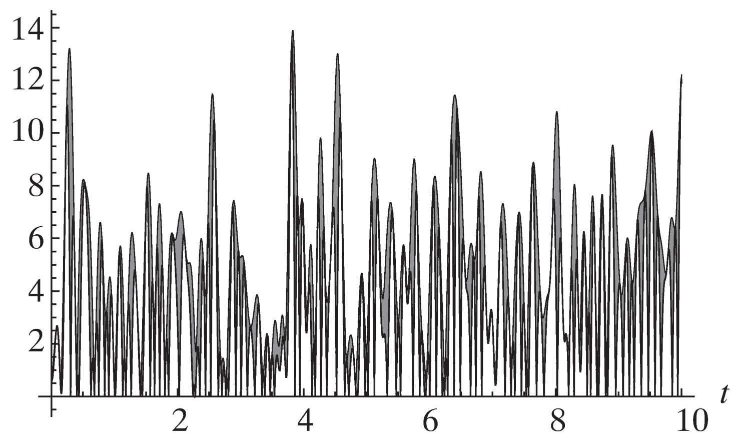

Figure 2: An example of $|f(t)|$(thin) and $A(t) = |(\mathcal{A}f)(t)|$(thick) (bad case)

We want to make this situation mathematically clear. We found rigorous inequalities about $\widehat{A^2}(\xi)$ $(A^2(t) = \{A(t)\}^2)$, which can be considered to explain this situation to some extent.

Part of our results was announced in [4] without proofs. The best possibility of the constants appearing in the inequalities is also discussed.

In the next section, we explain about Hilbert transform and analytic signals. In Section 3, we explain about the instantaneous amplitude, with several examples. In the following section, we give the main results. In Section 5, we give more examples, and in the last two sections, we give the proofs.

2 Hilbert transform and analytic signal

Let $\widehat{f}(\xi)$ be the Fourier transform of f:

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(t)e^{-i\xi t}\, dt.$$

The mapping $f(t) \mapsto \widehat{f}(\xi)$ can be considered as a transformation $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, where

$$L^p(K) := \left\{ f(t) \mid \|f\|_{L^p(K)} < \infty \right\},$$
$$\|f\|_{L^p(K)} := \left(\int_K |f(t)|^p\, dt \right)^{1/p}.$$

The Hilbert transform $\mathcal{H}f$ of $f \in L^2(\mathbb{R})$ is defined by

$$(\mathcal{H}f)^\wedge(\xi) : = -i(\operatorname{sgn}\xi)\widehat{f}(\xi), \tag{2-1}$$

$$\operatorname{sgn}\xi : = \begin{cases} 1, & \xi > 0, \\ -1, & \xi < 0. \end{cases} \quad \text{(Figure 3)} \tag{2-2}$$

The operator $\mathcal{H}$ is a unitary operator of $L^2(\mathbb{R})$, that is, $\mathcal{H} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $\langle \mathcal{H}f, \mathcal{H}g \rangle = \langle f, g \rangle$, and $\|\mathcal{H}f\| = \|f\|$, where $\langle f, g \rangle := \int_{\mathbb{R}} f(t)\overline{g(t)}\, dt$, $\overline{z}$ denotes the complex conjugate of z, and $\|f\| := \sqrt{\langle f, f \rangle} = \|f\|_{L^2(\mathbb{R})}$

If f is real-valued, which is equivalent to $\widehat{f}(-\xi) = \overline{\widehat{f}(\xi)}$, then $\mathcal{H}f$ is also real-valued, and $\mathcal{H}f \perp f$, that is $\langle \mathcal{H}f, f \rangle = 0$.

For a real-valued function (real signal) $f(t)$, we define its *analytic signal* by

$$(\mathcal{A}f)(t) := f(t) + i(\mathcal{H}f)(t). \tag{1.2}$$

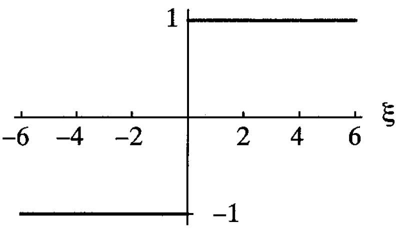

Figure 3: $\operatorname{sgn}\xi$

The analytic signal $(\mathcal{A}f)(t)$ has only positive frequencies:

$$\widehat{\mathcal{A}f}(\xi) = \begin{cases} 2\widehat{f}(\xi), & \xi > 0, \\ 0, & \xi < 0. \end{cases} \tag{1.3}$$

If $f(t)$ is real-valued, then the Fourier transform $\widehat{f}(\xi)$ satisfies $\widehat{f}(-\xi) = \overline{\widehat{f}(\xi)}$, and hence $(\mathcal{A}f)(t)$ does not lose any information of $f(t)$. As a matter of fact, we have $f(t) = \Re(\mathcal{A}f)(t)$, where $\Re z$ denotes the real part of z.

3 Instantaneous Amplitude

We define three important concepts about $f(t)$ using its analytic signal as follows.

The *instantaneous amplitude* (or amplitude envelope) of $f(t)$ is defined by

$$A(t) = |(\mathcal{A}f)(t)| = \sqrt{|f(t)|^2 + |(\mathcal{H}f)(t)|^2}. \tag{2.1}$$

The *instantaneous phase* of $f(t)$ is

$$\theta(t) = \arg(\mathcal{A}f)(t),$$

where $\arg z$ is the argument of z. That is,

$$(\mathcal{A}f)(t) = A(t)e^{i\theta(t)}.$$

The *instantaneous frequency* of $f(t)$ is $\omega(t) = \theta'(t)$. In this talk, we concentrate on the instantaneous amplitude.

For many signals in the real world, the instantaneous amplitude $A(t) = |(\mathcal{A}f)(t)|$ of $f(t)$ is *slowly varying*, and it represents a *coarse variation* of $f(t)$ like in Figure 1.

Further, the graph of $A(t)$ can be considered to be an "envelope" of the graph of $|f(t)|$. However, in some cases, the situation is not so good. $A(t)$ looks too rapidly varying like an example in Figure 2.

These phenomenon is rather well-known. However, mathematically, it is still a wonder why such a phenomena occurs. There is no clear boundary between the two cases. There is no rigorous definition of "envelope" nor "slowly varying". There are only many examples.

We give a simple example.

Example 1. Consider

$$f(t) = \cos 8t + \cos(8t + at) + \cos(8t - at)$$

for $0 < a < 8$. Though this $f(t)$ does not belong to $L^2(\mathbb{R})$, we can explain the situation very well by considering this function itself, rather than modifying it so that it belongs to $L^2(\mathbb{R})$. It is easy to extend the definition of the Hilbert transform to a class of distributions including $f(t)$. Note that the support of $\widehat{f}(\xi)$ does not include $\xi = 0$.

Since

$$\begin{aligned}\widehat{f}(\xi) = \pi\{\delta(\xi - 8) + \delta(\xi - 8 - a) + \delta(\xi - 8 + a)\\ + \delta(\xi + 8) + \delta(\xi + 8 + a) + \delta(\xi + 8 - a)\},\end{aligned}$$

we have

$$(\mathcal{A}f)(t) = e^{i8t} + e^{i(8+a)t} + e^{i(8-a)t},$$

and

$$A(t) = |(\mathcal{A}f)(t)| = |1 + e^{iat} + e^{-iat}| = |1 + 2\cos at|.$$

(See Figure 4.)

As is seen in the figures, If a is *small*, then $A(t) = |(\mathcal{A}f)(t)|$ seems to represent a coarse variation of $f(t)$, and the graph of $A(t)$ looks like a kind of "envelope" of the graph of $|f(t)|$. If a is *large*, the situation is different. The instantaneous amplitude $A(t) = |(\mathcal{A}f)(t)|$ is too rapidly varying and the graph of $A(t)$ seems inappropriate to call an "envelope" of the graph of $|f(t)|$.

For many signals in the real worlds, the situation seems near the case of *small a*, and the analytic signal is often used to extract a coarse variation of the signal.

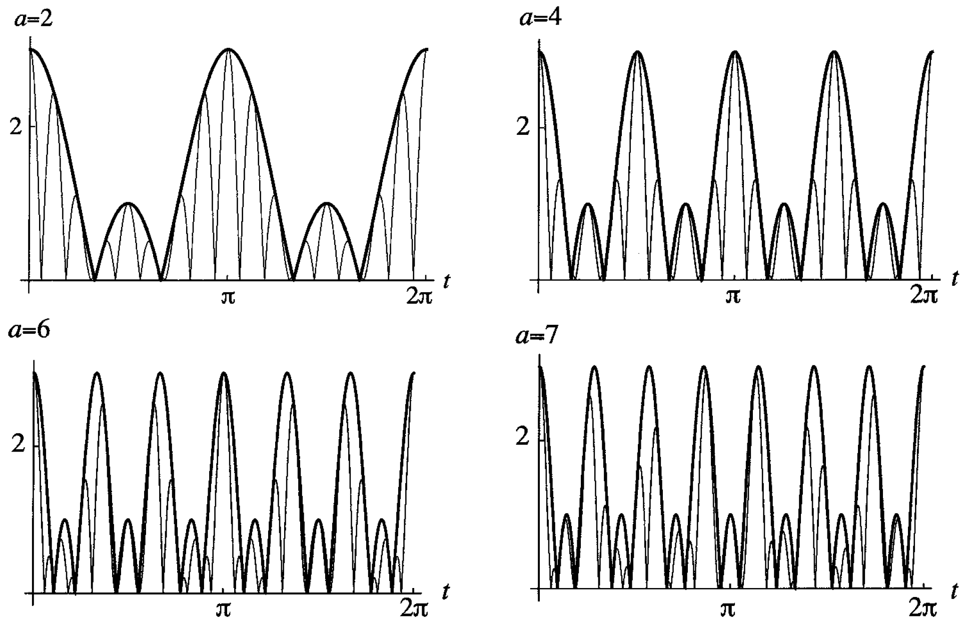

Figure 4: Example 1: $|f(t)|$(thin), $A(t) = |(\mathcal{A}f)(t)|$(thick): $a = 2, 4, 6, 7$.

We give some further examples in Figures 5–6. The last figure in each of Figures 5–6 is an enlarged part of the figure above it. These are examples where $\widehat{f}(\xi)$ is concentrated on $20 \leq |\xi| \leq 30$, with various rates of concentration.

Observation of many examples seems to suggest that

> if the frequency components of a function are concentrated on a narrow band, then the instantaneous amplitude varies mildly, and its graph looks like an "envelope" of the graph of the original function.

The statement above is a very vague, logically unclear statement. We want to give a *mathematical statement* which can be a supporting evidence for the above statement.

4 Main Results

First, we give a vague statement of our result.

Claim 2. Let $0 < w_1 \leq w_2$. If $\widehat{f}(\xi)$ is concentrated on $w_1 \leq |\xi| \leq w_2$, then $\widehat{A^2}(\xi)$ is concentrated on $|\xi| \leq w_2 - w_1$.

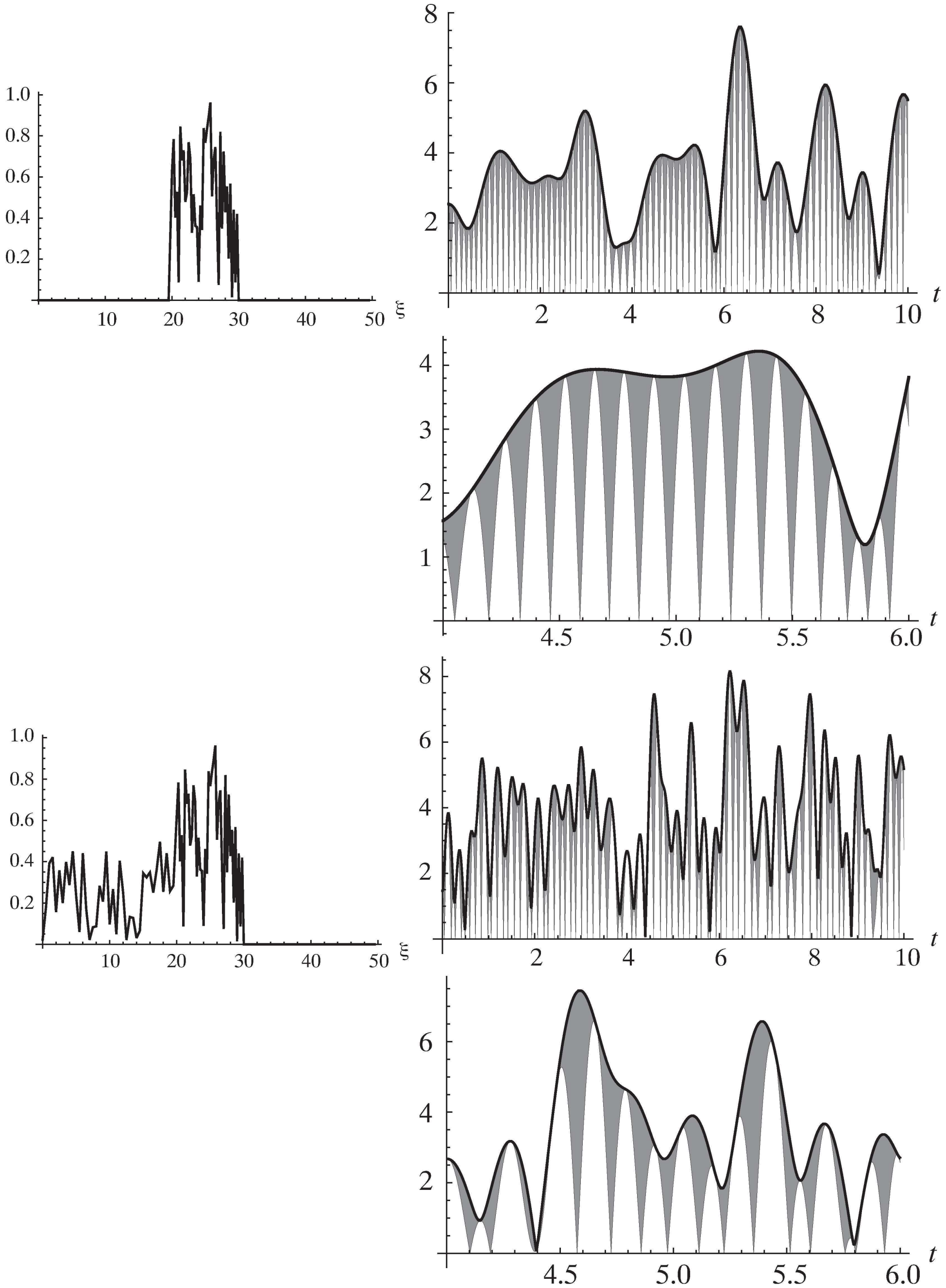

Figure 5: $|\widehat{f}(\xi)|$ and $A(t) = |(\mathcal{A}f)(t)|$(thick), $|f(t)|$(thin). No.1,2

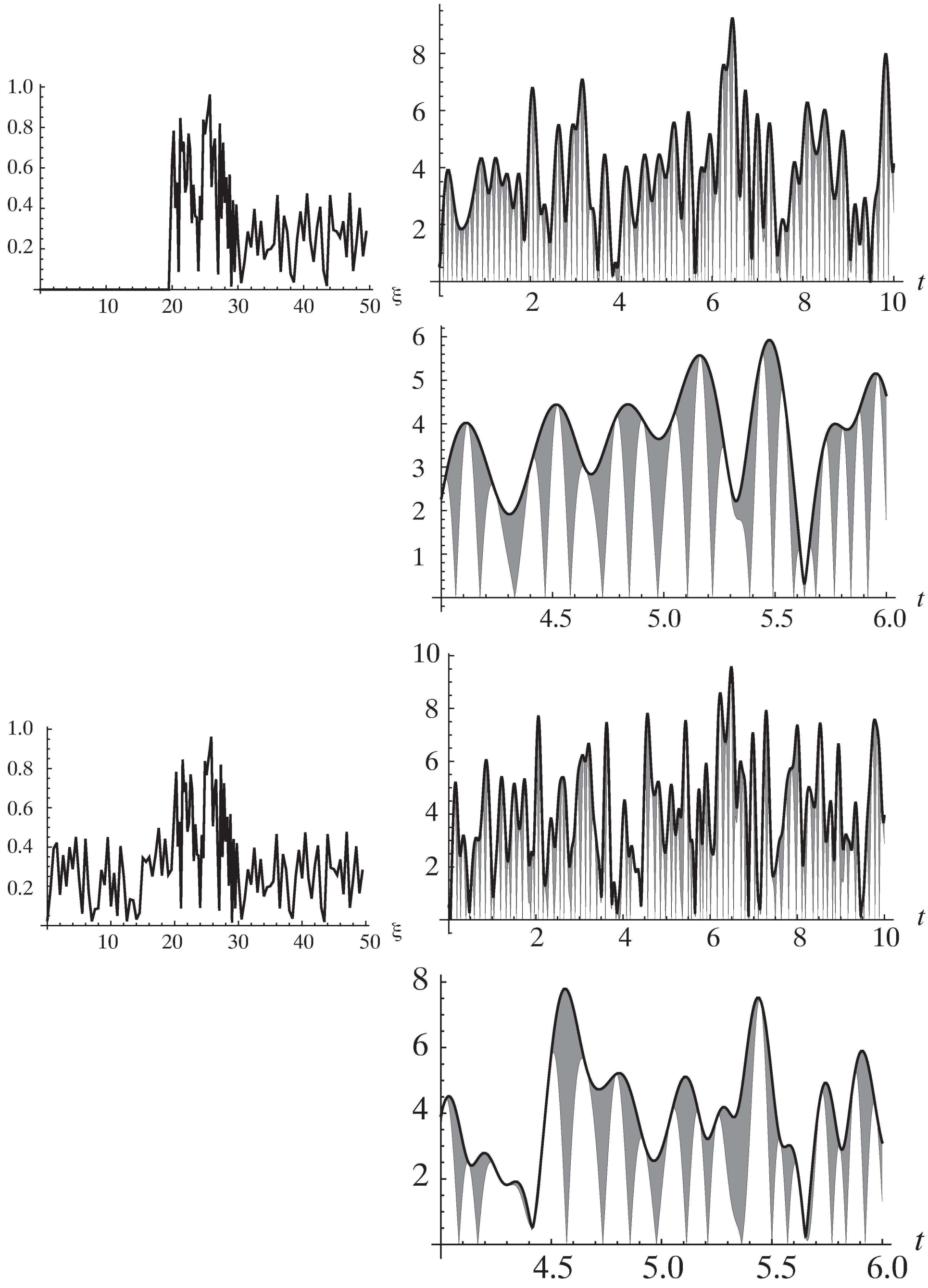

Figure 6: $|\widehat{f}(\xi)|$ and $A(t) = |(\mathcal{A}f)(t)|$(thick), $|f(t)|$(thin). No.3,4

This claim can be understood that if $\widehat{f}(\xi)$ is concentrated on $w_1 \leq |\xi| \leq w_2$, and if $w_2 - w_1$ is small compared with w_1, then $\widehat{A^2}(\xi)$ is concentrated on low frequency band, and hence $\{A(t)\}^2$ (and $A(t)$) varies mildly.

Now, we give precise mathematical statements. We give three types of inequalities.

Theorem 3. Let $f \in L^2(\mathbb{R})$ be real-valued, and set

$$A^2(t) = \{A(t)\}^2 = |(\mathcal{A}f)(t)|^2 = \{f(t)\}^2 + \{(\mathcal{H}f)(t)\}^2.$$

Then, $\widehat{A^2}(\xi)$ is a bounded continuous function, $\widehat{A^2}(\xi) \to 0$ ($|\xi| \to \infty$), and

$$|\widehat{A^2}(-\xi)| = |\widehat{A^2}(\xi)| \leq \widehat{A^2}(0) \left(= \frac{2}{\pi} \left\| \widehat{f} \right\|^2_{L^2(\mathbb{R}_+)} \right). \tag{4-1}$$

Let $0 < w_1 \leq w_2$, $J = [w_1, w_2]$, $|J| = w_2 - w_1$. We have the following.

(1)

$$\sup_{|\xi| \geq |J|} |\widehat{A^2}(\xi)| \leq \left(\frac{2}{\pi} \left\| \widehat{f} \right\|_{L^2(\mathbb{R}_+ \setminus J)} + \frac{2+\sqrt{2}}{\pi} \left\| \widehat{f} \right\|_{L^2(J)} \right) \left\| \widehat{f} \right\|_{L^2(\mathbb{R}_+ \setminus J)}, \tag{4-2}$$

$$\sup_{|\xi| \geq |J|} |\widehat{A^2}(\xi)| \leq \frac{2+\sqrt{2}}{\pi} \left\| \widehat{f} \right\|_{L^2(\mathbb{R}_+)} \left\| \widehat{f} \right\|_{L^2(\mathbb{R}_+ \setminus J)}. \tag{4-3}$$

Thus, if $\widehat{f}(\xi)$ is concentrated on $J \cup (-J)$, which can be considered that $\left\| \widehat{f} \right\|_{L^2(\mathbb{R}_+ \setminus J)}$ is small, then by the inequality $\sup_{|\xi| \geq |J|} |\widehat{A^2}(\xi)|$ is small, which means $\widehat{A^2}(\xi)$ is concentrated on $|\xi| \leq |J|$. Note that the inequality (4-3) does not follow from (4-2), since $K_1 \cap K_2 = \emptyset$ does not imply

$$\left\| \widehat{f} \right\|_{L^2(K_1)} + \left\| \widehat{f} \right\|_{L^2(K_2)} \leq \left\| \widehat{f} \right\|_{L^2(K_1 \cup K_2)},$$

but it implies

$$\left\| \widehat{f} \right\|_{L^2(K_1)} + \left\| \widehat{f} \right\|_{L^2(K_2)} \leq \sqrt{2} \left\| \widehat{f} \right\|_{L^2(K_1 \cup K_2)}. \tag{4-4}$$

(2) If $\widehat{f} \in L^1(\mathbb{R})$ further, then $\widehat{A^2} \in L^1(\mathbb{R})$ and

$$\begin{aligned} \int_{|J|}^{\infty} |\widehat{A^2}(\xi)| d\xi &\leq \left(\frac{1}{\pi} \left\| \widehat{f} \right\|_{L^1(\mathbb{R}_+ \setminus J)} + \frac{2}{\pi} \left\| \widehat{f} \right\|_{L^1(J)} \right) \left\| \widehat{f} \right\|_{L^1(\mathbb{R}_+ \setminus J)} \\ &\leq \frac{2}{\pi} \left\| \widehat{f} \right\|_{L^1(\mathbb{R}_+)} \left\| \widehat{f} \right\|_{L^1(\mathbb{R}_+ \setminus J)}. \end{aligned} \tag{4-5}$$

Thus, if $\widehat{f}(\xi)$ is concentrated on $J \cup (-J)$, which can be considered that $\left\|\widehat{f}\right\|_{L^1(\mathbb{R}_+\setminus J)}$ is small, then by the inequality $\int_{|J|}^{\infty} |\widehat{A^2}(\xi)| d\xi$ is small, which means $\widehat{A^2}(\xi)$ is concentrated on $|\xi| \leq |J|$.

(3) If $\widehat{f}$ is bounded on $\mathbb{R}$ further, then

$$\begin{aligned} \sup_{|\xi|\geq|J|} |\widehat{A^2}(\xi)| &\leq \left(\frac{2}{\pi}\left\|\widehat{f}\right\|_{L^1(\mathbb{R}_+\setminus J)} + \frac{4}{\pi}\left\|\widehat{f}\right\|_{L^1(J)}\right) \sup_{\xi\in\mathbb{R}_+\setminus J} |\widehat{f}(\xi)| \\ &\leq \frac{4}{\pi}\left\|\widehat{f}\right\|_{L^1(\mathbb{R}_+)} \sup_{\xi\in\mathbb{R}_+\setminus J} |\widehat{f}(\xi)|. \end{aligned} \tag{4-6}$$

Thus, if $\widehat{f}(\xi)$ is concentrated on $J\cup(-J)$, which can be considered that $\sup_{\xi\in\mathbb{R}_+\setminus J} |\widehat{f}(\xi)|$ is small, then by the inequality $\sup_{|\xi|\geq|J|} |\widehat{A^2}(\xi)|$ is small, which means $\widehat{A^2}(\xi)$ is concentrated on $|\xi| \leq |J|$.

By this theorem, we have a mathematical support for Claim 2, which can be understood as the following.

> If $\widehat{f}(\xi)$ is concentrated on $w_1 \leq |\xi| \leq w_2$, and if $w_2 - w_1$ is small compared with w_1, then $\{A(t)\}^2$ $(A(t))$ varies mildly.

We give further examples. The figures in Figures 7–9 show several examples with various spectrum. As the frequency components outside $20 \leq |\xi| \leq 30$ becomes larger, the graph of $|A(t)|$ becomes further apart from the "envelope" of $|f(t)|$.

5 Best possibility of the constants

Here, we consider the best possibility of the constants in the inequalities.

Theorem 4. (1) If the following inequality holds for every real $f \in L^2(\mathbb{R})$, then $C_1 \geq \dfrac{2}{\pi}$ and $C_2 \geq \dfrac{2\sqrt{2}}{\pi}$.

$$\sup_{|\xi|\geq|J|} |\widehat{A^2}(\xi)| \leq \left(C_1\left\|\widehat{f}\right\|_{L^2(\mathbb{R}_+\setminus J)} + C_2\left\|\widehat{f}\right\|_{L^2(J)}\right)\left\|\widehat{f}\right\|_{L^2(\mathbb{R}_+\setminus J)}. \tag{5-1}$$

Not that in Theorem 3, $C_2 = \dfrac{2+\sqrt{2}}{\pi}$.

(2) If the following inequality holds for every real $f \in L^2(\mathbb{R})$, then $C_1 \geq \dfrac{1}{\pi}$ and

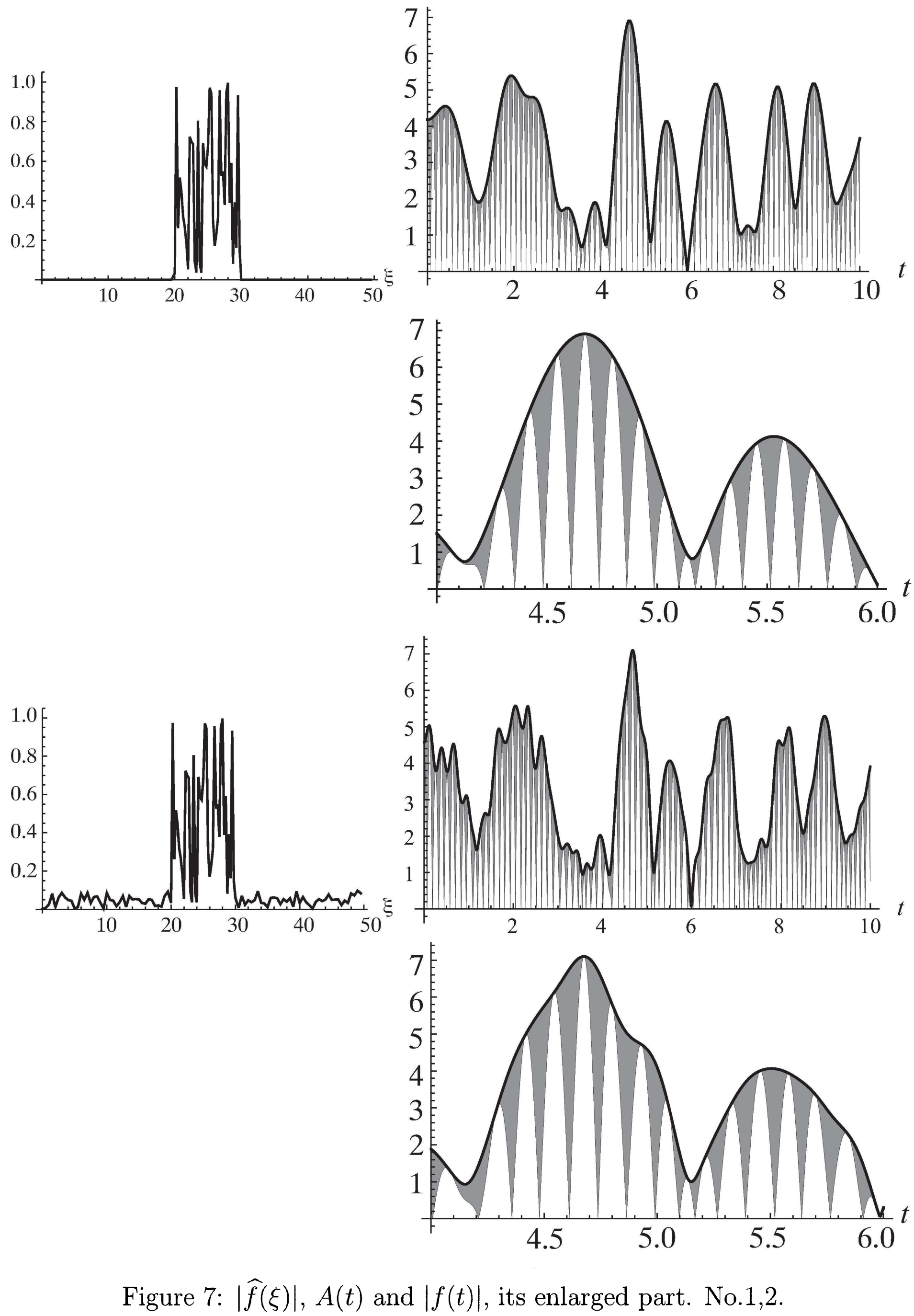

Figure 7: $|\widehat{f}(\xi)|$, $A(t)$ and $|f(t)|$, its enlarged part. No.1,2.

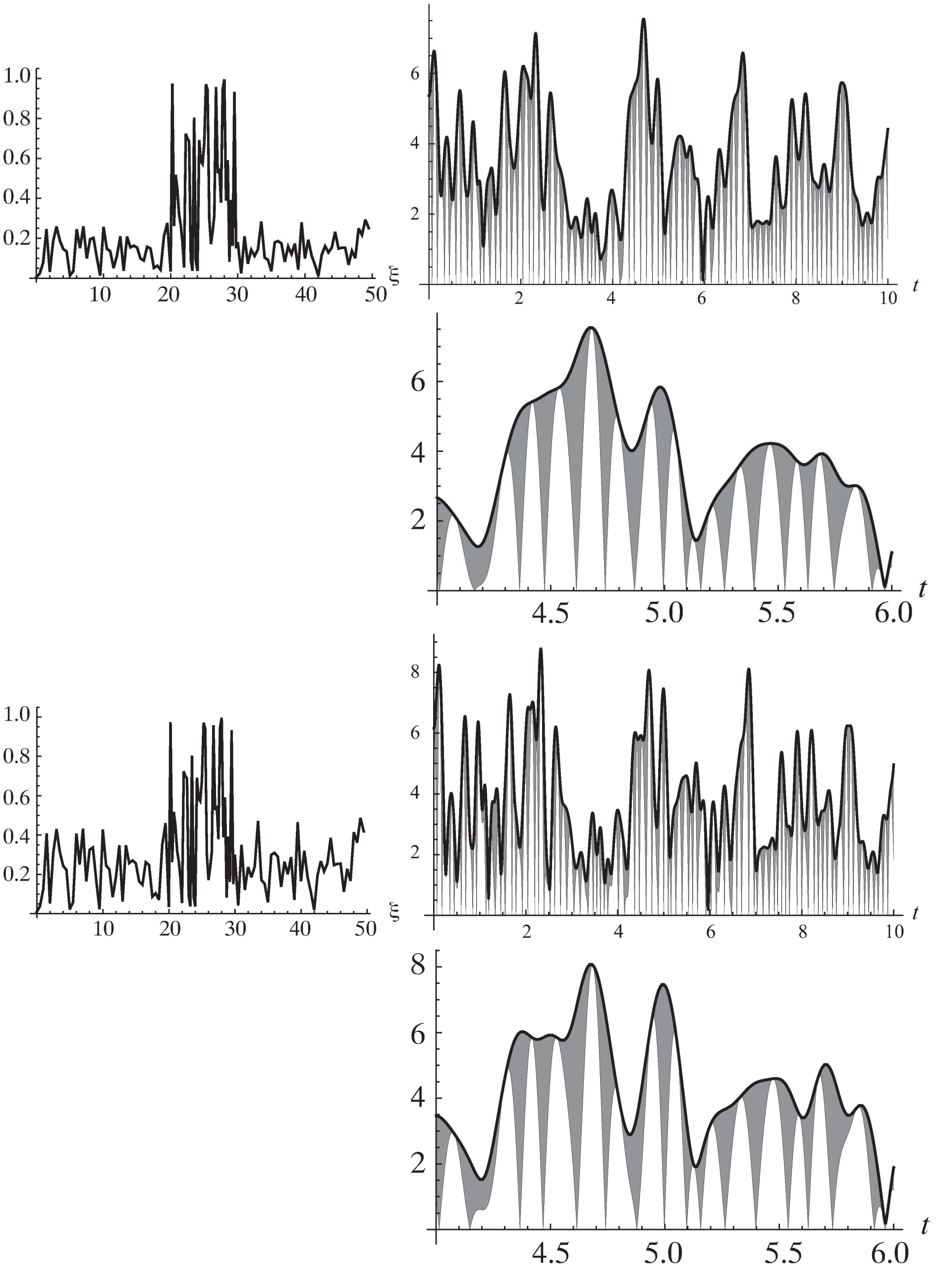

Figure 8: $|\widehat{f}(\xi)|$, $A(t)$ and $|f(t)|$, its enlarged part. No.3,4.

$C_2 \geq \dfrac{2}{\pi}$.

$$\int_{|J|}^{\infty} |\widehat{A^2}(\xi)|d\xi \leq \left(C_1 \left\|\widehat{f}\right\|_{L^1(\mathbb{R}_+\setminus J)} + C_2 \left\|\widehat{f}\right\|_{L^1(J)}\right) \left\|\widehat{f}\right\|_{L^1(\mathbb{R}_+\setminus J)}. \tag{5-2}$$

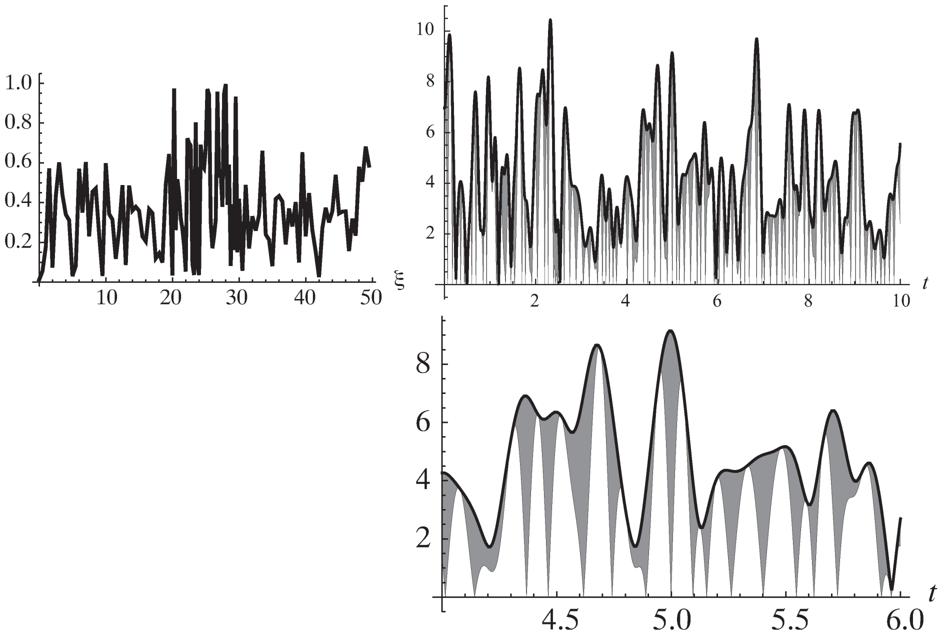

Figure 9: $|\widehat{f}(\xi)|$, $A(t)$ and $|f(t)|$, its enlarged part. No.5.

(3) If the following inequality holds for every real $f \in L^2(\mathbb{R})$, then $C_1 \geq \dfrac{2}{\pi}$ and $C_2 \geq \dfrac{4}{\pi}$.

$$\sup_{|\xi|\geq|J|} |\widehat{A^2}(\xi)| \leq \left(C_1 \left\| \widehat{f} \right\|_{L^1(\mathbb{R}_+ \setminus J)} + C_2 \left\| \widehat{f} \right\|_{L^1(J)} \right) \sup_{\xi \in \mathbb{R}_+ \setminus J} |\widehat{f}(\xi)|. \tag{5-3}$$

Remark 5. (1) We have considered only the mildness of the variation of $|A(t)|$ by $\widehat{A^2}(\xi)$. If $w_2 - w_1$ is quite smaller than w_1, then it seems that the graph of $A(t)$ is an "envelope" of the graph of $f(t)$. It is an important remained problem how can we give a good definition of an envelope and make a rigorous statement for the above observation.

Also, there is a possibility that the statement (1) of either Theorem 3 or Theorem 4, or possibly both, can be improved.

(2) We can extend our results to the general case as follows. For $J_1, J_2 \subset \mathbb{R}$, set $J_1 - J_2 := \{\, \eta_1 - \eta_2 \mid \eta_k \in J_k \, (k = 1, 2) \,\}$. Let $f_1, f_2 \in L^2(\mathbb{R})$, and set $E(t) := f_1(t)\overline{f_2(t)} \in L^1(\mathbb{R})$. Then we have similar inequalities as Theorem 3, for $\widehat{E}(\xi)$,

$\widehat{f_1}(\xi)$, and $\widehat{f_2}(\xi)$. Our case is when $f_1 = f_2 = f$ and $J_1 = J_2 = [w_1, w_2]$, where $J_1 - J_2 = [-(w_2 - w_1), w_2 - w_1]$.

6 Proof of Theorem 3

If $f \in L^2(\mathbb{R})$, then $A \in L^2(\mathbb{R})$, and hence $A^2 \in L^1(\mathbb{R})$, which implies that $\widehat{A^2}(\xi)$ is a bounded continuous function and $\widehat{A^2}(\xi) \to 0$ as $|\xi| \to \infty$. Since $A^2(t)$ is real-valued, we have $\widehat{A^2}(-\xi) = \overline{\widehat{A^2}(\xi)}$, and hence $|\widehat{A^2}(-\xi)| = |\widehat{A^2}(\xi)|$. Also, since $A^2(t) \geq 0$, we have

$$|\widehat{A^2}(\xi)| = \left| \int_{\mathbb{R}} A^2(t) e^{-i\xi t}\, dt \right| \leq \int_{\mathbb{R}} A^2(t)\, dt = \widehat{A^2}(0).$$

Set $\widehat{f}_+(\xi) := \begin{cases} \widehat{f}(\xi) & (\xi > 0), \\ 0 & (\xi < 0). \end{cases}$ Then, we have

$$\begin{aligned} \widehat{A^2}(\xi) &= \int_{\mathbb{R}} (\mathcal{A}f)(t) \overline{(\mathcal{A}f)(t) e^{i\xi t}}\, dt \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} (\mathcal{A}f)^\wedge(\omega) \overline{(\mathcal{A}f)^\wedge(\omega - \xi)}\, d\omega \\ &= \frac{2}{\pi} \int_{\mathbb{R}} \widehat{f}_+(\omega) \overline{\widehat{f}_+(\omega - \xi)}\, d\omega \qquad (6\text{-}1) \\ &= \frac{2}{\pi} \int_{\mathbb{R}} \widehat{f}_+(\omega + \xi) \overline{\widehat{f}_+(\omega)}\, d\omega. \qquad (6\text{-}2) \end{aligned}$$

We divide the integral in two ways. Consider $\xi > 0$. Note that $|\widehat{A^2}(-\xi)| = |\widehat{A^2}(\xi)|$. By (6-1), we have

$$\begin{aligned} |\widehat{A^2}(\xi)| &\leq \frac{2}{\pi} \left\{ \int_J |\widehat{f}_+(\omega)|\, |\widehat{f}_+(\omega - \xi)|\, d\omega + \int_{\mathbb{R}_+ \setminus J} |\widehat{f}_+(\omega)|\, |\widehat{f}_+(\omega - \xi)|\, d\omega \right\} \\ &=: \frac{2}{\pi}(I_1 + I_2). \end{aligned}$$

By (6-2), we also have

$$\begin{aligned} |\widehat{A^2}(\xi)| &\leq \frac{2}{\pi} \left\{ \int_J |\widehat{f}_+(\omega + \xi)|\, |\widehat{f}_+(\omega)|\, d\omega + \int_{\mathbb{R}_+ \setminus J} |\widehat{f}_+(\omega + \xi)|\, |\widehat{f}_+(\omega)|\, d\omega \right\} \\ &=: \frac{2}{\pi}(\widetilde{I}_1 + \widetilde{I}_2). \end{aligned}$$

(1) We can estimate I_1^2 and so on as follows.

$$I_1^2 \leq \left(\int_J |\widehat{f}_+(\omega)|^2\, d\omega \right) \left(\int_J |\widehat{f}_+(\omega - \xi)|^2\, d\omega \right)$$

$$
\begin{aligned}
&\le \left(\int_J |\widehat{f}(\omega)|^2\,d\omega\right)\left(\int_{(J-\xi)\cap\mathbb{R}_+} |\widehat{f}(\omega)|^2\,d\omega\right), \\
\widetilde{I_1}^2 &\le \left(\int_J |\widehat{f}_+(\omega+\xi)|^2\,d\omega\right)\left(\int_J |\widehat{f}_+(\omega)|^2\,d\omega\right) \\
&\le \left(\int_{J+\xi} |\widehat{f}(\omega)|^2\,d\omega\right)\left(\int_J |\widehat{f}(\omega)|^2\,d\omega\right).
\end{aligned}
$$

$$
\begin{aligned}
I_2^2 &\le \left(\int_{\mathbb{R}_+\setminus J} |\widehat{f}_+(\omega)|^2\,d\omega\right)\left(\int_{\mathbb{R}_+\setminus J} |\widehat{f}_+(\omega-\xi)|^2\,d\omega\right) \\
&\le \left(\int_{\mathbb{R}_+\setminus J} |\widehat{f}(\omega)|^2\,d\omega\right)\left(\int_{((\mathbb{R}_+\setminus J)-\xi)\cap\mathbb{R}_+} |\widehat{f}(\omega)|^2\,d\omega\right), \\
\widetilde{I_2}^2 &\le \left(\int_{\mathbb{R}_+\setminus J} |\widehat{f}_+(\omega+\xi)|^2\,d\omega\right)\left(\int_{\mathbb{R}_+\setminus J} |\widehat{f}_+(\omega)|^2\,d\omega\right) \\
&\le \left(\int_{(\mathbb{R}_+\setminus J)+\xi} |\widehat{f}(\omega)|^2\,d\omega\right)\left(\int_{\mathbb{R}_+\setminus J} |\widehat{f}(\omega)|^2\,d\omega\right).
\end{aligned}
$$

We write $L_M = \left\|\widehat{f}\right\|_{L^2(M)}$.

$$
\begin{aligned}
2|\widehat{A^2}(\xi)| &\le \frac{2}{\pi}(I_1 + \widetilde{I_1} + I_2 + \widetilde{I_2}) \\
&\le \frac{2}{\pi}\Big(L_J L_{(J-\xi)\cap\mathbb{R}_+} + L_J L_{J+\xi} + L_{\mathbb{R}_+\setminus J} L_{((\mathbb{R}_+\setminus J)-\xi)\cap\mathbb{R}_+} + L_{\mathbb{R}_+\setminus J} L_{(\mathbb{R}_+\setminus J)+\xi}\Big) \\
&\le \frac{2}{\pi}\Big\{L_J(L_{(J-\xi)\cap\mathbb{R}_+} + L_{J+\xi}) + (L_{((\mathbb{R}_+\setminus J)-\xi)\cap\mathbb{R}_+} + L_{(\mathbb{R}_+\setminus J)+\xi})L_{\mathbb{R}_+\setminus J}\Big\}.
\end{aligned}
$$

If $\xi \ge |J|$, then

$$
(J-\xi)\cap\mathbb{R}_+ \subset \mathbb{R}_+\setminus J, \qquad J+\xi \subset \mathbb{R}_+\setminus J, \tag{6-3}
$$

and the two intervals $(J-\xi)\cap\mathbb{R}_+$ and $J+\xi$ are disjoint. Hence

$$
(L_{(J-\xi)\cap\mathbb{R}_+} + L_{J+\xi})^2 \le 2(L^2_{(J-\xi)\cap\mathbb{R}_+} + L^2_{J+\xi}) \le 2L^2_{\mathbb{R}_+\setminus J}. \tag{6-4}
$$

Since $L_{((\mathbb{R}_+\setminus J)-\xi)\cap\mathbb{R}_+} + L_{(\mathbb{R}_+\setminus J)+\xi} \le 2L_{\mathbb{R}_+}$, we have

$$
\begin{aligned}
2|\widehat{A^2}(\xi)| &\le \frac{2}{\pi}(L_J\sqrt{2}L_{\mathbb{R}_+\setminus J} + 2L_{\mathbb{R}_+}L_{\mathbb{R}_+\setminus J}), \\
|\widehat{A^2}(\xi)| &\le \frac{2}{\pi}\left(L_{\mathbb{R}_+} + \frac{1}{\sqrt{2}}L_J\right)L_{\mathbb{R}_+\setminus J}. \qquad (6\text{-}5)
\end{aligned}
$$

Since ${L_{\mathbb{R}_+}}^2 = {L_{\mathbb{R}_+\setminus J}}^2 + {L_J}^2 \le (L_{\mathbb{R}_+\setminus J} + L_J)^2$, we have $L_{\mathbb{R}_+} \le L_{\mathbb{R}_+\setminus J} + L_J$, and hence

$$|\widehat{A^2}(\xi)| \le \frac{2}{\pi}\left(L_{\mathbb{R}_+\setminus J} + \frac{1+\sqrt{2}}{\sqrt{2}} L_J\right) L_{\mathbb{R}_+\setminus J}.$$

Hence, we have (4-2). Also from (6-5), since $L_J \le L_{\mathbb{R}_+}$, we have

$$|\widehat{A^2}(\xi)| \le \frac{2+\sqrt{2}}{\pi} L_{\mathbb{R}_+} L_{\mathbb{R}_+\setminus J}.$$

Hence, we have (4-3).

(2) We have

$$\begin{aligned}\int_{|J|}^{\infty} I_1\, d\xi &\le \int_J \left(\int_{|J|}^{\infty} |\widehat{f}_+(\omega-\xi)|\, d\xi\right) |\widehat{f}_+(\omega)|\, d\omega \\ &\le \int_J \left(\int_{0\le\xi\le\omega-|J|} |\widehat{f}(\xi)|\, d\xi\right) |\widehat{f}(\omega)|\, d\omega,\end{aligned}$$

since $w_2 - |J| = w_1$,

$$\le \int_0^{w_1} |\widehat{f}(\xi)|\, d\xi \int_J |\widehat{f}(\omega)|\, d\omega.$$

Also,

$$\begin{aligned}\int_{|J|}^{\infty} \widetilde{I}_1\, d\xi &\le \int_J \left(\int_{|J|}^{\infty} |\widehat{f}_+(\omega+\xi)|\, d\xi\right) |\widehat{f}_+(\omega)|\, d\omega \\ &\le \int_J \left(\int_{\omega+|J|}^{\infty} |\widehat{f}(\xi)|\, d\xi\right) |\widehat{f}(\omega)|\, d\omega,\end{aligned}$$

since $w_1 + |J| = w_2$,

$$\le \int_{w_2}^{\infty} |\widehat{f}(\xi)|\, d\xi \int_J |\widehat{f}(\omega)|\, d\omega.$$

We also have

$$\begin{aligned}\int_{|J|}^{\infty} I_2\, d\xi &\le \int_{\mathbb{R}_+\setminus J} \left(\int_{|J|}^{\infty} |\widehat{f}_+(\omega-\xi)|\, d\xi\right) |\widehat{f}_+(\omega)|\, d\omega \\ &\le \int_{\mathbb{R}_+\setminus J} \left(\int_{0\le\xi\le\omega-|J|} |\widehat{f}(\xi)|\, d\xi\right) |\widehat{f}(\omega)|\, d\omega.\end{aligned}$$

$$\int_{|J|}^{\infty} \widetilde{I}_2\, d\xi \le \int_{\mathbb{R}_+\setminus J} \left(\int_{|J|}^{\infty} |\widehat{f}_+(\omega+\xi)|\, d\xi\right) |\widehat{f}_+(\omega)|\, d\omega$$

$$\leq \int_{\mathbb{R}_+\setminus J}\left(\int_{\omega+|J|\leq\xi}|\widehat{f}(\xi)|\,d\xi\right)|\widehat{f}(\omega)|\,d\omega.$$

Thus,

$$\begin{aligned}
2\int_{|J|}^{\infty}|\widehat{A^2}(\xi)|\,d\xi &\leq \frac{2}{\pi}\int_{|J|}^{\infty}(I_1+\widetilde{I}_1+I_2+\widetilde{I}_2)\,d\xi \\
&\leq \frac{2}{\pi}\int_J|\widehat{f}(\omega)|\,d\omega\int_{\mathbb{R}_+\setminus J}|\widehat{f}(\xi)|\,d\xi \\
&\quad+\int_{\mathbb{R}_+}|\widehat{f}(\xi)|\,d\xi\int_{\mathbb{R}_+\setminus J}|\widehat{f}(\omega)|\,d\omega \\
&= \frac{2}{\pi}\left(\int_J|\widehat{f}(\omega)|\,d\omega+\int_{\mathbb{R}_+}|\widehat{f}(\xi)|\,d\xi\right)\int_{\mathbb{R}_+\setminus J}|\widehat{f}(\omega)|\,d\omega \\
&= \frac{2}{\pi}\left(2\int_J|\widehat{f}(\omega)|\,d\omega+\int_{\mathbb{R}_+\setminus J}|\widehat{f}(\xi)|\,d\xi\right)\int_{\mathbb{R}_+\setminus J}|\widehat{f}(\omega)|\,d\omega \\
&\leq \frac{4}{\pi}\int_{\mathbb{R}_+}|\widehat{f}(\omega)|\,d\omega\int_{\mathbb{R}_+\setminus J}|\widehat{f}(\omega)|\,d\omega.
\end{aligned}$$

Hence, we have (4-5).

(3) Here, there is no need to use $\widetilde{I}_1$, $\widetilde{I}_2$. If $\xi\geq|J|$, then

$$\begin{aligned}
I_1 &= \int_J|\widehat{f}_+(\omega)|\,|\widehat{f}_+(\omega-\xi)|\,d\omega \\
&\leq \left(\sup_{\omega\in J}|\widehat{f}_+(\omega-\xi)|\right)\int_J|\widehat{f}_+(\omega)|\,d\omega \\
&\leq \left\|\widehat{f}\right\|_{L^1(J)}\sup_{\omega\in\mathbb{R}_+\setminus J}|\widehat{f}(\omega)|. \\
I_2 &= \int_{\mathbb{R}_+\setminus J}|\widehat{f}_+(\omega)|\,|\widehat{f}_+(\omega-\xi)|\,d\omega \\
&\leq \left(\sup_{\omega\in\mathbb{R}_+\setminus J}|\widehat{f}_+(\omega)|\right)\int_{\mathbb{R}_+\setminus J}|\widehat{f}_+(\omega-\xi)|\,d\omega \\
&\leq \left\|\widehat{f}\right\|_{L^1(\mathbb{R}_+)}\sup_{\omega\in\mathbb{R}_+\setminus J}|\widehat{f}(\omega)|. \\
&= (\left\|\widehat{f}\right\|_{L^1(\mathbb{R}_+\setminus J)}+\left\|\widehat{f}\right\|_{L^1(J)})\sup_{\omega\in\mathbb{R}_+\setminus J}|\widehat{f}(\omega)|.
\end{aligned}$$

Hence, we have (4-6).

7 Proof of Theorem 2

Let $a \geq 0$ and $d \geq 1$. Consider $J = [3,4]$, $|J| = 1$. Define real-valued $f(t) = f_{a,d}(t)$ by

$$\widehat{f}_+(\xi) = \widehat{f_{a,d}}_+(\xi) = \begin{cases} 1 & (2 \leq \xi \leq 3) \\ a & (3 < \xi < 4) \\ 1 & (4 \leq \xi \leq 4+d) \end{cases} \qquad \text{(Figure 10).} \tag{7-1}$$

Then, by (6-1), we have

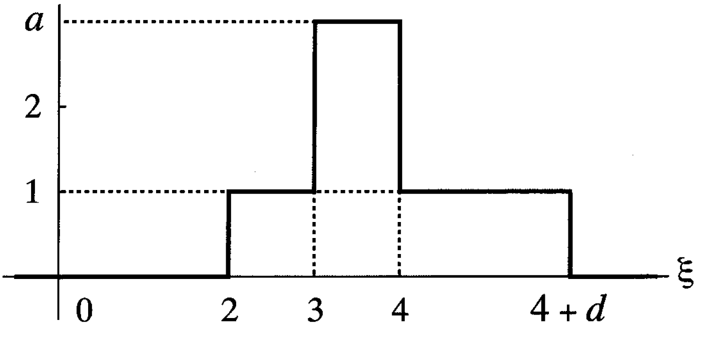

Figure 10: $\widehat{f_{a,d}}_+(\xi)$, $a \geq 0$, $d \geq 1$.

$$\begin{aligned}\widehat{A^2}(\xi) &= \frac{2}{\pi}\int_{\mathbb{R}} \widehat{f}_+(\omega)\widehat{f}_+(\omega-\xi)\,d\omega \\ &= \frac{2}{\pi}\left(\int_2^3 \widehat{f}_+(\omega-\xi)\,d\omega + a\int_3^4 \widehat{f}_+(\omega-\xi)\,d\omega + \int_4^{4+d} \widehat{f}_+(\omega-\xi)\,d\omega\right).\end{aligned}$$

Thus, we have

$$\begin{aligned}\widehat{A^2}(1) &= \frac{2}{\pi}\left(\int_2^3 \widehat{f}_+(\omega-1)\,d\omega + a\int_3^4 \widehat{f}_+(\omega-1)\,d\omega + \int_4^{4+d} \widehat{f}_+(\omega-1)\,d\omega\right) \\ &= \frac{2}{\pi}\left(\int_1^2 \widehat{f}_+(\omega)\,d\omega + a\int_2^3 \widehat{f}_+(\omega)\,d\omega + \int_3^{3+d} \widehat{f}_+(\omega)\,d\omega\right) \\ &= \frac{2}{\pi}(a + a + d - 1) = \frac{2}{\pi}(2a + d - 1).\end{aligned}$$

We also have $\left\|\widehat{f}\right\|_{L^2(\mathbb{R}_+\setminus J)} = \sqrt{1+d}$, $\left\|\widehat{f}\right\|_{L^2(J)} = a$, $\left\|\widehat{f}\right\|_{L^1(\mathbb{R}_+\setminus J)} = 1 + d$, and $\left\|\widehat{f}\right\|_{L^1(J)} = a$.

The inequality (1) reduces to

$$\frac{2}{\pi}(2a+d-1) \leq (C_1\sqrt{1+d}+C_2a)\sqrt{1+d}. \tag{7-2}$$

By letting $d \to \infty$, $a \to \infty$, we have $\frac{2}{\pi} \leq C_1$, $\frac{4}{\pi} \leq C_2\sqrt{1+d}$, and hence $C_1 \geq \frac{2}{\pi}$, $C_2 \geq \frac{2\sqrt{2}}{\pi}$.

Since

$$\int_{|J|}^{\infty} |\widehat{A^2}(\xi)|\, d\xi = \frac{1}{\pi}(d^2+1+2ad), \tag{7-3}$$

The inequality (2) reduces to

$$\frac{1}{\pi}(d^2+1+2ad) \leq \{C_1(1+d)+C_2a\}(1+d). \tag{7-4}$$

By letting $d \to \infty$, we have $C_1 \geq \frac{1}{\pi}$. By letting $a \to \infty$, we have $\frac{1}{\pi}2d \leq C_2(1+d)$, and hence by letting $d \to \infty$, we have $C_2 \geq \frac{2}{\pi}$.

The inequality (3) reduces to

$$\frac{2}{\pi}(2a+d-1) \leq \{C_1(1+d)+C_2a\}, \tag{7-5}$$

and hence we have $C_1 \geq \frac{2}{\pi}$, $C_2 \geq \frac{4}{\pi}$ similarly.

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