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A fluid-particle system related to Vlasov-Navier-Stokes equations

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Abstract

Preliminary results on the convergence to the Vlasov-Navier-Stokes equations of a system of particles interacting with a fluid are announced. Main emphasis is given to the difficulties that arise and hints for solutions are given.

1 Introduction

The aim of this work is to investigate a fluid-particle system which seems to converge, in the limit of infinitely many particles, to a Vlasov-Navier-Stokes (VNS) system. We restrict this preliminary investigation mainly to dimension $d=2$ and we shall assume to be on a torus $\mathbb{T}^{2}$ with periodic boundary conditions. The facts described in this note have only the character of a preliminary investigation and announcement of partial results.

Let $\epsilon\in(0,1)$ and $N\in \mathbb{N}$ be given, where N is the number of particles; consider the system:

$$
\begin{aligned}
&\frac{\partial u^{N}}{\partial t}=\Delta u^{N}-u^{N}\cdot\nabla u^{N}-\nabla\pi^{N}-\frac{c_{0}}{N}\sum_{i=1}^{N}\left(u_{\epsilon}^{N}\left(t,X_{t}^{i}\right)-V_{t}^{i}\right)\delta_{X_{t}^{i}}^{\epsilon}\\
&\frac{d}{dt}X_{t}^{i}=V_{t}^{i}\\
&\frac{1}{N}dV_{t}^{i}=\frac{c_{0}}{N}\left(u_{\epsilon}^{N}\left(t,X_{t}^{i}\right)-V_{t}^{i}\right)dt+\frac{1}{N^{2}}\sum_{j=1}^{N}K\left(X_{t}^{i}-X_{t}^{j}\right)dt+\frac{\sigma_{p}}{N}dW_{t}^{i}.
\end{aligned}
$$

The first equation is the usual Navier-Stokes system for the velocity and pressure $\left(u^{N},\pi^{N}\right)$ of a fluid, "forced" by the presence of N particles; a precise description of the interaction between particles and fluid is a difficult topic (just as an instance, see [2], [5], [8], [9], [16]), outside the scope of this preliminary note, hence we adopt a partially phenomenological description, where particles act as delta Dirac forces, with intensity proportional to the velocity difference between fluid and particle. For technical reasons, but also as a trace of the fact that particles occupy a volume, we use a smoothed delta Dirac $\delta_{X_{t}^{i}}^{\epsilon}$ to describe the force; and analogously the velocity difference is computed between the particle velocity V_t^i and a local average at particle center X_t^i of the fluid velocity, $u_\epsilon^N\left(t, X_t^i\right)$.

The smoothings used in the first equation above are given by classical mollifiers of the form $\theta_\epsilon^0(x) = \epsilon^{-d}\theta^0\left(\epsilon^{-1}x\right)$, where θ^0 is a smooth probability density with compact support which includes a neighbor of the origin, and are defined as

$$\delta_{X_t^i}^\epsilon(x) = \left(\theta_\epsilon^0 * \delta_{X_t^i}\right)(x) = \theta_\epsilon^0\left(x - X_t^i\right), \qquad u_\epsilon^N = \theta_\epsilon^0 * u^N.$$

The last two equations of the system above describe the Newtonian dynamics of particles and we assume the velocity V_t^i satisfies a stochastic differential equation driven by the Brownian motion W_t^i in $\mathbb{R}^d$; the Brownian motions W_t^i, $i = 1, ..., N$ are independent and defined on a probability space $(\Omega, \mathcal{F}, P)$.

We assume the particles have mass $\frac{1}{N}$; the force acting on particle i has three components: the Stokes drug force due to the fluid, an interaction force given by the interaction kernel K and a noise perturbation.

Remark 1 *Recall that Stokes drag force is given by $6\pi r\mu\mathbf{v}$ where r is particle radius, $\mathbf{v}$ is the relative velocity of particle and μ is viscosity. Hence the interpretation of the scalings in N chosen above is: the particle mass is of order $\frac{1}{N}$; the particle radius is of order $\frac{1}{N}$ and $c_0 \sim 6\pi\mu$. Particles with a mass density similar to the fluid should have mass of the order $\frac{1}{N^d}$, while here we assume it of order $\frac{1}{N}$, much bigger. This corresponds to a regime of sparse heavy particles.*

Example 2 *The interaction kernel is usually absent in classical formulations of VNS system. We include it here since it may be interesting in some applications. For instance, think to metastatic cancer cells flowing in the blood stream, an example where the condition of sparse heavy particles may be realistic. These cells do not only interact with the fluid but also between themselves and possibly with other special cells.*

Example 3 *Having in mind applications to biological fluids, an interesting variations could be to introduce a death-rate of the form $\lambda_t^i = g\left(u_\epsilon^N\left(t, X_t^i\right) - V_t^i\right)$, motivated by the fact that stress may induce cell death. Such additional term lead to the term $-g\left(u(t,x) - v\right)F(t,x,v)$ in the limit PDE.*

As we said above, our aim is proving convergence to a Vlasov-Navier-Stokes system.We would like to prove that, as $N \to \infty$ and $\epsilon \to 0$ with appropriate link between them, the pair

$$u^N(t,x), \qquad S_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i, V_t^i}$$

($S_t^N(dx, dv)$ is a time-dependent random probability measure, called empirical measure of the particle system) converges to the solution (u, F) of

$$\frac{\partial u}{\partial t} = \Delta u - u \cdot \nabla u - \nabla \pi - \int \left(u(t,x) - v\right) F(x,v)\, dv \tag{1}$$

$$\frac{\partial F}{\partial t}+v\cdot\nabla_x F+\mathrm{div}_v\left((u-v)\,F+(K*F)\,F\right)=\frac{\sigma_p^2}{2}\Delta_v F \tag{2}$$

Remark 4 *The term*

$$-\int\left(u\left(t,x\right)-v\right)F\left(x,v\right)dv=\int vF\left(x,v\right)dv-u\left(t,x\right)\int F\left(x,v\right)dv$$

is the so called Brinkman's force, usually denoted in the physical literature by "$j-u\rho$".

We shall see that proving this limit result is a difficult problem. Let us preliminarily describe a technical difficulty with the Navier-Stokes part.

Concerning the literature, there are results on particle systems related to VNS equations but only under special conditions, caused by the fact that a true fluid-particle interaction is imposed, see [1], [2], [6], [15]; and there are results on convergence of PDEs to PDEs, although motivated by particle arguments, see [7], [10], [11].

1.1 Difficulty with the Navier-Stokes forcing

Let us restrict here to $d=2$. The equation

$$\frac{\partial u^N}{\partial t}=\Delta u^N-u^N\cdot\nabla u^N-\nabla\pi^N-\frac{c_0}{N}\sum_{i=1}^N\left(u_\epsilon^N\left(t,X_t^i\right)-V_t^i\right)\delta_{X_t^i}^\epsilon$$

contains a subtle difficulty. If we put $\epsilon=0$, we force Navier-Stokes equations with an input which is worse than H^{-1} (recall that in two dimensions the delta Dirac is only in $H^{-1-\gamma}$ for every $\gamma>0$) and we pretend to speak of $u^N\left(t,X_t^i\right)$ (for $\epsilon=0$) which requires u^N to be continuous.

For $\epsilon>0$ we do not see this regularity issue; but we need uniform estimates in (ϵ,N) to pass to the limit, and thus, sooner or later, we meet the difficulty just described.

Remark 5 *The need for continuous-in-space velocity field in this area has been recognized also dealing with other questions, see [14] who assumes $u\in L^2\left(0,T;C\left(D\right)\right)$.*

Let us explain this difficulty also with the following argument. To simplify, assume we have the heat equation in place of the Navier-Stokes one and we have only one *fixed* point particle at position X_0:

$$\frac{\partial u}{\partial t}=\frac{1}{2}\Delta u+\left(u_\epsilon\left(t,X_0\right)-V_0\right)\delta_{X_0}^\epsilon$$

The solution with $u_0^N=0$ is

$$u\left(t,x\right)=\int_0^t\left(\int p_{t-s}\left(x-y\right)\delta_{X_0}^\epsilon\left(y\right)dy\right)\left(u_\epsilon\left(s,X_0\right)-V_0\right)ds$$

where $p_t(x)$ is the heat kernel. If we take the limit as $\epsilon \to 0$ we have

$$u(t,x) = \int_0^t \frac{1}{(2\pi(t-s))^{d/2}} e^{-\frac{|x-X_0|^2}{2(t-s)}} (u(s,X_0) - V_0)\, ds.$$

Already in $d=2$, for $x = X_0$, we see that $|u(t,X_0)| = +\infty$!

1.2 Why the problem should be solvable

Notice however that the conjectured limit equation, the Valsov-Navier-Stokes system is better: no delta Dirac appear there.

What is meaningless, as remarked in the previous section, is the model with a finite number N_0 of point particles of mass $\frac{1}{N_0}$, if we take the limit $\epsilon \to 0$. But this is not what we want to do: we want to take the limit of infinitely many particles, with infinitesimal interaction strength. We may hope that, in the limit as $\epsilon \to 0$, we may control the quantities because we also take $N \to \infty$ and the intensity of fluid-particle interaction is rescaled by $\frac{1}{N}$.

However, to realize this program, it is essential to prove that particles do not concentrate too much, otherwise we take the risk to have again, in the limit, concentrated masses of particles with finite interaction strength. Therefore, a main purpose of the estimates below is proving a form of non concentration.

2 Energy balance

Lemma 6 *Setting*

$$\mathcal{E}_t = \frac{1}{2}\int \left|u^N(t,x)\right|^2 dx + \frac{1}{2N}\sum_{i=1}^{N}\left|V_t^i\right|^2$$

if u^N is a regular solution then we have

$$d\mathcal{E}_t + \left(\int \left|\nabla u^N(t,x)\right|^2 dx + \frac{1}{N}\sum_{i=1}^{N}\left(u_\epsilon^N(t,X_t^i) - V_t^i\right)^2\right) dt$$
$$= \left(\frac{1}{N^2}\sum_{i,j=1}^{N} V_t^i K\left(X_t^i - X_t^j\right) + \frac{\sigma_p^2}{2}\right) dt + \frac{\sigma_p}{N}\sum_{i=1}^{N} V_t^i dW_t^i.$$

The proof is elementary by Itô formula. Notice that the previous result also gives us a control on

$$\frac{1}{N}\sum_{i=1}^{N} u_\epsilon^N\left(t,X_t^i\right)^2$$

because it is bounded (up to constants) by $\frac{1}{N}\sum_{i=1}^{N}\left|V_t^i\right|^2$ plus $\frac{1}{N}\sum_{i=1}^{N}\left(u_\epsilon^N\left(t,X_t^i\right)-V_t^i\right)^2$ that are both controlled (the second one integrated in time).

Using the previous a priori estimates one can prove, under the assumptions

$$u_0 \in \mathbb{L}_\sigma^2\left(\mathbb{T}^2\right)$$

$$E\left[\frac{1}{N}\sum_{i=1}^{N}\left(\left|X_0^i\right|^2+\left|V_0^i\right|^2\right)\right] \leq C$$

($\mathbb{L}_\sigma^2\left(\mathbb{T}^2\right)$ is the usual space of divergence free periodic zero mean vector fields on $\mathbb{T}^2$) existence and uniqueness of solutions (for finite N) such that

$$E\left[\sup_{t\in[0,T]}\int\left|u^N(t,x)\right|^2 dx\right] \leq C$$

$$E\left[\int_0^T\int\left|\nabla u^N(t,x)\right|^2 dxdt\right] \leq C$$

$$E\left[\sup_{t\in[0,T]}\frac{1}{N}\sum_{i=1}^{N}\left(\left|X_t^i\right|^2+\left|V_t^i\right|^2\right)\right] \leq C$$

$$E\left[\int_0^T\frac{1}{N}\sum_{i=1}^{N}\left(u_\epsilon^N\left(t,X_t^i\right)-V_t^i\right)^2 dt\right] \leq C$$

$$E\left[\int_0^T\frac{1}{N}\sum_{i=1}^{N}u_\epsilon^N\left(t,X_t^i\right)^2 dt\right] \leq C.$$

From these bounds, with relatively classical compactness theorems, one can show that the family of laws of $\left(u^N,S^N\right)$ are tight and thus there exist subsequences which converge in law; changing probability space it is possible to assume a.s. convergence in appropriate topologies. In the sequel, to understand the difficulties, we assume for simplicity such a.s. convergence. We do not want to give the details here, which will be included in a forthcoming technical work. Let us only mention that subsequences $\left(u^{N_k},S^{N_k}\right)$, on the new probability space, will have the property that

- u^{N_k} converges strongly in $L^2\left(0,T;L^2\left(\mathbb{T}^2\right)\right)$
- u^{N_k} converges weakly in $L^2\left(0,T;W^{1,2}\left(\mathbb{T}^2\right)\right)$ and weak star in $L^\infty\left(0,T;L^2\left(\mathbb{T}^2\right)\right)$
- S^{N_k} converges in the weak topology of measures uniformly in time.

In the sequel, when we informally discuss questions of convergence, we replace the subsequence $\left(u^{N_k},S^{N_k}\right)$ with the full sequence $\left(u^N,S^N\right)$ for notational simplicity.

3 A difficulty about passage to the limit in the Navier-Stokes system

Let us stress that the existence of a convergent subsequence $\left(u^{N_k}, S^{N_k}\right)$ (denoted below by $\left(u^N, S^N\right)$), in the topologies indicated at the end of the previous section, is true both if we keep $\epsilon > 0$ unchanged with N, or if we link it to N by choosing $\epsilon = \epsilon_N \to 0$. However, in the first case we can pass to the limit, in the second one we meet a relevant technical difficulty, that we now explain.

In weak form on divergence free smooth test vector fields ϕ, the Navier-Stokes system reads

$$\begin{aligned}&\left\langle u^N(t), \phi\right\rangle_x - \langle u_0, \phi\rangle_x + \int_0^t \left\langle \nabla u^N, \nabla\phi\right\rangle_x ds \\ &= \int_0^t \left\langle u^N \cdot \nabla\phi, u^N\right\rangle_x ds - \left\langle \frac{1}{N}\sum_{i=1}^N \left(u_\epsilon^N\left(t, X_t^i\right) - V_t^i\right)\delta_{X_t^i}^\epsilon(\cdot), \phi(\cdot)\right\rangle_x\end{aligned}$$

where $\langle f, g\rangle_x = \int_{\mathbb{T}^d} f(x)\cdot g(x)\, dx$ for suitable vector fields f, g, and $\cdot$ denotes scalar product in $\mathbb{R}^d$. The difficulty is only in the convergence of the last term, when u^N converges only in the usual topologies of weak solutions mentioned at the end of last section. What about the convergence of

$$u_\epsilon^N\left(t, X_t^i\right) = \left(\theta_\epsilon^0 * u_t^N\right)\left(X_t^i\right) \quad ?$$

It seems necessary to prove some convergence of u^N in the *uniform* topology. But uniform estimates are not among the a priori bounds.

Although not being the only one, a natural way to prove bounds in the uniform topology for u_t^N is by Sobolev embedding, hence investigating bounds on derivatives of u_t^N. Since we are on a torus and we restrict to $d = 2$, we use vorticity. The question then is: can we prove enstrophy type bounds? Consider thenthe vorticity equation, which in the case $d = 2$, for the vorticity function $\omega^N = \nabla^\perp \cdot u^N$, is

$$\frac{\partial\omega^N}{\partial t} = \Delta\omega^N - u^N \cdot \nabla\omega^N - \frac{c_0}{N}\sum_{i=1}^N \nabla^\perp \cdot \left(\left(u_\epsilon^N\left(t, X_t^i\right) - V_t^i\right)\delta_{X_t^i}^{\epsilon_N}\right). \tag{3}$$

A main conceptual remark is that *particles create vorticity.* Terms like $\partial_i \delta_{X_t^i}^{\epsilon_N}$ contribute diverging terms in N for $\epsilon = \epsilon_N \to 0$ and thus vorticity does not seem to be under control.

4 Summary of results

After this long introduction, let us state the two directions discussed below.

- First we develop a "two steps approach", which consists in two separate limit theorems.

  1. The first one is only the limit as $N \to \infty$, given a constant value of $\epsilon \in (0,1)$; it identifies a limit PDE$_\epsilon$;

  2. The second one is the limit $PDE_\epsilon \to PDE$ as $\epsilon \to 0$.

  Linking artificially (ϵ, N), there are sequences (ϵ_k, N_k) where $Particles_{(\epsilon_k,N_k)} \to PDE$ as $k \to \infty$. This is not the solution we were looking for, but it is important to know that at least this relatively simple two-step approach works.

- Second, we conjecture a local in time result of the form $Particles_{(\epsilon_N,N)} \to PDE$ as $N \to \infty$. It is based on local-in-time uniform-in-x estimates on u^N, jointly with estimates on no concentration of particles (these two facts proceed together). A full proof still requires to solve technical problems, so we limit ourselves to express a reasonable conjecture.

5 Two-steps approach

5.1 Preliminaries

The advantage of the "two-steps" or "separate limit" strategy is that it works with minimal ingredients: we do not need to prove no-concentration of particles; we do not need the noise to regularize; we can say something also in the case $d = 3$ (always on a torus $\mathbb{T}^3$, to simplify) . We assume, for simplicity of notations, $K = 0, \sigma_p = 0, c_0 = 1$. However the result remains true when $\sigma_p \neq 0$ and when K is bounded Lipschitz continuous and presumably also in some cases when $K = K_N$ is rescaled in a proper way.

For sake of clarity (also because here there is no average over the randomness), we restate the well posedness mentioned above for finite N and the energy bounds.

Lemma 7 *For every $\epsilon \in (0,1)$ and $N \in \mathbb{N}$, the system*

$$\frac{\partial u^N}{\partial t} = \Delta u^N - u^N \cdot \nabla u^N - \nabla \pi^N - \frac{1}{N} \sum_{i=1}^{N} \left(u_\epsilon^N \left(t, X_t^i\right) - V_t^i \right) \delta_{X_t^i}^\epsilon$$

$$\frac{d}{dt} X_t^i = V_t^i, \qquad \frac{d}{dt} V_t^i = u_\epsilon^N \left(t, X_t^i\right) - V_t^i$$

has a unique solution such that

$$\sup_{t \in [0,T]} \int \left| u^N(t,x) \right|^2 dx + \int_0^T \int \left| \nabla u^N(t,x) \right|^2 dx dt \leq C.$$

$$\sup_{t \in [0,T]} \frac{1}{N} \sum_{i=1}^{N} \left(\left| X_t^i \right|^2 + \left| V_t^i \right|^2 \right) \leq C.$$

5.2 First limit: $N \to \infty$, $\epsilon \in (0,1)$ given

We now consider the following *mollified* VNS system

$$\frac{\partial u}{\partial t} = \Delta u - u \cdot \nabla u - \nabla \pi - \theta_\epsilon^0 * \int (u_\epsilon (t, \cdot) - v) F(\cdot, dv) \tag{4}$$

$$\frac{\partial F}{\partial t} + v \cdot \nabla_x F + \operatorname{div}_v ((u_\epsilon - v) F) = 0 \tag{5}$$

where

$$u_\epsilon = \theta_\epsilon^0 * u.$$

Thanks to the mollification, we are allowed to investigate this system and convergence of the particle system when the density of particles is treated just as a measure, not as a density function. Let us give the appropriate definition. Denote by $\mathrm{Pr}_1 \left(\mathbb{T}^d \times \mathbb{R}^d\right)$ the set of all Borel probability measures μ on $\mathbb{T}^d \times \mathbb{R}^d$ such that

$$\int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |v| \, \mu(dx, dv) < \infty$$

we endow $\mathrm{Pr}_1 \left(\mathbb{T}^d \times \mathbb{R}^d\right)$ with the weak topology, with convergence of first moment. The notation $\theta_\epsilon^0 * \int (u_\epsilon (t, \cdot) - v) F (dv)$, when $F \in C\left([0,T]; \mathrm{Pr}_1 \left(\mathbb{T}^d \times \mathbb{R}^d\right)\right)$ and $u_\epsilon (t, \cdot)$ is measurable and bounded, stands for

$$\left(\theta_\epsilon^0 * \int (u_\epsilon (t, \cdot) - v) F(\cdot, dv)\right)(x) = \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \theta_\epsilon^0 (x - x') \left(u_\epsilon (t, x') - v'\right) F\left(t, dx', dv'\right).$$

Beside the notation $\langle f, g \rangle_x$ already introduced above, here we also write $\langle \mu, f \rangle_{x,v}$ for

$$\langle \mu, f \rangle_{x,v} = \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f(x, v) \, \mu(dx, dv).$$

Definition 8 *Let $u_0 \in \mathbb{L}_\sigma^2 \left(\mathbb{T}^2\right)$ and $F_0 \in \mathrm{Pr}_1 \left(\mathbb{T}^d \times \mathbb{R}^d\right)$ be given. A pair (u, F) is a solution of system (4)-(5) with initial condition (u_0, F_0) if*

$$u \in L^\infty \left(0, T; L^2 \left(\mathbb{T}^d\right)\right) \cap L^2 \left(0, T; W^{1,2} \left(\mathbb{T}^d\right)\right)$$
$$F \in C\left([0,T]; Pr_1 \left(\mathbb{T}^d \times \mathbb{R}^d\right)\right)$$

$$\langle u(t), \phi \rangle_x - \langle u_0, \phi \rangle_x + \int_0^t \langle \nabla u, \nabla \phi \rangle_x \, ds$$
$$= \int_0^t \langle u \cdot \nabla \phi, u \rangle_x \, ds - \int_0^t \left\langle \theta_\epsilon^0 * \int (u_\epsilon (s, \cdot) - v) F(s, \cdot, dv), \phi \right\rangle_x ds$$

$$\langle F(t), \varphi \rangle_{x,v} = \langle F_0, \varphi \rangle_{x,v} + \int_0^t \langle F(s), v \cdot \nabla_x \varphi \rangle_{x,v} \, ds + \int_0^t \langle F(s), (u_\epsilon (s) - v) \cdot \nabla_v \varphi \rangle_{x,v} \, ds$$

for all divergence free smooth fields $\phi : \mathbb{T}^d \to \mathbb{R}^d$ and all smooth functions $\varphi : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$.

Theorem 9 *Let $\epsilon \in (0,1)$ be given. Assume $u_0 \in \mathbb{L}^2_\sigma(\mathbb{T}^2)$, $\frac{1}{N}\sum_{i=1}^N \left(|X_0^i|^2 + |V_0^i|^2\right) \leq C$, and $S_0^N \to F_0$ in the weak sense of probability measures.*

1) Let $d=3$. Given $\epsilon \in (0,1)$, there exists a subsequence $N_k \to \infty$ such that the pair $\left(u^{N_k}, S^{N_k}\right)$ converges (in the sense described at the end of section 2) to a solution (u,F) of system (4)-(5) with initial condition (u_0, F_0).

2) Let $d=2$. System (4)-(5), with initial condition (u_0,F_0), has a unique solution (u,F) and, as $N \to \infty$, the pair $\left(u^N, S^N\right)$ converges to (u,F).

Remark 10 *In $d=3$ obviously we do not know uniqueness due to the Navier-Stokes part; hence the convergence holds only for certain subsequences (for every subsequence there is a sub-subsequence which converges).*

Remark 11 *Existence of a solution (u,F) of the limit system (with given $\epsilon \in (0,1)$) either can be proved directly or it follows from the convergence result itself, being based on a compactness argument. Uniqueness of (u,F) (for $d=2$) has to be proved directly.*

Let us give a few elements of the proof. From the estimates

$$\sup_{t\in[0,T]} \int \left|u^N(t,x)\right|^2 dx + \int_0^T \int \left|\nabla u^N(t,x)\right|^2 dxdt \leq C$$

it is classical (cf. [17]) to apply the compactness Aubin-Lions lemma. Due to the estimate

$$\sup_{t\in[0,T]} \frac{1}{N}\sum_{i=1}^N \left(|X_t^i|^2 + |V_t^i|^2\right) \leq C$$

one can use a criterion based on Wasserstein distance to prove compactness of $S^{N_k^{(0)}}$ in $C\left([0,T];\mathrm{Pr}_1\left(\mathbb{T}^d \times \mathbb{R}^d\right)\right)$. From these fact one has the existence of a subsequence $\left(u^{N_k}, S^{N_k}\right)$ which converges as described at the end of section 2. Call (u,F) the limit of such subsequence.

Taking the limit in the first four terms of the weak formulation of Navier-Stokes equations (see Definition 8) is classical (cf. [17]). Concerning the last term, we have to prove that

$$\lim_{k\to\infty} \left\langle \frac{1}{N_k}\sum_{i=1}^{N_k} \left(u_\epsilon^{N_k}\left(t, X_t^{i,N_k}\right) - V_t^{i,N_k}\right)\delta^\epsilon_{X_t^{i,N_k}}, \phi \right\rangle_x$$
$$= \left\langle \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} \theta_\epsilon^0\left(\cdot - x'\right)\left(u_\epsilon\left(t,x'\right) - v'\right) F\left(t, dx', dv'\right), \phi \right\rangle_x .$$

The term on the left-hand side is equal to

$$\begin{aligned}
&\int_{\mathbb{T}^d} \frac{1}{N_k} \sum_{i=1}^{N_k} \left(u_\epsilon^{N_k} \left(t, X_t^{i,N_k} \right) - V_t^{i,N_k} \right) \theta_\epsilon^0 \left(x - X_t^{i,N_k} \right) \phi(x)\, dx \\
&= \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(u_\epsilon^{N_k} \left(t, x' \right) - v' \right) \theta_\epsilon^0 \left(x - x' \right) \phi(x)\, S^{N_k} \left(dx', dv' \right) dx \\
&= \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(\left(\theta_\epsilon^0 * u^{N_k}(t) \right) (x') - v' \right) \left(\theta_\epsilon^{0-} * \phi \right) (x')\, S^{N_k} \left(dx', dv' \right)
\end{aligned}$$

where $\theta_\epsilon^{0-}(x) = \theta_\epsilon^0(-x)$. The term $\theta_\epsilon^0 * u^{N_k}(t)$ converges uniformly to $\theta_\epsilon^0 * u(t)$ for a.e. t (passing to a subsequence). With little additional care we can take the limit as $k \to \infty$ and get

$$\begin{aligned}
&\int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(\left(\theta_\epsilon^0 * u(t) \right) (x') - v' \right) \left(\theta_\epsilon^{0-} * \phi \right) (x')\, F \left(t, dx', dv' \right) \\
&= \left\langle \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \theta_\epsilon^0 \left(\cdot - x' \right) \left(u_\epsilon \left(t, x' \right) - v' \right) F \left(t, dx', dv' \right), \phi \right\rangle_x .
\end{aligned}$$

To prove that F satisfies the weak identity in Definition 8 we have first to derive an identity for S^{N_k}. By chain rule applied to $\varphi \left(X_t^{i,N_k}, V_t^{i,N_k} \right)$ we get

$$\left\langle S_t^{N_k}, \varphi \right\rangle_{x,v} = \left\langle S_0^{N_k}, \varphi \right\rangle_{x,v} + \int_0^t \left\langle S_s^{N_k}, v \cdot \nabla_x \varphi \right\rangle_{x,v} ds + \int_0^t \left\langle S_s^{N_k}, \left(u_\epsilon(s) - v \right) \cdot \nabla_v \varphi \right\rangle_{x,v} ds$$

(in the deterministic case it is a well known fact that the empirical measure is already a solution of the limit PDE; in the stochastic case, $\sigma_p \neq 0$, one has to apply Itô formula and an additional martingale term appears, which however, converges to zero). Then one can pass to the limit.

Finally, for $d = 2$ we have to prove uniqueness for the limit system (4)-(5). In principle, one of the difficulties is that we deal with solutions F which are only measures. However, we may use a well known method (see for instance [3]) based on Wasserstein distance $d_1(\mu, \nu)$ between $\mu, \nu \in \mathrm{Pr}_1 \left(\mathbb{T}^d \times \mathbb{R}^d \right)$. Assume that $(u, F), (u', F')$ are two solutions. One has

$$d_1 \left(F(t), F'(t) \right) \leq E \left[\left| X_t - X_t' \right| + \left| V_t - V_t' \right| \right]$$

where (X_t, V_t) satisfies

$$\begin{aligned}
\frac{d}{dt} X_t &= V_t \\
\frac{d}{dt} V_t &= u_\epsilon(t, X_t) - V_t
\end{aligned}$$

where u is the first component of the solution (u, F), and similarly for (X_t', V_t') (with respect to (u', F')). The initial conditions for these two problems are the same, (X_0, V_0), with law F_0. One can easily prove that

$$E\left[\left|X_t - X_t'\right| + \left|V_t - V_t'\right|\right] \leq C \int_0^t E\left[\left|X_s - X_s'\right| + \left|V_s - V_s'\right|\right] ds$$
$$+ C \int_0^t \left|u_\epsilon(s, X_s) - u_\epsilon'(s, X_s)\right| ds.$$

Then one has to repeat classical energy type computations of the 2-dimensional theory of Navier-Stokes equations (cf. [17]) to control $u - u'$ in the norms $L^\infty\left(0, T; L^2\left(\mathbb{T}^d\right)\right) \cap L^2\left(0, T; W^{1,2}\left(\mathbb{T}^d\right)\right)$, bounds to be used jointly with the previous one. The only non classical term is

$$\left|\left\langle \theta_\epsilon^0 * \int \left(u_\epsilon(t, \cdot) - v\right) F(\cdot, dv), u(t)\right\rangle_x - \left\langle \theta_\epsilon^0 * \int \left(u_\epsilon'(t, \cdot) - v\right) F'(\cdot, dv), u'(t)\right\rangle_x\right|$$

which is controlled by the previous norms of $u - u'$ plus the term

$$\left|\left\langle \theta_\epsilon^0 * \int \left(u_\epsilon(t, \cdot) - v\right)\left(F(\cdot, dv) - F'(\cdot, dv)\right), u(t)\right\rangle_x\right|$$
$$= \left|\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} \left(\theta_\epsilon^{0-} * u(t)\right)(x)\left(u_\epsilon(t, x) - v\right)\left(F(dx, dv) - F'(dx, dv)\right)\right|$$
$$\leq C \cdot d_1\left(F(t), F'(t)\right)$$

(the last bound requires some work, omitted here).These recursive estimates allow one to apply Gronwall lemma and prove that $(u, F) = (u', F')$.

5.3 Second limit: $\epsilon \to 0$

Until now we have proven that the fluid, coupled with the particles, converges, as $N \to \infty$ to system (4)-(5), where the mollification with $\epsilon \in (0, 1)$ survives. Called (u^ϵ, F^ϵ) the solution in $d = 2$ (or a solution in $d = 3$) of (4)-(5), it remains to investigate the limit as $\epsilon \to 0$.

The limit cannot be taken at the level of measure solutions F, or at least this looks very difficult. One can give a meaning to the weak formulation of the VNS (1)-(2) when F is only measure-valued, but at the price of imposing a priori that u is continuous bounded. This direction could be investigated but requires a fully original approach to VNS system which is beyond the scope of this note. And in addition, as remarked below, one should expect only local-in-time solutions.

Therefore let us consider the modified VNS system and the true one when F is a function. First, one has to prove that the modified VNS system has a solution (u^ϵ, F^ϵ) in appropriate function spaces; by the uniqueness result for measures proved above, it should be unique also in the weaker class of measure valued solutions. Then one should prove convergence of $(u^{\epsilon}, F^{\epsilon})$ to a solution (u, F) of system (1)-(2). We have checked that both these steps are plausible following the approach of [18]; see also [4]; however there are several details and results will appear in a forthcoming work.

When this is done, it is possible to extract suitable sequences (ϵ_k, N_k) such that $\left(u^{N_k}, S^{N_k}\right)$ converges to (u, F) as $k \to \infty$. Here, by $\left(u^{N_k}, S^{N_k}\right)$, we mean those obtained by the fluid-particle system with $\epsilon = \epsilon_k$.

6 Joint limit

6.1 Introduction

As described in Sections 1.1 and 3, uniform estimates on the velocity are required to pass to the limit simultaneously in $N \to \infty$ and $\epsilon \to 0$. A natural approach to prove uniform bounds on u^N is to get $W^{\varepsilon,2}$ (for $\varepsilon > 0$) bounds on the vorticity $\omega^N = \nabla^{\perp} \cdot u^N$, which satisfies equation (3). Bounds on the enstrophy are not sufficient, since they are bounds on the $W^{1,2}$-norm of u^N which do not imply uniform bounds on u^N. Hence we work with semigroups and look for bounds in more regular topologies. Notice however that enstrophy bounds meet the same difficulties we have with semigroups.

Denoting by $e^{t\Delta}$ the semigroup associated to the Laplacian operator in L^p or C^{α} spaces on the torus, we have

$$\frac{\partial \omega^N}{\partial t} = \Delta \omega^N - u^N \cdot \nabla \omega^N - \frac{c_0}{N} \sum_{i=1}^{N} \nabla^{\perp} \cdot \left(\left(u^N_{\epsilon_N}\left(t, X^i_t\right) - V^i_t \right) \delta^{\epsilon_N}_{X^i_t} \right).$$

$$\omega^N(t) = e^{t\Delta}\omega^N(0) - \int_0^t e^{(t-s)\Delta} u^N(s) \cdot \nabla \omega^N(s)\, ds$$
$$- \int_0^t e^{(t-s)\Delta} \nabla^{\perp} \cdot \frac{c_0}{N} \sum_{i=1}^{N} \left(u^N_{\epsilon_N}\left(s, X^i_s\right) - V^i_s \right) \delta^{\epsilon}_{X^i_s} ds.$$

We want to estimate $\omega^N(t)$ in $W^{2\alpha,2}\left(\mathbb{T}^d\right)$, hence we use the inequality

$$\left\| (I-\Delta)^{\alpha} \omega^N(t) \right\|_{L^2(\mathbb{T}^d)} \leq \left\| (I-\Delta)^{\alpha} \omega^N(0) \right\|_{L^2(\mathbb{T}^d)}$$
$$+ \int_0^t \left\| (I-\Delta)^{\alpha} e^{(t-s)\Delta} u^N(s) \cdot \nabla \omega^N(s) \right\|_{L^2(\mathbb{T}^d)} ds$$
$$+ \int_0^t \left\| (I-\Delta)^{\frac{1}{2}+\alpha} e^{(t-s)\Delta} \nabla^{\perp} (I-\Delta)^{-\frac{1}{2}} \cdot \frac{c_0}{N} \sum_{i=1}^{N} \left(u^N_{\epsilon_N}\left(s, X^i_s\right) - V^i_s \right) \delta^{\epsilon}_{X^i_s} \right\|_{L^2(\mathbb{T}^d)} ds. \quad (6)$$

Let us only concentrate on the term (6) without V_s^i, which is the source of the main difficulty. For every $T > 0$, denote by $\|\cdot\|_{T,\infty}$ the supremum norm over $[0,T] \times \mathbb{T}^d$. We have

$$\left| \frac{1}{N} \sum_{i=1}^{N} u_{\epsilon_N}^N \left(s, X_s^i\right) \delta_{X_s^i}^{\epsilon} (x) \right| \leq \left\| u_{\epsilon_N}^N \right\|_{T,\infty} F_s^{0,N} (x)$$

where $F_t^{0,N} = \theta_{\epsilon_N}^0 * \left(\frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i} \right)$ and therefore the term (6) without V_s^i is bounded above by ($\nabla^\perp (I - \Delta)^{-\frac{1}{2}}$ is a bounded operator in $L^2\left(\mathbb{T}^d\right)$)

$$\left\| \nabla^\perp (I-\Delta)^{-\frac{1}{2}} \right\|_{L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)} \left\| u_{\epsilon_N}^N \right\|_{T,\infty} \int_0^t \left\| (I-\Delta)^{\frac{1}{2}+\alpha} e^{(t-s)\Delta} \right\|_{L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)} \left\| F^{0,N}(s) \right\|_{L^2(\mathbb{T}^d)} ds$$

$$\leq C \left\| u_{\epsilon_N}^N \right\|_{T,\infty} T^{\frac{1}{2}-\alpha} \sup_{t \in [0,T]} \left\| F^{0,N} \right\|_{L^2(\mathbb{T}^d)} \tag{7}$$

because

$$\left\| (I-\Delta)^{\frac{1}{2}+\alpha} e^{(t-s)\Delta} \right\|_{L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)} \leq \frac{C}{(t-s)^{\frac{1}{2}+\alpha}}$$

by well known analytic semigroup estimates.

We need an estimate on $\left\| F^{0,N} \right\|_{L^2(\mathbb{T})^d}$: this is the property of no concentration of particles, as announced in Section 1.2.

6.2 No concentration of particles

Set $F_t^N = \theta_{\epsilon_N} * \left(\frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i, V_t^i} \right)$, where now $\theta_\epsilon = \theta_\epsilon (x, v)$ are suitable mollifiers in both variables, related to θ_ϵ^0. Here we need $\sigma_p \neq 0$. One has

$$dF_t^N = \left(\frac{\sigma_p^2}{2} \Delta_v F_t^N - \nabla_x \cdot \theta_{\epsilon_N} * \left(v S_t^N\right) \right) dt$$
$$- \left(\nabla_v \cdot \theta_{\epsilon_N} * \left(\left(u_{\epsilon_N}^N (t,x) - v \right) S_t^N \right) \right) dt + dM_t^N$$

where

$$M_t^N (x,v) = \frac{1}{N} \sum_{i=1}^{N} \int_0^t \nabla_v \theta_{\epsilon_N} \left(x - X_s^i, v - V_s^i \right) \sigma_p dB_s^i.$$

Hence

$$
\begin{aligned}
&\frac{1}{2} d \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(F_t^N\right)^2 dxdv + \frac{\sigma_p^2}{2} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left|\nabla_v F_t^N\right|^2 dxdvdt \\
&= \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(\theta_{\epsilon_N} * \left(v S_t^N\right)\right) \nabla_x F_t^N dxdvdt \\
&+ \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(\theta_{\epsilon_N} * \left(\left(u_{\epsilon_N}^N (t,x) - v\right) S_t^N\right)\right) \nabla_v F_t^N dxdvdt
\end{aligned}
$$

plus terms related to the martingale part that we do not discuss explicitly here. Let us see how to treat the most difficult term: since

$$
\begin{aligned}
&\left|\left(\theta_{\epsilon_N} * \left(u_{\epsilon_N}^N (t,x) S_t^N\right)\right)(x,v)\right| \\
&= \left|\int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \theta_{\epsilon_N} \left(x - x', v - v'\right) u_{\epsilon_N}^N \left(t, x'\right) S_t^N \left(dx', dv'\right)\right| \\
&\leq \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \theta_{\epsilon_N} \left(x - x', v - v'\right) \left|u_{\epsilon_N}^N \left(t, x'\right)\right| S_t^N \left(dx', dv'\right) \\
&\leq \left\|u_{\epsilon_N}^N\right\|_{T,\infty} \left(\theta_{\epsilon_N} * S_t^N\right)(x,v)
\end{aligned}
$$

we have

$$
\begin{aligned}
&\left|\int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(\theta_{\epsilon_N} * \left(u_{\epsilon_N}^N (t,x) S_t^N\right)\right) \nabla_v F_t^N dxdv\right| \\
&\leq \left\|u_{\epsilon_N}^N\right\|_{T,\infty} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} F_t^N \left|\nabla_v F_t^N\right| dxdv \\
&\leq \varepsilon \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left|\nabla_v F_t^N\right|^2 dxdv + \frac{\left\|u_{\epsilon_N}^N\right\|_{T,\infty}^2}{\varepsilon} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(F_t^N\right)^2 dxdv.
\end{aligned}
$$

Summarizing,

$$
\begin{aligned}
&\frac{1}{2} d \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(F_t^N\right)^2 dxdv + \frac{\sigma_p^2}{4} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left|\nabla_v F_t^N\right|^2 dxdvdt \\
&= \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(\theta_{\epsilon_N} * \left(v S_t^N\right)\right) \nabla_x F_t^N dxdvdt \\
&\qquad + \frac{\left\|u_{\epsilon_N}^N\right\|_{T,\infty}^2}{\varepsilon} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(F_t^N\right)^2 dxdv
\end{aligned}
$$

plus terms related to the martingale.

Heuristically, it seems that for small T, using (7), the previous estimates "close" and give a bound on

$$
\left\|u^N\right\|_{T,\infty} \text{ and } \sup_{t \in [0,T]} \left\|F_t^{0,N}\right\|_{L^2(\mathbb{T}^d)}.
$$

However, there are still several nontrivial technical problems to be overcome. In the previous section we needed an estimate on $\left\|F_t^{0,N}\right\|_{L^2(\mathbb{T}^d)}$. Here, in this section, we have shown a control on F_t^N, not $F_t^{0,N}$. One can prove the estimate

$$\int F_t^{0,N}(x)^2\,dx \leq C\int\int |v|^3 F_t^N(x,v)\,dxdv + C\int\int F_t^N(x,v)^4\,dxdv$$

(this is a variant of Lemma 1 of [4], which avoids $\|F_t^N\|_\infty$, since it looks too difficult to estimate $\|F_t^N\|_\infty$). But then the two quantities on the right-hand-side of this inequality have to be controlled. We presume that all these steps can be done but due to the complexity of these estimates, instead of formulating a result, we prefer to limit ourselves to state a conjecture.

Conjecture 12 *Assume $d = 2$, $u_0 \in W^{2\alpha,2}(\mathbb{T}^d)$, $\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\left(F_0^N\right)^2 dxdv \leq C$. Let $\left(u^N, X_t^i, V_t^i\right)$ be the solution of the fluid-particle interacting system, with $\epsilon = \epsilon_N \to 0$ as $N \to \infty$. Set $F_t^N = \theta_{\epsilon_N} * \left(\frac{1}{N}\sum_{i=1}^N \delta_{X_t^i,V_t^i}\right)$. Then, for small T, $\left(u^N, F^N\right)$ converges to the unique solution (u, F) of VNS system.*

The convergence should holds in several topologies, including the strong topology of $L^2\left(0,T;L^2\left(\mathbb{T}^d\right)\right)$ for u^N, and of $L^2\left(0,T;L^2\left(\mathbb{T}^d\times\mathbb{R}^2\right)\right)$ for F^N.

6.3 Open questions

A first main limitation of the results described here is the phenomenological description of the fluid-body interaction. We have already remarked in the Introduction about the difficulties met by more realistic models.

The two-step approach is complete and extendible to stochastic dynamics. On the contrary, the more interesting joint limit approach, even if true, contains two restrictions: the short time and the presence of noise in the particles - viscosity in the PDE. The short time is due, conceptually, to the vorticity produced by the immersed particles, which increases both with fluid velocity and particle density, therefore introducing a quadratic term in the equations. Blow-up due to quadratic terms is prevented by suitable conservation laws and we have an energy inequality but we miss a conservation law for vorticity, due to the vorticity production by particles.

Following [18], section 4.1, it could be that uniform-in-x estimates on u^N can be replaced by estimates in L^4, which are global; correspondingly, an L^4-control on F^N is needed. Here and for other purposes, we see the importance of a major problem: F^N does not satisfy a continuity equation, but an identity with weaker geometric properties.

As a remark, as soon as we restrict to local in time results, it seems possible to extend the result of the joint limit to the 3D case, by working in spaces of sufficiently regular solutions u; see a related problem in [12]. This stresses once more the fact that presumably we have not taken in full advantage the properties of 2D fluids.

The viscosity has been used in our approach to obtain an L^2-estimate on F_t^N, conceptually fundamental as a mean to prove no concentration of particles. However, for the limit equation for F, L^p-estimates are easily obtained in terms of the L^p-norm of initial conditions, without need of any Laplacian (see for instance [13]). This could be a signature of the fact that noise is not needed to prove L^2-estimate on F_t^N.

Finally, the a priori estimate on

$$\int_0^T \frac{1}{N}\sum_{i=1}^N \left(u_\epsilon^N\left(t,X_t^i\right)-V_t^i\right)^2 dt = \int_0^T \int \left(u_\epsilon^N\left(t,x\right)-v\right)^2 S_t^N\left(dx,dv\right) dt$$

obtained by the energy bound looks promising to control the difficult quadratic terms, but in all computations they seem to be coupled with other terms not under control.

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