数理解析研究所講究録
第2058巻 2017年 26-42

A note on the decay estimates for the compressible Navier-Stokes-Poisson system in critical Besov spaces

Noboru CHIKAMI* and Raphaël DANCHIN†

Abstract

This is a survey on the Cauchy problem for the Navier-Stokes-Poisson system in the critical regularity framework. Under a suitable additional condition involving only the low frequencies of the data, we establish optimal decay estimates in the L^2-critical framework for the global solutions around small perturbations of a linearly stable constant state.

1 Problem and formulation

This is a survey paper on [5]. We consider the Cauchy problem of the compressible Navier-Stokes equations coupled with a Poisson equation, in the whole space $\mathbb{R}^n$ with $n \geq 2$. The system reads

$$\begin{cases} \partial_t\rho + \mathrm{div}\,(\rho u) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla P & \\ \qquad = \mu\Delta u + (\lambda+\mu)\nabla \mathrm{div}\, u - \kappa\rho\nabla\psi, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ -\Delta\psi = \rho - \bar\rho, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ (\rho,u)|_{t=0} = (\rho_0, u_0), & x \in \mathbb{R}^n. \end{cases} \tag{1}$$

Above, the unknown functions $\rho = \rho(t,x) \in \mathbb{R}_+$, $u = u(t,x) \in \mathbb{R}^n$ and $\psi = \psi(t,x) \in \mathbb{R}$ represent the fluid density, the velocity field and the potential force, respectively. The pressure $P = P(\rho)$ is given by a smooth function only depending on ρ. The Lamé coefficients μ and λ are assumed to be constant (just for simplicity) and to satisfy

$$\mu > 0 \quad \text{and} \quad \lambda + 2\mu > 0, \tag{2}$$

so that the operator $\mu\Delta + (\lambda+\mu)\nabla\mathrm{div}$ is elliptic. Finally, we assume that ρ tends to some background constant density $\bar\rho > 0$ at infinity.

System (1) (that we shall sometimes designate by NSP) is often referred to as the *compressible Navier-Stokes-Poisson equations with a Coulomb potential.* The first equation represents the mass conservation law, the second one corresponds to the momentum balance, and the third equation is a Poisson type elliptic equation that determines the potential given by

*Mathematical Institute Tohoku University, Sendai 980-8578, Japan.
e-mail: `sb0m23@math.tohoku.ac.jp`
†Université Paris-Est, LAMA, UMR 8050 and Institut Universitaire de France, 61 avenue du Général de Gaulle, 94010 Créteil Cedex, France.
e-mail: `danchin@univ-paris12.fr`

the electric or gravitational field. If $\kappa > 0$ (repulsive case) then (1) describes the transport of charged particles under the electric field of electrostatic potential force (cf. Markowich-Ringhofer-Schmeiser [14]). When $\kappa < 0$ (attractive case), it models the dynamics of a self-gravitating gaseous star (cf. Chandrasekhar [2]). As may be seen by solving the linearized equations around the equilibrium $(\rho, u) = (\bar{\rho}, 0)$ in Fourier variables, the case $\kappa > 0$ is linearly stable, while the case $\kappa < 0$ is unstable. We shall focus on the repulsive case $\kappa > 0$ in this paper.

2 Statement of Results

Only a few works have been dedicated to the study of (1) in the so-called critical regularity framework. By *critical* regularity, we mean that the solutions are looked for in a functional space having the same invariance by time and space dilations as (1) itself, namely $(\psi, \rho, u) \to (\psi_\ell, \rho_\ell, u_\ell)$ for all $\ell > 0$, with

$$\psi_\ell(t,x) := \ell^{-2}\psi(\ell^2 t, \ell x), \quad \rho_\ell(t,x) := \rho(\ell^2 t, \ell x) \text{ and } u_\ell(t,x) := \ell u(\ell^2 t, \ell x). \tag{3}$$

The idea of critical framework for the standard compressible Navier-Stokes system (that is $\kappa = 0$) has been successfully employed by many authors (see e.g. [7, 4, 3, 9]).

In the case of (1) with $\kappa > 0$, Hao-Li [10] first adapted the method of [7], to prove global existence in dimension $n \geq 3$, where the initial data satisfies critical regularity with somewhat strong low frequency assumption. Later, still in dimension $n \geq 3$, Zheng [16] weakened the regularity on the velocity and extended the global existence result to the L^p critical framework. The result in [16] has been extended to any dimensions $n \geq 2$ by [5], and the large data local theory for (1) in critical framework is established by [6].

However, the long time behavior of the above solutions for (1) have not been fully investigated. In contrast, when $\kappa = 0$, there have been a number of results concerning the decay estimates for the solutions of barotropic compressible Navier-Stokes system. Matsumura-Nishida [15] considered the global classical solution and proved the optimal decay rates for the (1) with $\kappa = 0$ for data with high Sobolev regularities. Okita recently showed that a similar decay estimate holds for the critical solution, under an additional assumption that the data belongs to L^1, which was further extended to the L^p critical framework to any dimension $n \geq 2$.

As for decay estimates of solutions for (1) when $\kappa > 0$, little is known for critical solution. Li-Matsumura-Zhang [13] proved the global existence of a classical solution and time decay estimates in the three-dimensional case under the assumption that data are close to the constant equilibrium state. However, the decay results in [13] do not cover the critical solutions constructed in [5, 10, 16] as it treats data with high Sobolev regularity. It is our aim in this paper to prove optimal decay estimates for the critical global solutions of (1), in the spirit of those of Okita in [12] or Danchin [8] for the barotropic Navier-Stokes equations.

2.1 Notation

Before writing out our main statements, we need to introduce some notation. First of all, we will denote by C harmless generic 'constants' that may change from line to line, and we agree that the notation $A \cong B$ means that we have $C^{-1}A \leq B \leq CA$.

Next, we need to introduce some functional spaces. Let L^p $(1 \le p \le \infty)$ be the standard Lebesgue space on $\mathbb{R}^n$, and ℓ^p be the corresponding sequence space. To define Besov spaces, we start with a dyadic decomposition of unity $\{\phi_j\}_{j\in\mathbb{Z}}$ in the Fourier space generated by some non-negative radially symmetric function $\widehat{\phi} \in \mathcal{S}$, that satisfies

$$\operatorname{supp}\widehat{\phi} \subset \{\xi \in \mathbb{R}^n; 3/4 < |\xi| < 8/3\},$$
$$\widehat{\phi}_j := \widehat{\phi}(2^{-j}\cdot),\ j \in \mathbb{Z} \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \widehat{\phi}_j(\xi) = 1,\ \xi \neq 0.$$

We set $\widehat{\Phi}(\xi) := 1 - \sum_{j\ge 1} \widehat{\phi}_j(\xi)$ and $\widehat{\Phi}_j := \widehat{\Phi}(2^{-j}\cdot)$.

Definition 2.1 (Besov spaces). *Let $\mathcal{S}'$ be the space of all tempered distributions. For $s \in \mathbb{R}$ and $1 \le p \le \infty$ we define the homogeneous Besov space $\dot{B}^s_{p,1}$ to be*

$$\dot{B}^s_{p,1} := \{u \in \mathcal{S}'_h\ ;\ \|u\|_{\dot{B}^s_{p,1}} < \infty\},$$

$$\textit{with}\ \ \mathcal{S}'_h := \Big\{u \in \mathcal{S}'\ ;\ \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u = u \textit{ in } \mathcal{S}'\Big\} \quad \textit{and} \quad \|u\|_{\dot{B}^s_{p,1}} := \sum_{j\in\mathbb{Z}} 2^{js} \|\dot{\Delta}_j u\|_{L^p}.$$

Let us recall that if $s \le n/p$ then $\dot{B}^s_{p,1}$ is a Banach space, and that the set $\mathcal{S}_0$ of functions in $\mathcal{S}$ with Fourier transform supported away from the origin is dense. More properties of Besov spaces may be found in e.g. [1].

Having fixed some $k_0 \in \mathbb{Z}$, we denote by $u_L := \dot{S}_{k_0} u := \Phi_{k_0} * u$ the low frequencies of u, and by $u_H := u - u_L$ the high frequencies of u. We shall also need the notation

$$\|u\|^L_{\dot{B}^s_{p,1}} := \sum_{j\le k_0} 2^{js} \|\dot{\Delta}_j u\|_{L^p} \quad \text{and} \quad \|u\|^H_{\dot{B}^\sigma_{p,1}} := \sum_{j\ge k_0-1} 2^{j\sigma} \|\dot{\Delta}_j u\|_{L^p}. \tag{4}$$

Note the (intentional) small overlap between low and high frequencies, ensuring that

$$\|u_L\|_{\dot{B}^s_{p,1}} \le C\|u\|^L_{\dot{B}^s_{p,1}} \quad \text{and} \quad \|u_H\|_{\dot{B}^\sigma_{p,1}} \le C\|u\|^H_{\dot{B}^\sigma_{p,1}}. \tag{5}$$

2.2 Main results

We introduce the set $E_p(T)$ of tempered distributions (a, u) satisfying

$$\begin{aligned}
&a_L \in \widetilde{\mathcal{C}}([0,T]; \dot{B}^{\frac{n}{2}-2}_{2,1}),\ u_L \in \widetilde{\mathcal{C}}([0,T]; \dot{B}^{\frac{n}{2}-1}_{2,1}),\ (\nabla a, \nabla^2 u)_L \in L^1(0,T; \dot{B}^{\frac{n}{2}-1}_{2,1}),\\
&a_H \in \widetilde{\mathcal{C}}([0,T]; \dot{B}^{\frac{n}{p}}_{p,1}) \cap L^1(0,T; \dot{B}^{\frac{n}{p}}_{p,1}),\\
&u_H \in \widetilde{\mathcal{C}}([0,T]; \dot{B}^{\frac{n}{p}-1}_{p,1}) \quad \text{and} \quad \nabla^2 u_H \in L^1(0,T; \dot{B}^{\frac{n}{p}-1}_{p,1}).
\end{aligned} \tag{6}$$

We shall denote

$$\begin{aligned}
X(T) := \|a_L\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{n}{2}-2}_{2,1})} + \|u_L\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{n}{2}-1}_{2,1})} + \|(\nabla a, \nabla^2 u)_L\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})}\\
+ \|(\nabla a, u)\|^H_{\widetilde{L}^\infty_T(\dot{B}^{\frac{n}{p}-1}_{p,1})} + \|(\nabla a, \nabla^2 u)\|^H_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})}
\end{aligned} \tag{7}$$

with the convention that $\|\cdot\|_{L^r_T(Y)}$ designates the norm of L^r functions (or essentially bounded functions if $r=+\infty$) on the interval $[0,T]$ with values in the Banach space Y, black and that $\|(f,g)\|_{L^r_T(Y)} := \|f\|_{L^r_T(Y)} + \|g\|_{L^r_T(Y)}$.

The norms $\|\cdot\|_{\widetilde{L}^\infty_t(\dot{B}^s_{2,1})}$ that are slightly stronger than the norms with no tilde are defined in (44). Finally, in (6) we agreed that

$$\widetilde{\mathcal{C}}([0,T];\dot{B}^s_{2,1}) := \{v\in \mathcal{C}([0,T];\dot{B}^s_{2,1})\,;\,\|v\|_{\widetilde{L}^\infty_T(\dot{B}^s_{2,1})}<\infty\}.$$

Note that owing to (5) and Bernstein inequality, $X(T)$ is equivalent to the 'natural' norm of $E_p(T)$ stemming directly from Definition (6).

The following result of global solvability is proved in [5].

Theorem 2.2 ([5]). *Assume that $\kappa>0$ and that $P'(\bar\rho)>0$. Let $n\geq 2$ and*

$$p\in\begin{cases}[2,4) & \text{if } \; n=2\\ [2,4] & \text{if } \; n=3\\ \left[2,\dfrac{2n}{n-2}\right] & \text{if } \; n\geq 4.\end{cases} \tag{8}$$

Consider initial data (ρ_0,u_0) satisfying $\inf_x\rho_0(x)>0$ and such that $a_0:=(\rho_0-\bar\rho)\in\dot{B}^{\frac{n}{p}}_{p,1}$ and $u_0\in\dot{B}^{\frac{n}{p}-1}_{p,1}$.

There exists an integer k_0 depending only on κ, μ and λ and some small enough constant $c=c(n,p,\mu,\lambda,\kappa,P)$ such that if we have in addition

$$X_0 := \|a_{0L}\|_{\dot{B}^{\frac{n}{2}-2}_{2,1}} + \|u_{0L}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|(\nabla a_0,u_0)\|^H_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \leq c, \tag{9}$$

then there exists a unique global solution $(\rho,u,\nabla\psi)$ to (1) satisfying for all $T>0$, $(a:=\rho-\bar\rho,u)\in E_p(T)$ and $\nabla^2\psi$ has the same regularity as a. Moreover, for some constant $C=C(n,p,\mu,\lambda,\kappa,P)$, we have

$$X(T)\leq CX_0 \quad \text{for all } \; T\geq 0. \tag{10}$$

Under additional assumptions on the low frequencies of the initial data, one may obtain time-decay estimates that are very similar to those of the standard compressible Navier-Stokes equations. For simplicity, we focus on the result for L^2-based critical Besov spaces.

Theorem 2.3 ([5]). *Let the data (a_0,u_0) satisfy the assumptions of Theorem 2.2 with $p=2$ and assume for simplicity that $P'(1)=1$ and that $\kappa=1$. There exists a positive constant c so that if in addition*

$$D_0 := \|a_0\|^L_{\dot{B}^{-\frac{n}{2}-1}_{2,\infty}} + \|u_0\|^L_{\dot{B}^{-\frac{n}{2}}_{2,\infty}} \leq c \tag{11}$$

then the global solution (a,u) given by Theorem 2.2 satisfies for all $t\geq 0$,

$$D(t)\leq C\big(\|a_0\|^L_{\dot{B}^{-\frac{n}{2}-1}_{2,\infty}} + \|u_0\|^L_{\dot{B}^{-\frac{n}{2}}_{2,\infty}} + \|a_0\|^H_{\dot{B}^{\frac{n}{2}}_{2,1}} + \|u_0\|^H_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\big) \tag{12}$$

with, denoting $\langle t\rangle := (1+t)$ *and* $\alpha := \frac{n}{2}+\frac{1}{2}-\varepsilon$ *for sufficiently small* $\varepsilon > 0$, *black and agreeing that the notation* $\langle\tau\rangle^\sigma f$ *(with* $f\in\{a,\nabla a,u,\nabla u\}$ *and* $\sigma\in\mathbb{R}$*) designates the function* $(\tau,x)\mapsto\langle\tau\rangle^\sigma f(\tau,x)$,

$$D(t) := \sup_{s\in[\varepsilon-\frac{n}{2},\frac{n}{2}+1]}\Big(\|\langle\tau\rangle^{\frac{n}{4}+\frac{s}{2}}a\|^L_{L^\infty_t(\dot B^{s-1}_{2,1})}+\|\langle\tau\rangle^{\frac{n}{4}+\frac{s}{2}}u\|^L_{L^\infty_t(\dot B^{s}_{2,1})}\Big)$$
$$+\|\langle\tau\rangle^{\alpha}(\nabla a,u)\|^H_{\widetilde{L}^\infty_t(\dot B^{\frac{n}{2}-1}_{2,1})}+\|\tau\nabla u\|^H_{\widetilde{L}^\infty_t(\dot B^{\frac{n}{2}}_{2,1})}. \tag{13}$$

The rest of the survey is dedicated to the proof of Theorem 2.3. For the proof of Theorem 2.2, we refer to [5, 16]. In Appendix, for the reader convenience, we list some results concerning product and commutator in the Besov spaces.

3 Linear analysis

In the case where $\rho-\bar\rho$ is in $\mathcal{S}_0$ then the last equation of (1) allows to compute $\nabla\psi$ from ρ by the formula

$$\nabla\psi=\nabla(-\Delta)^{-1}(\rho-\bar\rho).$$

As $\mathcal{S}_0$ is dense in $\dot B^s_{p,1}$ whenever $s\le n/p$, we deduce that in the functional setting of e.g. Theorem 2.2 and if ρ is positive, then System (1) may be equivalently written as

$$\begin{cases}\partial_t a+\bar\rho\,\mathrm{div}\,u=-\mathrm{div}\,(au),\\ \partial_t u-\dfrac{1}{\bar\rho}\mathcal{L}u+\dfrac{P'(\bar\rho)}{\bar\rho}\nabla a+\kappa\nabla(-\Delta)^{-1}a\\ \qquad =-u\cdot\nabla u+\Big(\dfrac{P'(\bar\rho)}{\bar\rho}-\dfrac{P'(\bar\rho(1+a))}{\bar\rho(1+a)}\Big)\nabla a-\dfrac{1}{\bar\rho}\Big(\dfrac{a}{a+\bar\rho}\Big)\mathcal{L}u,\end{cases}\tag{14}$$

where we denoted $a:=\rho-\bar\rho$ and $\mathcal{L}:=\mu\Delta+(\lambda+\mu)\nabla\mathrm{div}$.

The following change of variables allows us to normalize the coefficients of the linear terms to 1 (except for those pertaining to the viscous stress tensor):

$$\widetilde a(t,x)=\frac{1}{\bar\rho}a\Big(\frac{1}{\sqrt{\kappa\bar\rho}}t,\frac{c}{\sqrt{\kappa\bar\rho}}x\Big),\quad \widetilde u(t,x)=\frac{1}{c}u\Big(\frac{1}{\sqrt{\kappa\bar\rho}}t,\frac{c}{\sqrt{\kappa\bar\rho}}x\Big)\quad\text{with }\ c:=\sqrt{P'(\bar\rho)}.\tag{15}$$

Then System (14) is transformed to

$$\begin{cases}\partial_t\widetilde a+\mathrm{div}\,\widetilde u=F(\widetilde a,\widetilde u),\\ \partial_t\widetilde u-\widetilde{\mathcal{L}}\widetilde u+\nabla\widetilde a+\nabla(-\Delta)^{-1}\widetilde a=G(\widetilde a,\widetilde u),\end{cases}\tag{16}$$

with $\widetilde{\mathcal{L}}:=\frac{1}{c^2}\sqrt{\frac{\kappa}{\bar\rho}}\mathcal{L}=\widetilde\mu\Delta+(\widetilde\lambda+\widetilde\mu)\nabla\mathrm{div}$,

$$\begin{aligned}&F(\widetilde a,\widetilde u):=-\mathrm{div}\,(\widetilde a\widetilde u),\\ \text{and}\quad &G(\widetilde a,\widetilde u):=-\widetilde u\cdot\nabla\widetilde u+\Big(1-\frac{P'(\bar\rho(1+\widetilde a))}{c^2(1+\widetilde a)}\Big)\nabla\widetilde a-\Big(\frac{\widetilde a}{\widetilde a+1}\Big)\widetilde{\mathcal{L}}\widetilde u.\end{aligned}\tag{17}$$

From now on, we drop the tilde on $(\widetilde a,\widetilde u)$ as well as on $\widetilde\lambda$ and $\widetilde\mu$, and consider the normalized system (16). Let us decompose u into $u=w+\mathcal{P}^\perp u$, with $w:=\mathcal{P}u$ where $\mathcal{P}$ and $\mathcal{P}^\perp$

are the projectors onto divergence-free and potential vector-fields, respectively (hence $w := (\mathrm{Id} + \nabla\mathrm{div}\,(-\Delta)^{-1})u$). Setting $v := \Lambda^{-1}\mathrm{div}\,u = \Lambda^{-1}\mathrm{div}\,\mathcal{P}^{\perp}u$ with $\Lambda^{-1} := (-\Delta)^{-1/2}$, the system for (a, v, w) reads

$$\begin{cases} \partial_t a + \Lambda v = F, \\ \partial_t v - \nu\Delta v - \Lambda a - \Lambda^{-1}a = \Lambda^{-1}\mathrm{div}\,G, \\ \partial_t w - \mu\Delta w = \mathcal{P}G, \end{cases} \tag{18}$$

where $\Lambda := (-\Delta)^{1/2}$, $\nu := \lambda + 2\mu$, and F and G have been defined in (17).

At the linear level, the interaction between the velocity and the density only involves the compressible part of the velocity, namely v. The incompressible part w, as for it, satisfies a mere heat equation. In the second equation of (18), we immediately notice that $\Lambda a + \Lambda^{-1}a$ should have the same regularity as $\nu\Delta v$, that is $L^1(0,T;\dot{B}^{\frac{n}{p}-1}_{p,1})$. As such, it suffices to estimate a in $\dot{B}^{\frac{n}{p}-2}_{p,1} \cap \dot{B}^{\frac{n}{p}}_{p,1}$, i.e., $\Lambda^{-1}a_L \in \dot{B}^{\frac{n}{p}-1}_{p,1}$ while $a_H \in \dot{B}^{\frac{n}{p}}_{p,1}$. This turns out to be useful observation when estimating the decay rates in low frequency as we shall see soon.

4 Proof of Theorem 2.3

We refer to [5] for the proof of Theorem 2.2. We focus on the proof of Theorem 2.3, which is divided into three steps, corresponding to the three terms of the time weighted functional defined in (13). Recall the following elementary inequality.

Lemma 4.1 ([5, 8]). *For any $a, b > 0$ with $\max(a,b) > 1$, there exists a positive constant C such that:*

$$\int_0^t \langle\tau\rangle^{-a}\langle t-\tau\rangle^{-b}\,d\tau \le C\langle t\rangle^{-\min(a,b)} \quad \textit{for all} \quad t \ge 0.$$

Step 1: Bounds for the low frequencies. As pointed out in the previous section, for low frequencies, it suffices to bound (a, v, w) where $v := \Lambda^{-1}\mathrm{div}\,u$ and $w = \Lambda^{-1}\mathrm{rot}\ u$. At the linear level, the incompressible component w satisfies a mere heat equation, while (a, v) fulfills

$$\begin{cases} \partial_t a + \Lambda v = F, \\ \partial_t v - \nu\Delta v - \Lambda a - \Lambda^{-1}a = \Lambda^{-1}G. \end{cases} \tag{19}$$

In the low-frequency regime, we may also expect to work at the same level of regularity for $\widetilde{a} := \Lambda^{-1}a$, v and w, so that it is natural to consider the following linear system:

$$\begin{cases} \partial_t \widetilde{a} + v = \Lambda^{-1}F, \\ \partial_t v - \nu\Delta v - (1+\Lambda^2)\widetilde{a} = \Lambda^{-1}\mathrm{div}\,G. \end{cases} \tag{20}$$

The Green matrix $H(t,\cdot)$ corresponding to the semi-group $e^{t\widetilde{A}}$ of (20) may be deduced from the Green matrix $G(t,\cdot)$ of (19) by the relation

$$\widehat{H}(t,\xi) = \begin{pmatrix} |\xi|^{-1} & 0 \\ 0 & 1 \end{pmatrix} \widehat{G}(t,\xi) \begin{pmatrix} |\xi| & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the expression of $\widehat{G}(t,\xi)$ given in Prop. 5.2, we discover that

$$\widehat{H}(t,\xi) = \frac{1}{\lambda_+ - \lambda_-}\begin{pmatrix} \lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t} & -(e^{\lambda_+ t} - e^{\lambda_- t}) \\ (|\xi|^2+1)(e^{\lambda_+ t} - e^{\lambda_- t}) & \lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t} \end{pmatrix},$$

where $\lambda_\pm(\xi) := -\frac{1}{2}\nu|\xi|^2\Big(1 \pm \sqrt{1 - \frac{4(|\xi|^2+1)}{\nu^2|\xi|^4}}\Big)$.

One can thus easily conclude that for all $k_0 \in \mathbb{Z}$, there exist positive constants c_0 and C depending only on k_0 such that

$$|\widehat{H}(t,\xi)| \le Ce^{-c_0 t|\xi|^2} \quad \text{for all} \quad |\xi| \le 2^{k_0}. \tag{21}$$

Combining (21) along with the parabolic estimate for the incompressible part, we may obtain the linear decay estimate via Fourier-Plancherel theorem and the localization property of $\dot{\Delta}_j$. Denoting by e^{tB} the semi-group associated to System (30) written in terms of $(\Lambda^{-1}a, u)$, we get for all $s > -n/2$,

$$\sup_{t\ge0}\langle t\rangle^{\frac{n}{4}+\frac{s}{2}}\|e^{tB}(\Lambda^{-1}a,u)\|^L_{\dot{B}^s_{2,1}} \le C_s\|(\Lambda^{-1}a_0,u_0)\|^L_{\dot{B}^{-\frac{n}{2}}_{2,\infty}}. \tag{22}$$

Since it is expected that a small solution to (1) behaves asymptotically like a linear solution, the above estimate gives us some clues on the decay rate for the nonlinear problem. More concretely, rewriting (1) as (16), using (22) and Duhamel formula, we see that the solution (a,u) to (1) fulfills for all $s > -n/2$ and $t \ge 0$,

$$\|(\Lambda^{-1}a,u)(t)\|^L_{\dot{B}^s_{2,1}} \le C\Big(\langle t\rangle^{-\frac{n}{4}-\frac{s}{2}}\|(\Lambda^{-1}a_0,u_0)\|^L_{\dot{B}^{-\frac{n}{2}}_{2,\infty}} + \int_0^t \langle t-\tau\rangle^{-\frac{n}{4}-\frac{s}{2}}\|(\Lambda^{-1}F,G)(\tau)\|^L_{\dot{B}^{-\frac{n}{2}}_{2,\infty}}\,d\tau\Big). \tag{23}$$

We claim that if $s \in (-\frac{n}{2}, \frac{n}{2}+1]$, then we have

$$\int_0^t \langle t-\tau\rangle^{-\frac{n}{4}-\frac{s}{2}}\|(\Lambda^{-1}F,G)(\tau)\|^L_{\dot{B}^{-\frac{n}{2}}_{2,\infty}}\,d\tau \le C\langle t\rangle^{-\frac{n}{4}-\frac{s}{2}}(D^2(t)+X^2(t)) \tag{24}$$

with X and D defined in (7) and (13), respectively.

Note that, in light of the following inequality:

$$\|h\|^L_{\dot{B}^{-\frac{n}{2}}_{2,\infty}} \le C\|h\|_{\dot{B}^0_{1,\infty}} \le C\|h\|_{L^1}, \tag{25}$$

it is sufficient to prove (24) with $\|(\Lambda^{-1}F,G)\|_{L^1}$ instead of $\|(\Lambda^{-1}F,G)\|^L_{\dot{B}^{-\frac{n}{2}}_{2,\infty}}$.

For bounding $\Lambda^{-1}F = -\Lambda^{-1}\mathrm{div}\,(au)$, we use the fact that $\Lambda^{-1}\mathrm{div}$ is continuous on $\dot{B}^{-\frac{n}{2}}_{2,\infty}$ (being a homogeneous multiplier of degree 0). Hence, owing to (25), it suffices to bound $\|au\|_{L^1}$. Now, from Cauchy-Schwarz inequality, the definition of $D(t)$ and Lemma 4.1, one may write, if $-\frac{n}{2} < s \le \frac{n}{2}+1$,

$$\begin{aligned}\int_0^t \langle t-\tau\rangle^{-\frac{n}{4}-\frac{s}{2}}\|(au)(\tau)\|_{L^1}d\tau &\le \Big(\sup_{0\le\tau\le t}\langle\tau\rangle^{\frac{n}{4}}\|u(\tau)\|_{L^2}\Big)\Big(\sup_{0\le\tau\le t}\langle\tau\rangle^{\frac{n}{4}+\frac{1}{2}}\|a(\tau)\|_{L^2}\Big)\\ &\qquad\times\int_0^t\langle t-\tau\rangle^{-\frac{n}{4}-\frac{s}{2}}\langle\tau\rangle^{-\frac{n}{2}-\frac{1}{2}}d\tau\\ &\le C\langle t\rangle^{-\frac{n}{4}-\frac{s}{2}}\Big(\sup_{0\le\tau\le t}\langle\tau\rangle^{\frac{n}{4}}\|u(\tau)\|_{L^2}\Big)\Big(\sup_{0\le\tau\le t}\langle\tau\rangle^{\frac{n}{4}+\frac{1}{2}}\|a(\tau)\|_{L^2}\Big).\end{aligned}$$

We claim that

$$\langle\tau\rangle^{\frac{n}{4}}\|u(\tau)\|_{L^2}\le CD(\tau). \tag{26}$$

Indeed we have

$$\langle\tau\rangle^{\frac{n}{4}}\|u(\tau)\|_{L^2}\le\langle\tau\rangle^{\frac{n}{4}}\|u_L(\tau)\|_{L^2}+\langle\tau\rangle^{\frac{n}{4}}\|u_H(\tau)\|_{L^2}.$$

On one hand, according to the definition of D and to $\alpha\ge n/4$, one may write

$$\begin{aligned}\langle\tau\rangle^{\frac{n}{4}}\|u_H(\tau)\|_{L^2}&\le\langle\tau\rangle^{\frac{n}{4}}\sum_{k\ge k_0}\|\dot\Delta_k u(\tau)\|_{L^2}\\ &\le 2^{-k_0(\frac{n}{2}-1)}\langle\tau\rangle^{\frac{n}{4}}\sum_{k>k_0}2^{k(\frac{n}{2}-1)}\|\dot\Delta_k u(\tau)\|_{L^2}\\ &\le 2^{-k_0(\frac{n}{2}-1)}\langle\tau\rangle^{\alpha}\|u(\tau)\|^H_{\dot B^{\frac{n}{2}-1}_{2,1}}.\end{aligned}$$

On the other hand, taking $s=0$ in the definition of D and using that $\dot B^0_{2,1}\hookrightarrow L^2$ yields $\langle\tau\rangle^{\frac{n}{4}}\|u(\tau)\|^L_{L^2}\le CD(\tau)$.

Regarding $\sup_{0\le\tau\le t}\langle\tau\rangle^{\frac{n}{4}+\frac{1}{2}}\|a(\tau)\|_{L^2}$, let us use that $\alpha\ge\frac{n}{4}+\frac{1}{2}$ if $\varepsilon>0$ is taken small enough in the definition of α. Now, because

$$\langle\tau\rangle^{\frac{n}{4}+\frac{1}{2}}\|a_L(\tau)\|_{L^2}\le\langle\tau\rangle^{\frac{n}{4}+\frac{1}{2}}\|\Lambda^{-1}a_L(\tau)\|_{\dot B^1_{2,1}}$$

and

$$\begin{aligned}\langle\tau\rangle^{\frac{n}{4}+\frac{1}{2}}\|a_H(\tau)\|_{L^2}&\le\langle\tau\rangle^{\frac{n}{4}+\frac{1}{2}}\sum_{k\ge k_0}\|\dot\Delta_k a(\tau)\|_{L^2}\\ &\le 2^{-k_0\frac{n}{2}}\langle\tau\rangle^{\frac{n}{4}+\frac{1}{2}}\sum_{k\ge k_0}2^{k\frac{n}{2}}\|\dot\Delta_k a(\tau)\|_{L^2}\\ &\le 2^{-k_0\frac{n}{2}}\langle\tau\rangle^{\alpha}\|a(\tau)\|^H_{\dot B^{\frac{n}{2}}_{2,1}},\end{aligned}$$

we conclude that

$$\langle\tau\rangle^{\frac{n}{4}+\frac{1}{2}}\|a(\tau)\|_{L^2}\le CD(\tau). \tag{27}$$

Therefore, we have

$$\int_0^t\langle t-\tau\rangle^{-\frac{n}{4}-\frac{s}{2}}\|(au)(\tau)\|_{L^1}\,d\tau\le CD^2(t). \tag{28}$$

Bounding the second term of G is similar: whenever k is a smooth function vanishing at 0, we have

$$\begin{aligned}\int_0^t \langle t-\tau\rangle^{-\frac n4-\frac s2}\|(k(a)\nabla a)(\tau)\|_{L^1}d\tau &\leq \Big(\sup_{0\leq\tau\leq t}\langle\tau\rangle^{\frac n4+\frac12}\|a(\tau)\|_{L^2}\Big)\Big(\sup_{0\leq\tau\leq t}\langle\tau\rangle^{\frac n4+\frac12}\|\nabla a(\tau)\|_{L^2}\Big)\\ &\qquad\times\int_0^t\langle t-\tau\rangle^{-\frac n4-\frac s2}\langle\tau\rangle^{-\frac n2-1}\,d\tau\\ &\leq \langle t\rangle^{-\frac n4-\frac s2}D^2(t)\end{aligned}$$

as a consequence of (27), of the fact that $\frac n4+\frac s2\leq\frac n2+1$ for all $s\leq\frac n2+1$ and of

$$\begin{aligned}\langle\tau\rangle^{\frac n4+\frac12}\|\nabla a_H(\tau)\|_{L^2} &\leq \langle\tau\rangle^{\frac n4+\frac12}\sum_{k\geq k_0}\|\dot\Delta_k\nabla a(\tau)\|_{L^2}\\ &\leq 2^{-k_0(\frac n2-1)}\langle\tau\rangle^{\alpha}\sum_{k\geq k_0}2^{k(\frac n2-1)}\|\dot\Delta_k\nabla a(\tau)\|_{L^2}\\ &= 2^{-k_0(\frac n2-1)}\langle\tau\rangle^{\alpha}\|\nabla a(\tau)\|^H_{\dot B^{\frac n2-1}_{2,1}}.\end{aligned}$$

To handle the term with $u\cdot\nabla u$, we use the decomposition:

$$u\cdot\nabla u = u\cdot\nabla u_L + u\cdot\nabla u_H.$$

The term $u\cdot\nabla u_L$ may be treated similarly as the previous term au. Indeed,

$$\begin{aligned}\int_0^t \langle t-\tau\rangle^{-\frac n4-\frac s2}\|(u\cdot\nabla u_L)(\tau)\|_{L^1}\,d\tau &\leq \Big(\sup_{0\leq\tau\leq t}\langle\tau\rangle^{\frac n4}\|u(\tau)\|_{L^2}\Big)\Big(\sup_{0\leq\tau\leq t}\langle\tau\rangle^{\frac n4+\frac12}\|\nabla u_L(\tau)\|_{L^2}\Big)\\ &\qquad\times\int_0^t\langle t-\tau\rangle^{-\frac n4-\frac s2}\langle\tau\rangle^{-\frac n2-\frac12}\,d\tau\\ &\leq C\langle t\rangle^{-\frac n4-\frac s2}D^2(t),\end{aligned}$$

where we used the inequality (26) and the fact that, by definition of $D(t)$,

$$\sup_{0\leq\tau\leq t}\langle\tau\rangle^{\frac n4+\frac12}\|\nabla u_L(\tau)\|_{L^2}\leq CD(t).$$

The term $u\cdot\nabla u_H$ has to be treated differently since in low dimension, we do not have the control of $\|\nabla u_H(\tau)\|_{L^2}$ itself, only on $\tau^\beta\|\nabla u_H(\tau)\|_{L^2}$ for some appropriate $\beta>0$. More precisely, if $2\leq n\leq 4$ then, by interpolation,

$$\|\nabla u_H(t)\|_{L^2}\leq C\|\nabla u_H(t)\|^{\frac n4}_{\dot B^{\frac n2-2}_{2,1}}\|\nabla u_H(t)\|^{1-\frac n4}_{\dot B^{\frac n2}_{2,1}}\leq Ct^{-\frac n4(\alpha-1)-1}D(t)$$

and if $n\geq5$, just by embedding, we have

$$\|\nabla u_H(t)\|_{L^2}\leq C\|\nabla u_H(t)\|_{\dot B^{\frac n2-2}_{2,1}}\leq Ct^{-\alpha}D(t).$$

Therefore, if $t\geq2$ then we can write for $\beta:=\min(\alpha,\frac n4(\alpha-1)+1)$

$$\begin{aligned}&\int_1^t \langle t-\tau\rangle^{-\frac n4-\frac s2}\|(u\cdot\nabla u_H)(\tau)\|_{L^1}d\tau\\ &\qquad\leq C\int_1^t\langle t-\tau\rangle^{-\frac n4-\frac s2}\langle\tau\rangle^{-\frac n4-\beta}\big(\langle\tau\rangle^{\frac n4}\|u(\tau)\|_{L^2}\,\tau^\beta\|\nabla u_H(\tau)\|_{L^2}\big)\,d\tau\\ &\qquad\leq C\langle t\rangle^{-\frac n4-\frac s2}D^2(t),\end{aligned}$$

because we have $\frac{s}{2} \leq \min(\alpha, \frac{n}{4}(\alpha-1)+1)$ for all $s \leq 1+\frac{n}{2}$ and $\alpha = \frac{n}{2}+\frac{1}{2}-\varepsilon$ with small enough ε.

Obviously, thanks to (26), we may write (still for $t \geq 2$),

$$\int_0^1 \langle t-\tau\rangle^{-\frac{n}{4}-\frac{s}{2}} \|(u\cdot\nabla u_H)(\tau)\|_{L^1} d\tau \leq \int_0^1 \langle t-\tau\rangle^{-\frac{n}{4}-\frac{s}{2}} \|u(\tau)\|_{L^2} \|\nabla u_H(\tau)\|_{L^2}\, d\tau$$
$$\leq C\langle t\rangle^{-\frac{n}{4}-\frac{s}{2}} D(t) X(t),$$

and thus $u\cdot\nabla u_H$ satisfies the estimate (24) if $t \geq 2$. The case $t \leq 2$ is easy as $\langle t\rangle \cong 1$ and $\langle t-\tau\rangle \cong 1$ for $0 \leq \tau \leq t \leq 2$ and one may write

$$\int_0^t \|u\cdot\nabla u_H\|_{L^1} d\tau \leq \|u\|_{L^\infty_t(L^2)} \|\nabla u_H\|_{L^1_t(L^2)} \leq CD(t)X(t).$$

The last term of G may be written $I(a)\widetilde{\mathcal{L}}u_L + I(a)\widetilde{\mathcal{L}}u_H$ for some smooth function I vanishing at 0. Now we have

$$\int_0^t \langle t-\tau\rangle^{-\frac{n}{4}-\frac{s}{2}} \|(I(a)\widetilde{\mathcal{L}}u_L)(\tau)\|_{L^1}\, d\tau$$
$$\leq \Big(\sup_{0\leq\tau\leq t} \langle\tau\rangle^{\frac{n}{4}+\frac{1}{2}} \|a(\tau)\|_{L^2}\Big)\Big(\sup_{0\leq\tau\leq t} \langle\tau\rangle^{\frac{n}{4}+1} \|\nabla^2 u_L(\tau)\|_{L^2}\Big) \int_0^t \langle t-\tau\rangle^{-\frac{n}{4}-\frac{s}{2}} \langle\tau\rangle^{-\frac{n}{2}-\frac{3}{2}} d\tau.$$

Hence, thanks to Lemma 4.1 and (27), $I(a)\widetilde{\mathcal{L}}u_L$ fulfills (24).

Finally, to bound $I(a)\widetilde{\mathcal{L}}u_H$, we use the fact that, by interpolation if $2 \leq n \leq 6$,

$$\|\nabla^2 u_H(t)\|_{L^2} \leq C\|\nabla^2 u_H(t)\|^{\frac{n}{4}-\frac{1}{2}}_{\dot{B}^{\frac{n}{2}-3}_{2,1}} \|\nabla^2 u_H(t)\|^{\frac{3}{2}-\frac{n}{4}}_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \leq Ct^{-((\frac{n}{4}-\frac{1}{2})\alpha+\frac{3}{2}-\frac{n}{4})} D(t)$$

and just by embedding if $n \geq 7$,

$$\|\nabla^2 u_H(t)\|_{L^2} \leq C\|\nabla^2 u_H(t)\|_{\dot{B}^{\frac{n}{2}-3}_{2,1}} \leq Ct^{-\alpha} D(t).$$

Therefore, if $t \geq 2$ then we can write for $\gamma := \min(\alpha, (\frac{n}{4}-\frac{1}{2})\alpha+\frac{3}{2}-\frac{n}{4})$:

$$\int_1^t \langle t-\tau\rangle^{-\frac{n}{4}-\frac{s}{2}} \|(I(a)\widetilde{\mathcal{L}}u_H)(\tau)\|_{L^1} d\tau$$
$$\leq C\int_1^t \langle t-\tau\rangle^{-\frac{n}{4}-\frac{s}{2}} \langle\tau\rangle^{-\frac{n}{4}-\frac{1}{2}-\gamma} \big(\langle\tau\rangle^{\frac{n}{4}+\frac{1}{2}} \|a(\tau)\|_{L^2}\, \tau^\gamma \|\nabla^2 u_H(\tau)\|_{L^2}\big)\, d\tau$$
$$\leq C\langle t\rangle^{-\frac{n}{4}-\frac{s}{2}} D^2(t),$$

where we have used the fact that $\frac{s}{2} \leq \gamma+\frac{1}{2}$ for all $s \leq 1+\frac{n}{2}$, if $\varepsilon > 0$ has been chosen small enough in the definition of α.

As it is clear that thanks to (27), we have for $t \geq 2$,

$$\int_0^1 \langle t-\tau\rangle^{-\frac{n}{4}-\frac{s}{2}} \|(I(a)\widetilde{\mathcal{L}}u_H)(\tau)(\tau)\|_{L^1} d\tau \leq \int_0^1 \langle t-\tau\rangle^{-\frac{n}{4}-\frac{s}{2}} \|a(\tau)\|_{L^2} \|\nabla^2 u_H(\tau)\|_{L^2}\, d\tau$$
$$\leq C\langle t\rangle^{-\frac{n}{4}-\frac{s}{2}} D(t) X(t),$$

the term $(I(a)\widetilde{\mathcal{L}}u_H)(\tau)$ satisfies the estimate (24) if $t \geq 2$. The easy case $t \leq 2$ is left to the reader, which completes the proof of (24).

Combining with (23), we conclude that for all $t \geq 0$ and $s \in (-\frac{n}{2}, \frac{n}{2}+1]$, we have for some constant C depending continuously on s,

$$\langle t\rangle^{\frac{n}{4}+\frac{s}{2}}\|(a,u)\|^L_{\dot{B}^s_{2,1}} \leq C(D_0 + D^2(t) + X^2(t)). \tag{29}$$

Step 2: Bounds for the high frequencies of $(\nabla a, u)$. Recall that the solution given by Theorem 2.2 satisfies

$$\begin{cases} \partial_t a + u\cdot\nabla a + \operatorname{div} u = \widetilde{F}, \\ \partial_t u + u\cdot\nabla u - \mu\Delta u - (\lambda+\mu)\nabla\operatorname{div} u + \nabla a + \nabla(-\Delta)^{-1}a = \widetilde{G}, \end{cases} \tag{30}$$

where $\widetilde{F}$ and $\widetilde{G}$ are defined by

$$\widetilde{F} = -a\operatorname{div} u \quad\text{and}\quad \widetilde{G} = -u\cdot\nabla u + \left(1 - \frac{P'(\bar\rho(1+a))}{c^2(1+a)}\right)\nabla a - \left(\frac{a}{a+1}\right)\widetilde{\mathcal{L}}u.$$

We next want to establish bounds for the second term of $D(t)$. Recall that Theorem 2.2 already ensures that

$$\|(\nabla a, u)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{2}-1}_{2,1})} \leq CX(0) \quad \text{for all} \quad t \geq 0. \tag{31}$$

Therefore, it suffices to bound $\|\tau^\alpha(\nabla a, u)\|_{\widetilde{L}^\infty(2,t;\dot{B}^{\frac{n}{2}-1}_{2,1})}$ for, say, $t \geq 2$.

Denote $a_k := \dot{\Delta}_k a$, $u_k := \dot{\Delta}_k u$ and so on, and set

$$\mathcal{E}_k^2 := (\nu+\nu^{-1})\|u_k\|^2_{L^2} + \nu\|\nabla a_k\|^2_{L^2} + 2(u_k\,|\,\nabla a_k)_{L^2}.$$

By an appropriate energy method including the convection term in the spirit of [7], we may see that there exist an integer k_0 and two positive real numbers c_0 and C (all of them depending only on ν) so that for all $k \geq k_0$, we have

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}_k^2 + c_0\mathcal{E}_k^2 \leq C\Big(\|(\nabla\widetilde{F}_k, \widetilde{G}_k)\|_{L^2} + \|R_k(u,u)\|_{L^2} + \|\widetilde{R}_k(u,a)\|_{L^2} + \|\nabla u\|_{L^\infty}\mathcal{E}_k\Big)\mathcal{E}_k \tag{32}$$

with $F := -a\operatorname{div} u$, $G := -k(a)\nabla a - I(a)\mathcal{L}u$, $R_k(u,u) := \dot{\Delta}_k(u\cdot\nabla u) - u\cdot\nabla\dot{\Delta}_k u$ and $\widetilde{R}^i_k(u,a) := \partial_i\dot{\Delta}_k(u\cdot\nabla a) - u\cdot\nabla\partial_i\dot{\Delta}_k a$ for $i = 1,\cdots,n$. See [5] for the details of this proof.

Performing a time integration yields

$$e^{c_0 t}\mathcal{E}_k(t) \leq \mathcal{E}_k(0) + C\int_0^t e^{c_0\tau}(\|(\nabla\widetilde{F}_k, \widetilde{G}_k)\|_{L^2} + \|R_k(u,u)\|_{L^2} + \|\widetilde{R}_k(u,a)\|_{L^2} + \|\nabla u\|_{L^\infty}\mathcal{E}_k)d\tau.$$

Multiplying both sides by $t^\alpha e^{-c_0 t}2^{k(\frac{n}{2}-1)}$, taking the supremum on $[2,t]$, and summing up over $k \geq k_0$, we thus get

$$\|\tau^\alpha(\nabla a, u)\|^H_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{2}-1}_{2,1})} \leq C\Bigg(\|(\nabla a_0, u_0)\|^H_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{2}-1}_{2,1})} + \sum_{k\geq k_0}\sup_{0\leq\tau\leq t}\left(\tau^\alpha\int_0^\tau e^{c_0(\tau'-\tau)}2^{k(\frac{n}{2}-1)}S_k\,d\tau'\right)\Bigg) \tag{33}$$

with $S_k := \sum_{i=1}^4 S_k^i$ and

$$\begin{aligned} S_k^1 &:= \|(\nabla\widetilde{F}_k, \widetilde{G}_k)\|_{L^2}, \quad & S_k^2 &:= \|R_k(u,u)\|_{L^2}, \\ S_k^3 &:= \|\widetilde{R}_k(u,a)\|_{L^2}, \quad & S_k^4 &:= \|\nabla u\|_{L^\infty}\|(\dot\Delta_k \nabla a, \dot\Delta_k u)\|_{L^2}. \end{aligned}$$

Here, note that if k_0 is large enough then

$$\mathcal{E}_k \cong \|(\dot\Delta_k \nabla a, \dot\Delta_k u)\|_{L^2} \quad \text{for all } \ k \geq k_0. \tag{34}$$

To bound the supremum on $[2,t]$, we split the integral on $[0,\tau]$ into integrals on $[0,1]$ and $[1,\tau]$, respectively. We first handle the $[0,1]$ part of the integral : for $0 \leq \tau' \leq 1$ (hence $\tau' \leq \frac{\tau}{2}$), we have

$$\begin{aligned} \sum_{k\geq k_0} \sup_{2\leq\tau\leq t} \Big(\tau^\alpha \int_0^1 e^{c_0(\tau'-\tau)} 2^{k(\frac{n}{2}-1)} S_k(\tau')\, d\tau'\Big) &\leq C \sum_{k\geq k_0} \sup_{2\leq\tau\leq t} \tau^\alpha e^{-\frac{c_0}{2}\tau} \int_0^1 2^{k(\frac{n}{2}-1)} S_k(\tau')\, d\tau' \\ &\leq C \sum_{k\geq k_0} \int_0^1 2^{k(\frac{n}{2}-1)} S_k(\tau')\, d\tau'. \end{aligned}$$

Hence, bounding $\nabla\widetilde{F}$ and $\widetilde{G}$ as in the proof of Theorem 2.2 leads to

$$\sum_{k\geq k_0} \sup_{2\leq\tau\leq t} \Big(\tau^\alpha \int_0^1 e^{c_0(\tau'-\tau)} 2^{k(\frac{n}{2}-1)} S_k(\tau')\, d\tau'\Big) \leq C X^2(1). \tag{35}$$

To handle the $[1,\tau]$ part of the integral for $2 \leq \tau \leq t$, we shall use repeatedly the following inequality

$$\|\tau\nabla u\|_{\widetilde{L_t^\infty}(\dot B^{\frac{n}{2}}_{2,1})} \leq C D(t), \tag{36}$$

which is obvious for the high frequencies of u.

To estimate $S_k^1 = \|(\nabla\widetilde{F}_k, \widetilde{G}_k)\|_{L^2}$, we notice that

$$\begin{aligned} &\sum_{k\geq k_0} \sup_{2\leq\tau\leq t} \Big(\tau^\alpha \int_1^\tau e^{c_0(\tau'-\tau)} 2^{k(\frac{n}{2}-1)} S_k^1(\tau')\, d\tau'\Big) \\ &\quad \leq \sum_{k\geq k_0} \Big(\sup_{2\leq\tau\leq t} 2^{k(\frac{n}{2}-1)} \Big(\sup_{1\leq\tau'\leq\tau} (\tau')^\alpha S_k^1(\tau')\Big)\, \tau^\alpha \int_1^\tau (\tau')^{-\alpha} e^{c_0(\tau'-\tau)}\, d\tau'\Big) \end{aligned}$$

and a variant of the proof of Lemma 4.1 guarantees that

$$\tau^\alpha \int_1^\tau (\tau')^{-\alpha} e^{c_0(\tau'-\tau)}\, d\tau' \leq C. \tag{37}$$

Hence

$$\sum_{k\geq k_0} \sup_{2\leq\tau\leq t} \Big(\tau^\alpha \int_1^\tau e^{c_0(\tau'-\tau)} 2^{k(\frac{n}{2}-1)} S_k^1(\tau')\, d\tau'\Big) \leq C \|\tau^\alpha(\nabla\widetilde{F}, \widetilde{G})\|_{\widetilde{L_t^\infty}(\dot B^{\frac{n}{2}-1}_{2,1})}. \tag{38}$$

Now, product laws in tilde spaces give

$$\|\tau^\alpha \nabla\widetilde{F}\|^H_{\widetilde{L_t^\infty}(\dot B^{\frac{n}{2}-1}_{2,1})} \leq C \|\tau^{\alpha-1} a\|_{\widetilde{L_t^\infty}(\dot B^{\frac{n}{2}}_{2,1})} \|\tau \operatorname{div} u\|_{\widetilde{L_t^\infty}(\dot B^{\frac{n}{2}}_{2,1})}.$$

The high frequencies of the first term of the r.h.s. are obviously bounded by $D(t)$. As for the low frequencies, we notice that

$$\|\tau^{\alpha-1}a\|^{L}_{\widetilde{L_t^\infty}(\dot B^{\frac n2}_{2,1})} \leq C\|\tau^{\alpha-1}a\|^{L}_{L_t^\infty(\dot B^{\frac n2-2\varepsilon}_{2,1})} \leq CD(t) \tag{39}$$

provided $\alpha \leq \frac n2 + \frac 32 - \varepsilon$. Therefore, using (36), we get

$$\|\tau^\alpha \nabla \widetilde F\|^{H}_{\widetilde{L_t^\infty}(\dot B^{\frac n2-1}_{2,1})} \leq CD^2(t).$$

Next, we have

$$\|\tau^\alpha(k(a)\nabla a_H)\|_{\widetilde{L_t^\infty}(\dot B^{\frac n2-1}_{2,1})} \leq C\|a\|_{\widetilde{L_t^\infty}(\dot B^{\frac n2}_{2,1})}\|\tau^\alpha a_H\|_{\widetilde{L_t^\infty}(\dot B^{\frac n2}_{2,1})} \leq CX(t)D(t)$$

and according to (39),

$$\|\tau^\alpha(k(a)\nabla a_L)\|_{\widetilde{L_t^\infty}(\dot B^{\frac n2-1}_{2,1})} \leq C\|\tau a\|_{\widetilde{L_t^\infty}(\dot B^{\frac n2}_{2,1})}\|\tau^{\alpha-1} a_L\|_{\widetilde{L_t^\infty}(\dot B^{\frac n2}_{2,1})} \leq D^2(t).$$

We also see that

$$\|\tau^\alpha I(a)\widetilde{\mathcal L}u\|_{\widetilde{L_t^\infty}(\dot B^{\frac n2-1}_{2,1})} \leq C\|\tau\nabla^2 u\|_{\widetilde{L_t^\infty}(\dot B^{\frac n2-1}_{2,1})}\Big(\|\tau^{\alpha-1}a_L\|_{\widetilde{L_t^\infty}(\dot B^{\frac n2}_{2,1})} + \|\tau^{\alpha-1}a_H\|_{\widetilde{L_t^\infty}(\dot B^{\frac n2}_{2,1})}\Big).$$

The first term of the r.h.s. may be bounded by virtue of (36), and it is also clear that the last term is bounded by $D(t)$. As for the second one, we use again (39). Resuming to (38), we end up with

$$\sum_{k\geq k_0}\sup_{2\leq\tau\leq t}\Big(\tau^\alpha\int_1^\tau e^{c_0(\tau'-\tau)}2^{k(\frac n2-1)}S_k^1(\tau')\,d\tau'\Big) \leq CD^2(t).$$

The terms S_k^2, S_k^3 and S_k^4 may be treated along the same lines. For details, see [5].

Putting all estimates together, we conclude that

$$\sum_{k\geq k_0}\sup_{2\leq\tau\leq t}\Big(\tau^\alpha\int_1^\tau e^{c_0(\tau'-\tau)}2^{k(\frac n2-1)}S_k(\tau')\,d\tau'\Big) \leq C(D(t)X(t)+D^2(t)).$$

Then plugging this latter inequality, (31) and (35) into (33) yields

$$\|\langle\tau\rangle^\alpha(\nabla a,u)\|^{H}_{\widetilde{L_t^\infty}(\dot B^{\frac n2-1}_{2,1})} \leq C\Big(\|(\nabla a_0,u_0)\|^{H}_{\widetilde{L_t^\infty}(\dot B^{\frac n2-1}_{2,1})} + X^2(t)+D^2(t)\Big). \tag{40}$$

Step 3: Decay estimates and gain of regularity for the high frequencies of ∇u.

To complete the proof of Theorem 2.3, there only remains to bound $\|\tau\nabla u\|^{H}_{\widetilde{L_t^\infty}(\dot B^{\frac n2}_{2,1})}$. To this end, we shall use that the velocity u satisfies

$$\partial_t u - \widetilde{\mathcal L}u = f := -(1+k(a))\nabla a - u\cdot\nabla u - I(a)\widetilde{\mathcal L}u - \nabla(-\Delta)^{-1}a, \tag{41}$$

whence

$$\partial_t(t\widetilde{\mathcal L}u) - \widetilde{\mathcal L}(t\mathcal Lu) = \widetilde{\mathcal L}u + t\widetilde{\mathcal L}F.$$

Because the maximal regularity estimates for the Lamé semi-group are the same as for the heat semi-group, we deduce that

$$\|t\widetilde{\mathcal{L}}u\|^{H}_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2-1})}\leq C\big(\|\widetilde{\mathcal{L}}u\|^{H}_{L_t^1(\dot B_{2,1}^{\frac n2-1})}+\|t\widetilde{\mathcal{L}}f\|^{H}_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2-3})}\big)$$

whence, using the bounds given in Theorem 2.2,

$$\|t\nabla u\|^{H}_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2})}\leq C\big(X(0)+\|\tau f\|^{H}_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2-1})}\big).\tag{42}$$

In order to bound the first term of the r.h.s. of (41), we note that, as $\alpha\geq1$, we have

$$\|\tau\nabla a\|^{H}_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2-1})}\leq C\|\langle\tau\rangle^\alpha\nabla a\|^{H}_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2-1})}.$$

Product and composition estimates give

$$\begin{aligned}
\|\tau(k(a)\nabla a)\|^{H}_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2-1})}&\leq C\|\tau^{\frac12}a\|^2_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2})},\\
\|\tau(u\cdot\nabla u)\|^{H}_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2-1})}&\leq C\|u\|_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2-1})}\|\tau\nabla u\|_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2})},\\
\|\tau(I(a)\mathcal{L}u)\|^{H}_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2-1})}&\leq C\|a\|_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2})}\|\tau\nabla^2u\|_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2-1})}
\end{aligned}$$

and lastly

$$\|\tau(\nabla(-\Delta)^{-1}a)\|^{H}_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2-1})}\leq C\|\langle\tau\rangle^\alpha\nabla a\|^{H}_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2-1})}.$$

Therefore, reverting to (42), we get

$$\|t\nabla u\|^{H}_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2})}\leq C\Big(X(0)+D(t)X(t)+D^2(t)+\|\langle\tau\rangle^\alpha\nabla a\|^{H}_{\widetilde{L_t^\infty}(\dot B_{2,1}^{\frac n2-1})}\Big).$$

Finally, bounding the last term according to (40), and adding up the final inequality to (29) and (40) yields

$$D(t)\leq C\big(D_0+\|(\nabla a_0,u_0)\|^{H}_{\dot B_{2,1}^{\frac n2-1}}+X^2(t)+D^2(t)\big).$$

As Theorem 2.2 ensures that $X(t)$ is small, one can now conclude that the decay estimate is fulfilled for all times if D_0 and $\|(\nabla a_0,u_0)\|^{H}_{\dot B_{2,1}^{\frac n2-1}}$ are sufficiently small.

5 Appendix

For the reader convenience, we here recall some technical results without proof that were needed in the previous sections.

5.1 Regularity estimates for the linear heat equation

Consider

$$\begin{cases}\partial_tu-\mu\Delta u=f, & t>0,\quad x\in\mathbb{R}^n,\\ u|_{t=0}=u_0, & x\in\mathbb{R}^n,\end{cases}\tag{43}$$

where $\mu>0$ and u_0,f are given.

It is known that optimal regularity estimates in Besov spaces for (43) have to be stated in terms of the following norms:

$$\|u\|_{\widetilde{L_t^r}(\dot{B}^{\sigma}_{p,1})} := \sum_{j\in\mathbb{Z}} 2^{j\sigma}\|\dot{\Delta}_j u\|_{L^r(0,t;L^p)}. \tag{44}$$

Proposition 5.1 ([1, 8]). *Let $u:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}$ satisfy the heat equation* (43). *Then for all $1\le p\le\infty$, $s\in\mathbb{R}$, $1\le q_1\le q_2\le\infty$, we have :*

$$\|u\|_{\widetilde{L_t^{q_1}}(\dot{B}_{p,1}^{s+\frac{2}{q_1}})} \le C\big(\|u_0\|_{\dot{B}^s_{p,1}} + \|f\|_{\widetilde{L_t^{q_2}}(\dot{B}_{p,1}^{s-2+\frac{2}{q_2}})}\big) \quad \text{for all } \ t\ge 0. \tag{45}$$

5.2 Green matrix of the linearized NSP

Here we compute the green matrix of the following (reduced) linearized NSP system:

$$\begin{cases} \partial_t a + \Lambda v = 0, \\ \partial_t v - \nu\Delta v - \Lambda a - \Lambda^{-1} a = 0. \end{cases} \tag{46}$$

Proposition 5.2 ([5]). *In Fourier variables, the green matrix $\mathcal{G}$ of System* (46) *is given by*

$$\widehat{\mathcal{G}}(t,\xi) := \begin{pmatrix} \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} & -\Big(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\Big)|\xi| \\ \Big(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\Big)(|\xi| + |\xi|^{-1}) & \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} \end{pmatrix} \tag{47}$$

where

$$\lambda_\pm(\xi) := -\frac{1}{2}\nu|\xi|^2\Big(1\pm\sqrt{1-\frac{4(|\xi|^2+1)}{\nu^2|\xi|^4}}\Big).$$

5.3 Estimates for product, composition and commutators

For any couple (u,v) of tempered distributions, we have the following (formal) decomposition of uv:

$$\begin{aligned} uv &= \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}u\ \dot{\Delta}_j v + \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}v\ \dot{\Delta}_j u + \sum_{j\in\mathbb{Z}}\sum_{|k-j|\le 1} \dot{\Delta}_j u\,\dot{\Delta}_k v \\ &=: T_u v + T_v u + R(u,v). \end{aligned} \tag{48}$$

Clearly, the first two terms are defined for any couple (u,v) in $\mathcal{S}'$ as the series is locally finite in the Fourier space. As for the last so-called remainder term, it is also defined if, roughly speaking, the sum of the regularity indices of u and of v is positive. This is detailed in the following lemma, the proof of which may be found in e.g. [1, 8].

Lemma 5.3. *Let $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$ and $t<0$. We have*

$$\|T_u v\|_{\dot{B}^s_{p,r}} \le C\|u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}} \quad \text{and} \quad \|T_u v\|_{\dot{B}^{s+t}_{p,r}} \le C\|u\|_{\dot{B}^t_{\infty,\infty}}\|v\|_{\dot{B}^s_{p,r}}. \tag{49}$$

Let $(s_j,p_j,r_j)\in\mathbb{R}\times[1,\infty]^2$ for $j=1,2$. We have

- *if $s_1+s_2>0$, $\frac{1}{p}:=\frac{1}{p_1}+\frac{1}{p_2}\le 1$ and $\frac{1}{r}:=\frac{1}{r_1}+\frac{1}{r_2}\le 1$ then*

$$\|R(u,v)\|_{\dot{B}^{s_1+s_2}_{p,r}} \le C\|u\|_{\dot{B}^{s_1}_{p_1,r_1}}\|v\|_{\dot{B}^{s_2}_{p_2,r_2}}; \tag{50}$$

- *if* $s_1+s_2=0$, $\frac{1}{p}:=\frac{1}{p_1}+\frac{1}{p_2}\le 1$ *and* $\frac{1}{r_1}+\frac{1}{r_2}\ge 1$ *then*

$$\|R(u,v)\|_{\dot B^0_{p,\infty}}\le C\|u\|_{\dot B^{s_1}_{p_1,r_1}}\|v\|_{\dot B^{s_2}_{p_2,r_2}}. \tag{51}$$

As a corollary of the above lemma, we have the following general product estimate.

Lemma 5.4. *Let* $\delta\ge 0$ *and* $-\min\left(\frac{n}{p},\frac{n}{p'}\right)<\sigma\le\frac{n}{p}-\delta$. *Then we have*

$$\|uv\|_{\dot B^\sigma_{p,1}}\le C\|u\|_{\dot B^{\frac{n}{p}-\delta}_{p,1}}\|v\|_{\dot B^{\sigma+\delta}_{p,1}}.$$

Lemma 5.5 ([5]). *Let I be an open interval of $\mathbb{R}$ containing* 0, *and* $F:I\to\mathbb{R}$, *a smooth function vanishing at 0. Then for any* $s>0$, $1\le p\le\infty$ *and interval J compactly supported in I there exists a constant C such that*

$$\|F(a)\|_{\dot B^s_{p,1}}\le C\|a\|_{\dot B^s_{p,1}}\quad\text{for any}\quad a\in\dot B^s_{p,1}\quad\text{valued in}\quad J.$$

In the case $s>-\min(n/p,n/p')$ *if in addition to the above hypotheses we have* $a\in\dot B^{\frac{n}{p}}_{p,1}$ *then* $F(a)\in\dot B^{\frac{n}{p}}_{p,1}\cap\dot B^s_{p,1}$ *and*

$$\|F(a)\|_{\dot B^s_{p,1}}\le\|a\|_{\dot B^s_{p,1}}\left(|F'(0)|+C\|a\|_{\dot B^{\frac{n}{p}}_{p,1}}\right).$$

The following commutator estimates are classical (see e.g. [1] and the references therein).

Lemma 5.6. *Let* $1\le p\le\infty$ *and* $-\left(\frac{n}{p},\frac{n}{p'}\right)<s\le 1+\frac{n}{p}$. *Then we have*

$$\sum_{j\in\mathbb{Z}}2^{js}\|[u\cdot\nabla,\dot\Delta_j]a\|_{L^p}\le C\|\nabla u\|_{\dot B^{\frac{n}{p}}_{p,1}}\|a\|_{\dot B^s_{p,1}},$$

$$\sum_{j\in\mathbb{Z}}2^{j(s-1)}\|[u\cdot\nabla,\partial_i\dot\Delta_j]a\|_{L^p}\le C\|\nabla u\|_{\dot B^{\frac{n}{p}}_{p,1}}\|\nabla a\|_{\dot B^{s-1}_{p,1}},\qquad i=1,\cdots,n.$$

References

[1] Bahouri, H., Chemin, J.-Y. and Danchin, R., Fourier analysis and nonlinear partial differential equations, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* (Springer), **343** (2011).

[2] Chandrasekhar, S., An introduction to the study of stellar structure, *Dover Publications Inc., New York, N. Y.*, 1957.

[3] Charve, F. and Danchin, R., A global existence result for the compressible Navier-Stokes equations in the critical L^p framework, *Arch. Ration. Mech. Anal.* **198** (2010), 233–271.

[4] Chen, Q. and Miao, C., Zhang, Z., Well-posedness in critical spaces for the compressible Navier-Stokes equations with density dependent viscosities, *Rev. Mat. Iberoam.*, **26** (2010), 915–946.

[5] Chikami, N. and Danchin, R., On the global existence and time decay estimates in critical spaces for the Navier-Stokes-Poisson system, preprint.

[6] Chikami, N. and Ogawa, T., On the well-posedness of the compressible Navier-Stokes-Poisson system in Besov spaces, *to appear in Journal of Evolution Equations.*

[7] Danchin, R., Global existence in critical spaces for compressible Navier-Stokes equations, *Invent. Math.*, **141** (2000), 579–614.

[8] Danchin, R., Fourier analysis methods for the compressible Navier-Stokes equations, *arXiv:1507.02637.*

[9] Haspot, B., Well-posedness in critical spaces for the system of compressible Navier-Stokes in larger spaces, *J. Differential Equations.* **251** (2011), 2262–2295.

[10] Hao, C. and Li, H.-L., Global existence for compressible Navier-Stokes-Poisson equations in three and higher dimensions, *J. Diff. Eq.* **246** (2009), 4791–4812.

[11] Hoff, D., Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of heat-conducting fluids, *Arch. Ration. Mech. Anal.* **139** (1997), 303–354.

[12] Okita, M., Optimal decay rate for strong solutions in critical spaces to the compressible Navier-Stokes equations, *J. Differential Equations* **257** (2014), 3850–3867.

[13] Li, H.-L., Matsumura, A., and Zhang, G., Optimal decay rate of the compressible Navier-Stokes-Poisson system in $\mathbb{R}^3$, *Arch. Ration. Mech. Anal.* **196** (2010), 681–713.

[14] Markowich, P. A., Ringhofer, C. A., and Schmeiser, C., Semiconductor equations, *Springer-Verlag, Vienna,* 1990.

[15] Matsumura, A. and Nishida, T., The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids, *Proc. Japan Acad. Ser. A Math. Sci.* **55** (1979), 337–342.

[16] Zheng, X., Global well-posedness for the compressible Navier-Stokes-Poisson system in the L^p framework, *Nonlinear Anal.* **75** (2012), 4156–4175.