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Gelfand-Shilov 空間における連続ウェーブレット変換について

Wavelet Transforms on Gelfand-Shilov Spaces

松江工業高等専門学校 数理科学科 福田 尚広 (Naohiro Fukuda)
National Institute of Technology, Matsue College
筑波大学大学院数理物質科学研究科 木下 保 (Tamotu Kinoshita)
Institute of Mathematics, University of Tsukuba
筑波大学大学院数理物質科学研究科 芳野 和久 (Kazuhisa Yoshino)
Institute of Mathematics, University of Tsukuba

1 Wavelet Transform

We are concerned with convolution operators. The FBI transform

$$Tf(x,\xi;h)=\frac{1}{2^{1/2}(\pi h)^{3/4}}\int_{\mathbf{R}_y} e^{i(x-y)\xi/h}e^{-(x-y)^2/2h}f(y)dy$$

provides an alternative approach to analytic wave front sets in the microlocal analysis, which is developed independently by Sato-Kashiwara-Kawai. Thanks to the term $e^{ix\xi/h}$, the FBI transform can be rewritten as the convolution operator with the function of coherent state. If we remove $e^{ix\xi/h}$ and replace $e^{-iy\xi/h}$ by $e^{-iy\xi}$, it becomes the short time Fourier transform (STFT). Then, the parameter h plays a role of the size of the window $e^{-(x-y)^2/2h}$. The wavelet transform is also a convolution operator with different parameters. Let $\mathbf{A}:=\mathbf{R}_+\times\mathbf{R}$ denote the $ax+b$ group, endowed with the multiplication $(a,b)(a',b')=(aa',ab'+b)$ The left-invariant Haar measure on $\mathbf{A}$ is $d\mu=\frac{da}{a^2}db$. Let U be the unitary representation of $\mathbf{A}$ on $L^2(\mathbf{R})$ defined by

$$\Big(U(a,b)f\Big)(x)=\frac{1}{\sqrt{a}}f\left(\frac{x-b}{a}\right).$$

Based on this representation, the theory of the harmonic analysis on groups gives a generalized Fourier transform, that is wavelet. The wavelet transform of $f\in L^2(\mathbf{R})$ with respect to the analyzing wavelet $\psi\in L^2(\mathbf{R})$ satisfying the admissible condition

$$C_\psi:=\int_{\mathbf{R}}\frac{|\hat{\psi}(\xi)|^2}{|\xi|}d\xi<\infty,$$

is defined by

$$W_\psi f(a,b) = \frac{1}{\sqrt{C_\psi}} \int_{\mathbf{R}} f(x)\overline{\psi_{a,b}(x)}dx,$$

where

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right) \quad \text{for} \quad (a,b) \in \mathbf{A}.$$

The inverse wavelet transform of $F \in L^2(\mathbf{R}_+ \times \mathbf{R})$ with respect to the analyzing wavelet $\psi \in L^2(\mathbf{R})$ is defined by

$$M_\psi F(x) = \frac{1}{\sqrt{C_\psi}} \int_{\mathbf{R}_+} \int_{\mathbf{R}} F(a,b)\psi_{a,b}(x)\frac{dbda}{a^2} \quad (x \in \mathbf{R}).$$

Remark: If ψ is real-valued, we have the equality

$$\int_{-\infty}^{0} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi = \int_{0}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi \ (< \infty) \tag{1}$$

which gives the reconstruction formula $f = M_\psi W_\psi f$ and

$$\|W_\psi f\|_{L^2(\mathbf{A})} = \|f\|_{L^2(\mathbf{R})}.$$

These always hold when the set of a is $\mathbf{R}$ instead of $\mathbf{R}_+$. In general, without (1) the Cauchy-Schwarz inequality gives

$$\|W_\psi f\|_{L^2(\mathbf{A})} \leq C\|f\|_{L^2(\mathbf{R})}.$$

If we regard $\mathbf{A} := \mathbf{R}_+ \times \mathbf{R}$ as not group but set, we have to rewrite this estimate as

$$\|a^{-1}W_\psi f\|_{L^2(\mathbf{R}_+ \times \mathbf{R})} \leq C\|f\|_{L^2(\mathbf{R})}.$$

This is regarded as the continuity property in L^2.

Time - Frequency localization depends on window size. There is a general relationship between a and frequency:

Wide window (Stretched wavelet with large a)

Poor time localization and *Good* frequency localization.

Coarse features $\Rightarrow$ Low frequency

For the wide window, we can not find out the high frequency, which is averaged by integration.

Narrow window (Compressed wavelet with small a)
Good time localization and *Poor* frequency localization.
Rapidly changing details $\Rightarrow$ High frequency
For the narrow window, we can not find out the low frequency, which behaves very slowly.

Remark: Even if we use the narrow window, we can not detect the high frequency, which exists only locally in the frequency space. There is a limit to the detection with window due to the uncertainty principle, which says that the window sizes of time space and frequency space have an inverse proportionality.

Indeed, both STFT and wavelet transform use a window having an inverse proportionality. For STFT, the window size can be changed, but must be fixed and applied to all frequencies. A more flexible approach in which window size varies across frequencies would be desirable. So, the wavelet transform utilizes different window sizes for each frequency, as $a \sim |\xi|^{-1}$. That is just the auto focus property of wavelets. The wavelet transform is an improved version rather than a simplified version of STFT.

2 Application

We consider the Cauchy problem on $[0,T] \times \mathbf{R}_x$

$$\begin{cases} \partial_t^2 u - A(t)\partial_x^2 u = 0, \\ u(0,x) = u_0(x), \;\; \partial_t u(0,x) = u_1(x), \end{cases} \tag{2}$$

where the coefficient $A(t)$ satisfies the weakly hyperbolic condition

$$A(t) \geq 0 \;\; \text{for} \;\; t \in [0,T].$$

Let us denote by $G^s(\mathbf{R})$ $(1 \leq s < \infty)$ the space of Gevrey functions $f(x)$ satisfying

$$\sup_{x \in K} |\partial_x^n f(x)| \leq C_K r_K^n n!^s$$

for any compact set $K \subset \mathbf{R}$, $n \in \mathbf{N}$. [2] gave the assumption that $A \in C^{\alpha}[0,T]$ $(0 \le \alpha \le 1)$ and proved the well-posedness in G^s for

$$1 \le s < 1 + \frac{\alpha}{2}. \tag{3}$$

Counter Example: [3] gave the following example of the ill-posedness: Define that $T_0 = 0$, $T_j = \sum_{n=1}^{j} 2^{-(n-1)/20}$ $(j \ge 1)$,

$$A(t) = 2^{-j/10}\Theta\left((2^{21j/20}(t - T_j)\right) \quad \text{for} \quad t \in [T_j, T_{j+1}] \ (j \ge 0),$$

where

$$\Theta(\tau) = \frac{2 - 2\cos 2\pi\tau}{2 + 3\Gamma^3 \sin 2\pi\tau + (\Gamma - 9\Gamma^2)\cos 2\pi\tau}$$

and

$$\Gamma = (1 + 2\sqrt{7})^{1/3} - \frac{3}{(1 + 2\sqrt{7})^{1/3}}.$$

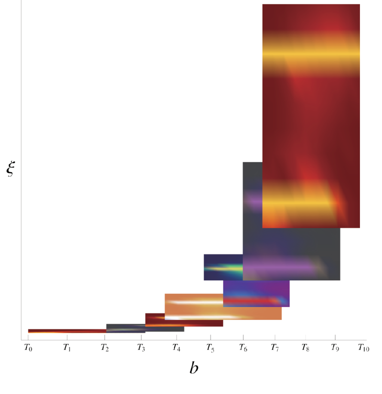
Then, the Cauchy problem (2) with $A(t) \in C^0[0,T]$ which is non-negative and degenerates at $t = T_j$ $(j \ge 0)$, is ill-posed in G^s for $s > 11/10 \sim 1$.

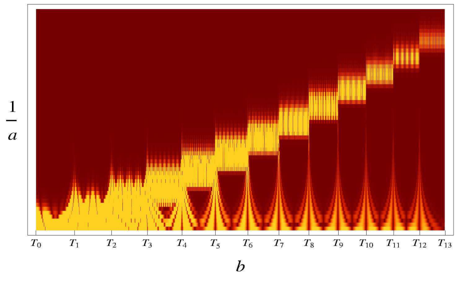
To know the behaviour of the coefficient concerned with the frequency, the standard Fourier transform is not good, because the coefficients are usually not defined in the whole interval $\mathbf{R}_t$. Therefore, it is natural to consider STFT:

$$T_w A(\xi, b) = \int_{\mathbf{R}} e^{-it\xi} A(t) \overline{w(t - b)} dt$$

and the wavelet transform:

$$W_\psi A(a, b) = \frac{1}{\sqrt{a}} \int_{\mathbf{R}} A(t) \overline{\psi\left(\frac{t - b}{a}\right)} dt.$$





The slopes of both figures indicate that a peak moves toward the blow-up point T_∞ as the frequency increases, which possibly causes the ill-posedness.

Remark 2.1 *If a equals some power of 2, the form of the counter example resembles the wavelet. Generally for a function $F\left(\frac{t-b'}{a'}\right)$, the wavelet transform with $\psi\left(\frac{t-b}{a}\right)$ detects $a \sim a'$ and $b \sim b'$. The above figure means that $a \sim 2^{-21j/20}$ and $b = T_j$ are conspicuous since $A(t) = 20^{-j/10}\Theta\left(\frac{t-T_j}{2^{-21j/20}}\right)$ for each interval.*

Remark 2.2 *Amplitudes of oscillating coefficients are flattened by the degeneracy. Regularities depend on not only frequency but also amplitude (degeneracy). For example, according to [1] let us consider*

$$\text{high frequency small amplitude} \quad f_1(t) = \begin{cases} 0 & for \quad t = 0, \\ \dfrac{\sin(\log|t|)}{(\log|t|)^2 + 1} & otherwise, \end{cases}$$

$$\text{higher frequency smaller amplitude} \quad f_2(t) = \begin{cases} 0 & for \quad t = 0, \\ \dfrac{\sin(\exp\frac{1}{|t|})}{\exp\frac{1}{|t|}} & otherwise. \end{cases}$$

Then, we find that

- *f_1 belongs not $\cup_{0<\alpha<1} C^\alpha$ but BV,*
- *f_2 belongs to not BV but $\cap_{0<\alpha<1} C^\alpha$.*

The counter example corresponds to the case of f_1, so the regularity is only C^0 (around $t = T_\infty$) and not BV.

STFT would require some graphs to adjust the brightness of the spectrogram. On the other hand, such an arrangement is not necessary for the wavelet transform. For this case, the wavelet transform will be useful.

3 Gelfand-Shilov Space and Continuity

From the point of view of the uncertainty principle, we are interested in better Time-Frequency localization. In this sense, the Schwartz space $\mathcal{S}$ will be preferable, because it has an arbitrary polynomial decay in both time and frequency spaces. For instance, very famous Mexican hat wavelet belongs to the Schwartz space $\mathcal{S}$. But in fact, the Mexican hat wavelet like the Gaussian satisfies an exponential decay. Therefore, we shall introduce the Gelfand-Shilov space which is an interpolation between arbitrary polynomial decay and exponetial decay, that is sub-exponetial decay in both time and frequency spaces. For positive constants μ, ν and h such that $\nu + \mu \geq 1$, we define the Banach Gelfand-Shilov space

$$\mathcal{S}^{\mu}_{\nu,h}(\mathbf{R}) = \left\{ f \in \mathcal{S} \ ; \ \|x^{\alpha}\partial^{\beta}_{x} f(x)\|_{L^{\infty}(\mathbf{R})} \leq C h^{\alpha+\beta} \alpha!^{\nu} \beta!^{\mu} \ \text{ for all } \alpha, \beta \in \mathbf{N} \right\}$$

with the norm

$$\|f\|_{\mathcal{S}^{\mu}_{\nu,h}(\mathbf{R})} = \sup_{\alpha,\beta \in \mathbf{N}} \frac{\|x^{\alpha}\partial^{\beta}_{x} f(x)\|_{L^{\infty}(\mathbf{R})}}{h^{\alpha+\beta}\alpha!^{\nu}\beta!^{\mu}},$$

and the (non-Banach) Gelfand-Shilov space $\mathcal{S}^{\mu}_{\nu}(\mathbf{R})$

$$\mathcal{S}^{\mu}_{\nu}(\mathbf{R}) = \operatorname{ind}\lim_{h>0} \mathcal{S}^{\mu}_{\nu,h}(\mathbf{R})$$

with the inductive limit topology. The Gelfand-Shilov spaces have often appeared in the study of functional analysis and PDE's (see [8], etc.). For the discrete wavelet case requiring strong additional conditions, [4] constructed the wavelets belonging to the Gelfand-Shilov spaces.

	A_x	G^s_x	C^r_x
A_ξ	nonexistence	nonexistence	Battle-Lemarié, Daubechies
G^s_ξ	Meyer	[FKU]	[HWW]
C^r_ξ	Meyer	[FKU]	[HWW]

As for the continuous wavelet transform requiring only the admissible condition, there are many possibilities to choose analyzing wavelets. Recently, [9] proved some estimates concerned with the continuity (boundedness) property of wavelet transforms in the (non-Banach) Gelfand-Shilov type of space $\mathcal{S}_{\nu}^{\mu,+}(\mathbf{R})$ which is restricted to the half space $\xi > 0$ as the Hardy space. For example, the Bessel wavelet $\psi(x)$ defined by $\hat{\psi}(\xi) = e^{-\xi - \xi^{-1}}$ for $\xi > 0$ and $= 0$ for $\xi \leq 0$ belongs to $\mathcal{S}_{2}^{1,+}(\mathbf{R})$. In fact, we know that $\psi(x) = \frac{1}{\pi\sqrt{1-ix}} K_1(2\sqrt{1-ix})$, where K_1 is the first modified Bessel function of the second kind (see [6]).

In this paper we assume vanishing moment conditions for not only ψ but also f. Paying the attention to the parameter h, we try to derive some detailed estimates. Our purpose is to show the continuity (boundedness) property of wavelet transforms in the (Banach) Gelfand-Shilov space $\mathcal{S}_{\nu,h}^{\mu}(\mathbf{R})$. Moreover, we also compute the wavelet transforms of concrete functions in the Gelfand-Shilov spaces and show the optimality of our results.

Lemma 3.1 *There exists $C > 0$ and $h_0 > 0$ such that*

$$\|e^{h_0|x|^{1/\nu}} f\|_{L^\infty(\mathbf{R})} + \|e^{h_0|\xi|^{1/\mu}} \hat{f}\|_{L^\infty(\mathbf{R})} \leq C,$$

if and only if $f \in \mathcal{S}_{\nu,h}^{\mu}(\mathbf{R})$.

Taking Lemma 3.1 into account, we also introduce the Banach Gelfand-Shilov space combining with the infinite vanishing moments condition $|\hat{f}(\xi)| \leq Ce^{-h|\xi|^{-1/\delta}}$,

$$S_{\nu,h}^{\mu,\delta}(\mathbf{R}) = \Big\{ f \in \mathcal{S};\, \|e^{h|x|^{1/\nu}} f\|_{L^\infty} + \|e^{h\max\{|\xi|^{1/\mu}, |\xi|^{-1/\delta}\}} \hat{f}\|_{L^\infty} < \infty \Big\}.$$

We remark that $\mathcal{S}_{\nu,h}^{\mu}(\mathbf{R})$ (without the infinite vanishing moments condition) corresponds to $S_{\nu,h}^{\mu,\delta}(\mathbf{R})$ with $\delta = \infty$, i.e.,

$$S_{\nu,h}^{\mu,\infty}(\mathbf{R}) = \Big\{ f \in \mathcal{S}\ ;\ \|e^{h|x|^{1/\nu}} f\|_{L^\infty} + \|e^{h|\xi|^{1/\mu}} \hat{f}\|_{L^\infty} < \infty \Big\}.$$

Then, we get the following theorem (see [5]):

Theorem 3.2 *Let μ, ν, h and δ be positive constants such that $\mu + \nu \geq 1$. Define that $d(\lambda) = \lambda(\lambda - 1)^{-1+1/\lambda}$. Then for the wavelet transform W_ψ with $\psi \in S_{\nu,h}^{\mu,\delta}(\mathbf{R})$, the following estimates hold:*

(*i*) if $\nu > 1$

$$\left\| \frac{e^{h|b/(a+1)|^{1/\nu}}}{a^{1/2}+1} W_\psi f \right\|_{L^\infty(\mathbf{R}_+\times\mathbf{R})} \leq C \left\| e^{h|x|^{1/\nu}} f \right\|_{L^\infty(\mathbf{R})},$$

$(i)'$ if $\nu \leq 1$

$$\left\| e^{h' 2^{1-1/\nu} |b/(a+1)|^{1/\nu}} W_\psi f \right\|_{L^\infty(\mathbf{R}_+ \times \mathbf{R})} \leq C \left\| e^{h|x|^2} f \right\|_{L^\infty(\mathbf{R})} \quad (0 < h' < h),$$

(ii) if $\mu > 1$

$$\left\| \frac{a^{1/2} e^{hd(\delta/\mu+1)^{1/\mu} a^{-1/(\mu+\delta)}}}{a+1} W_\psi f \right\|_{L^\infty(\mathbf{R}_+ \times \mathbf{R})} \leq C \left\| e^{h|\xi|^{1/\mu}} \hat{f} \right\|_{L^\infty(\mathbf{R})},$$

(iii) if $\mu > 1$

$$\left\| \frac{a^{1/2} e^{hd(\delta/\mu+1)^{1/\mu} (\max\{a, a^{-1}\})^{1/(\mu+\delta)}}}{a+1} W_\psi f \right\|_{L^\infty(\mathbf{R}_+ \times \mathbf{R})} \leq C \left\| e^{h \max\{|\xi|^{1/\mu}, |\xi|^{-1/\delta}\}} \hat{f} \right\|_{L^\infty(\mathbf{R})}.$$

Example: Let us consider the Mexican hat wavelet

$$\psi(x) = \frac{2}{\pi^{1/4}\sqrt{3}}(1 - x^2)e^{-x^2/2}, \quad \hat{\psi}(\xi) = \frac{2\sqrt{2\pi}}{\pi^{1/4}\sqrt{3}} \xi^2 e^{-\xi^2/2}.$$

We see that $\psi \in S^{1/2,\infty}_{1/2,h}(\mathbf{R})$ with $0 < h < 1/2$. In particular when $f(x) = e^{-x^2/2}$, we can get

$$W_\psi f(a, b) = \frac{2\sqrt{2}\pi^{1/4} a^{5/2}(a^2 - 1 - b^2)}{\sqrt{3C_\psi}(a^2+1)^{5/2}} e^{-b^2/(2a^2+2)}.$$

Then, $(i)'$ in Theorem 3.2 becomes

$$\left\| e^{h' 2^{-1} |b/(a+1)|^2} W_\psi f \right\|_{L^\infty(\mathbf{R}_+ \times \mathbf{R})} \leq C \left\| e^{h|x|^2} f \right\|_{L^\infty(\mathbf{R})},$$

where $0 < h' < h$. This implies that the exponent in $(i)'$ is almost optimal with respect to a and b, since

$$h' 2^{-1} |b/(a+1)|^2 \sim \frac{1}{2} \cdot b^2/(2a^2 + 2).$$

Thus, we see that $(i)'$ can not be improved anymore.

Moreover, we define the following weighted $L^\infty(\mathbf{R}_+ \times \mathbf{R})$ space which is a subspace of $L^2(\mathbf{R}_+ \times \mathbf{R})$ as far as h is positive:

$$V^{\mu,\delta}_{\nu,h}(\mathbf{R}_+ \times \mathbf{R})$$

$$=\left\{F\in L^2(\mathbf{R}_+\times\mathbf{R});\ \left\|e^{h\max\{|b/(a+1)|^{1/\nu},\ a^{1/\mu},\ a^{-1/\delta}\}}F\right\|_{L^\infty(\mathbf{R}_+\times\mathbf{R})}<\infty\right\}.$$

We remark that when $\mu=\infty$

$$V^{\infty,\delta}_{\nu,h}(\mathbf{R}_+\times\mathbf{R})$$
$$=\left\{F\in L^2(\mathbf{R}_+\times\mathbf{R});\ \left\|e^{h\max\{|b/(a+1)|^{1/\nu},\ a^{-1/\delta}\}}F\right\|_{L^\infty(\mathbf{R}_+\times\mathbf{R})}<\infty\right\}.$$

Theorem 3.2 gives the following continuity results:

Corollary 3.3 *Let $\mu>1$, $\nu>1$, $h>0$ and $\delta>0$. Then for $\psi\in S^{\mu,\delta}_{\nu,h}(\mathbf{R})$, the wavelet transform*

$$S^{\mu,\infty}_{\nu,h}(\mathbf{R})\ni f\mapsto W_\psi f\in V^{\infty,\mu+\delta}_{\nu,h}(\mathbf{R}_+\times\mathbf{R})$$

is continuous. If f also satisfies the infinite vanishing moments condition, the wavelet transform

$$S^{\mu,\delta}_{\nu,h}(\mathbf{R})\ni f\mapsto W_\psi f\in V^{\mu+\delta,\mu+\delta}_{\nu,h}(\mathbf{R}_+\times\mathbf{R})$$

is continuous.

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