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On derivation of incompressible fluid systems with heat equation[1]

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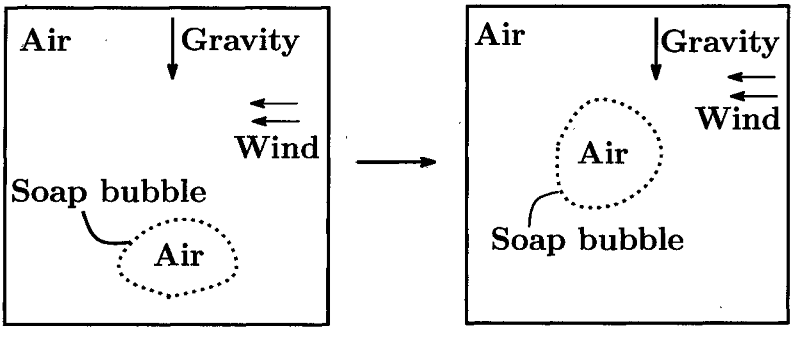
1 Introduction

We are interested in a mathematical model of a soap bubble in air:



Surface flow and surface tension play an important role in a soap bubble in air. The flow is often called *surface flow.* One can consider surface flow as fluid flow on an evolving or moving surface. An evolving surface means that the surface is moving or the shape of the surface is changing along with the time. Soap bubble is soft matter, so fluid-flow in the bubble is affected by the temperature of the fluid. Therefore we have to consider the fluid with heat effect on an evolving surface. This paper focuses on fluid flow on a bubble. Especially, we consider the incompressible fluid on an evolving surface. In this paper we report two incompressible fluid systems with heat equation on an evolving surface, which derived by an energetic variational approach. Moreover, we study fundamental properties of some operators on surfaces. The argument in the Appendix (I) in Koba, etc [8] is not right, so Section 7 gives its explanation and correction.

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The outline of the paper is as follows: In Section 2, we introduce two fluid systems on a manifold derived by Arnold [2] and Taylor [13]. In Section 3, we introduce two incompressible fluid systems on an evolving surface, which derived by our energetic variational approaches. In Section 4, we state historical results on surface flow and some papers related to this paper. In Section 5, we study fundamental properties of some operators on surfaces. In Section 6, we introduce several Laplace operators on surfaces. In Section 7, we study the viscous term of our systems.

2 Known Results(Incompressible fluid systems on a manifold)

Let us now introduce Arnold's system and Taylor's system on a manifold. Let $\mathcal{M}$ be a 2-dimensional closed Riemannian manifold, and let $\mu > 0$ be the viscosity coefficients of the fluid on $\mathcal{M}$. Let u be the velocity of the fluid on $\mathcal{M}$, and let p be a pressure associated with u. Assume that u is a 1-form on $\mathcal{M}$ and that p is a function on $\mathcal{M}$.

Arnold [2] applied the Lie group of diffeomorphisms to derive the following inviscid incompressible fluid system on a manifold $\mathcal{M}$:

$$(\mathrm{E})_{\mathcal{M}} \begin{cases} \partial_t u + \nabla_u u + \mathrm{grad}_{\mathcal{M}} p = 0, \\ \mathrm{div}_{\mathcal{M}} u = 0. \end{cases}$$

Here $\nabla_u u$ is covariant derivative along with the velocity u and $\mathrm{grad}_{\mathcal{M}}$ is gradient operator on $\mathcal{M}$.

Taylor [13] introduced the following viscous incompressible fluid system, derived from their physical sense, on a manifold $\mathcal{M}$:

$$(\mathrm{NS})_{\mathcal{M}} \begin{cases} \partial_t u + \nabla_u u + \mathrm{grad}_{\mathcal{M}} p = \mu(\Delta_{\mathcal{M}} u + K u), \\ \mathrm{div}_{\mathcal{M}} u = 0. \end{cases}$$

Mitsumatsu and Yano [9] also derived the system $(\mathrm{NS})_{\mathcal{M}}$ by using their energetic variational approach. Arnaudon and Cruzeiro [1] applied stochastic variational approach to derive the system $(\mathrm{NS})_{\mathcal{M}}$. Here $\Delta_{\mathcal{M}}$ is the Borhner-Laplacian, K is the Gaussian curvature (the Ricci curvature), and $\mathrm{div}_{\mathcal{M}}$ is divergence operator on $\mathcal{M}$.

3 Main Results(Incompressible fluid systems on surface)

Let us state our main results. Let $\Gamma(t)$ be a surface in $\mathbb{R}^3$ depending on time $t \in [0, T)$ for some $T \in (0, \infty]$. Let $w = w(x,t) = {}^t(w_1, w_2, w_3)$ be the motion velocity of $\Gamma(t)$. Let $u = u(x,t) = {}^t(u_1, u_2, u_3)$ be a relative velocity on $\Gamma(t)$. The velocity $v = v(x,t) =$

${}^t(v_1, v_2, v_3) := u + w$ is called a *total velocity* of the fluid on $\Gamma(t)$. The symbol $\sigma = \sigma(x,t)$ denotes a *total pressure* or a *pressure associated with* v. Let $\rho = \rho(x,t)$, $\theta(x,t)$, $\mu = \mu(x,t)$, and $\kappa = \kappa(x,t)$ be the density, temperature, viscosity coefficient, and thermal coefficient of the fluid on $\Gamma(t)$, respectively. We assume that $\Gamma(t)$ is a 2-dimensional closed manifold for each fixed $t \in [0,T)$ and that

$$\int_{\Gamma(t)} H_\Gamma(n \cdot w) d\mathcal{H}^2_x = 0,$$

where $H_\Gamma = H_\Gamma(x,t)$ is the mean curvature and $n = n(x,t) = {}^t(n_1, n_2, n_3)$ is the unit outer normal vector at $x \in \Gamma(t)$.

Applying energetic variational approaches similar to those in Koba-Liu-Giga [8] and Koba [7], we can derive the two incompressible fluid system with heat equation:

$$\begin{cases} D_t\rho = 0 & \text{on } \mathcal{S}_T, \\ \mathrm{div}_\Gamma v = 0 & \text{on } \mathcal{S}_T, \\ \rho D_t v + \mathrm{grad}_\Gamma \sigma + \sigma H_\Gamma n = \mathrm{div}_\Gamma\{2\mu D_\Gamma(v)\} + \rho F & \text{on } \mathcal{S}_T, \\ \rho D_t \theta = -\mathrm{div}_\Gamma q_\theta & \text{on } \mathcal{S}_T, \end{cases} \tag{3.1}$$

$$\begin{cases} D_t^\Gamma\rho = 0 & \text{on } \mathcal{S}_T, \\ \mathrm{div}_\Gamma v = 0 & \text{on } \mathcal{S}_T, \\ P_\Gamma \rho D_t^\Gamma v + \mathrm{grad}_\Gamma \sigma = P_\Gamma \mathrm{div}_\Gamma\{2\mu D_\Gamma(v)\} + P_\Gamma \rho F & \text{on } \mathcal{S}_T, \\ \rho D_t^\Gamma \theta = -\mathrm{div}_\Gamma q_\theta & \text{on } \mathcal{S}_T, \end{cases} \tag{3.2}$$

where $\mathcal{S}_T = \bigcup_{0<t<T}\{\Gamma(t) \times \{t\}\}$. Here D_t is material derivative, div_Γ is surface divergence, grad_Γ is surface gradient, $D_\Gamma(v)$ is a projected strain rate, $F = {}^t(F_1, F_2, F_3)$ is the exterior force or gravity, q_θ is the heat flux defined by $q_\theta = -\kappa \mathrm{grad}_\Gamma \theta$, D_t^Γ is surface material derivative, and P_Γ is an orthogonal projection to a tangent space. See Sections 5-7 for some operators on surfaces.

Note that the tangential incompressible fluid system (3.2) is different from Taylor's system $(NS)_\mathcal{M}$ when $v \cdot n \neq 0$. See Section 7 for the case when $v \cdot n = 0$.

Now we investigate some energies of the system (3.1). Multiplying (3.1) by v, and then integration by parts on the surface, we check that for $0 < s < t < T$,

$$\int_{\Gamma(t)} \frac{1}{2}\rho|v|^2 \, d\mathcal{H}^2_x + \int_s^t \int_{\Gamma(\tau)} 2\mu|D_\Gamma(v)|^2 \, d\mathcal{H}^2_x d\tau$$
$$= \int_{\Gamma(s)} \frac{1}{2}\rho|v|^2 \, d\mathcal{H}^2_x + \int_s^t \int_{\Gamma(\tau)} \rho F \cdot v \, d\mathcal{H}^2_x d\tau.$$

We call $\rho|v|^2/2$ the *kinetic energy*, $\mu|D_\Gamma(v)|^2$ the *energy dissipation due to the viscosity*, and $\rho F \cdot v$ the *work done by F*. In [7] and [8], they used such energy densities to derive their fluid systems.

4 Note(Surface Flow and Fluid-Flow on Evolving Surfaces)

Let us introduce some historical results on surface flow. Boussinesq [4] first considered the existence of fluid-flow on a surface. Scriven [10] introduced their surface stress tensor, which is called the *the Boussinesq-Scriven law.* Slattery [11] studied some properties of the stress tensor determined by the Boussinesq-Scriven law. After that many researchers have studied surface flow, interface flow, and two-phase flow with surface viscosity and surface tension. See Slattery-Sagis-Oh [12] for details.

Next we state some papers on fluid systems on an evolving surface. Dzuik and Elliott [5] made use of the transport theorem on an evolving surface and their surface flux to make several fluid systems on the evolving surface. Bothe-Prüss [3] applied the Boussinesq-Scriven law to make a two-phase flow system with surface viscosity and surface tension. Koba-Giga-Liu [8] used their energetic variational approach to derive incompressible fluid systems on an evolving surface. Koba [7] applied their energetic variational approach and the first and second laws of thermodynamics to derive their compressible fluid systems on an evolving surface. Note that, in [7] and [8], they did not use the Boussinesq-Scriven law directly in order to derive their fluid systems on an evolving surface.

The paper [7] gave a mathematical justification of the Boussinesq-Scriven law from an energetic point of view. However, there exist other possibilities for surface stress tensor, for example, third viscosity problem.

5 Fundamental Properties of Several Operators on Surfaces

We study fundamental properties of several operators on surfaces. We first introduce some notation. Let $\Gamma(t)(=\{\Gamma(t)\}_{0<t<T})$ be a smooth evolving 2-dimensional surface in $\mathbb{R}^3$. By $n = n(x,t) = {}^t(n_1, n_2, n_3)$ we mean the unit outer normal vector at $x \in \Gamma(t)$. Let P_Γ be an orthogonal projection to a tangent space defined by

$$P_\Gamma = P_\Gamma(x,t) = I_{3\times 3} - n \otimes n = \begin{pmatrix} 1 - n_1 n_1 & -n_1 n_2 & -n_1 n_3 \\ -n_2 n_1 & 1 - n_2 n_2 & -n_2 n_3 \\ -n_3 n_1 & -n_3 n_2 & 1 - n_3 n_3 \end{pmatrix} = (\delta_{ij} - n_i n_j)_{3\times 3}.$$

Let $g = g(x,t)$ be a smooth function. For each $j = 1,2,3$,

$$\partial_j g := \frac{\partial g}{\partial x_j},$$
$$\partial_j^\Gamma g := \partial_j g - n_j(n_1\partial_1 g + n_2\partial_2 g + n_3\partial_3 g) = \sum_{i=1}^{3}(\delta_{ij} - n_i n_j)\partial_i g.$$

We call the operator ∂_j^Γ a *differential operator* on the surface $\Gamma(t)$. Moreover, we define

$$\begin{aligned}\nabla &:= {}^t(\partial_1, \partial_2, \partial_3),\\ \nabla_\Gamma &:= {}^t(\partial_1^\Gamma, \partial_2^\Gamma, \partial_3^\Gamma),\\ \Delta &:= \partial_1^2 + \partial_2^2 + \partial_3^2,\\ \Delta_\Gamma &:= (\partial_1^\Gamma)^2 + (\partial_2^\Gamma)^2 + (\partial_3^\Gamma)^2.\end{aligned}$$

Next we introduce gradient, divergence, and curl operators. For smooth functions $f = f(x,t) = {}^t(f_1, f_2, f_3)$, $g = g(x,t)$, and $F = F(x,t) = (F_{ij})_{3\times 3}$,

$$\mathrm{grad} g := \nabla g = \begin{pmatrix}\partial_1 g\\ \partial_2 g\\ \partial_3 g\end{pmatrix}, \quad \mathrm{grad}_\Gamma g := \nabla_\Gamma g = \begin{pmatrix}\partial_1^\Gamma g\\ \partial_2^\Gamma g\\ \partial_3^\Gamma g\end{pmatrix},$$

$$\mathrm{grad} f := \nabla f = \begin{pmatrix}\partial_1 f_1 & \partial_2 f_1 & \partial_3 f_1\\ \partial_1 f_2 & \partial_2 f_2 & \partial_3 f_2\\ \partial_1 f_3 & \partial_2 f_3 & \partial_3 f_3\end{pmatrix}, \quad \mathrm{grad}_\Gamma f := \nabla_\Gamma f = \begin{pmatrix}\partial_1^\Gamma f_1 & \partial_2^\Gamma f_1 & \partial_3^\Gamma f_1\\ \partial_1^\Gamma f_2 & \partial_2^\Gamma f_2 & \partial_3^\Gamma f_2\\ \partial_1^\Gamma f_3 & \partial_2^\Gamma f_3 & \partial_3^\Gamma f_3\end{pmatrix},$$

$$\mathrm{div} f := \nabla\cdot f = \partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3, \quad \mathrm{div}_\Gamma f := \nabla_\Gamma \cdot f = \partial_1^\Gamma f_1 + \partial_2^\Gamma f_2 + \partial_3^\Gamma f_3,$$

$$\mathrm{div} F := \begin{pmatrix}\partial_1 F_{11} + \partial_2 F_{12} + \partial_3 F_{13}\\ \partial_1 F_{21} + \partial_2 F_{22} + \partial_3 F_{23}\\ \partial_1 F_{31} + \partial_2 F_{32} + \partial_3 F_{33}\end{pmatrix}, \quad \mathrm{div}_\Gamma F := \begin{pmatrix}\partial_1^\Gamma F_{11} + \partial_2^\Gamma F_{12} + \partial_3^\Gamma F_{13}\\ \partial_1^\Gamma F_{21} + \partial_2^\Gamma F_{22} + \partial_3^\Gamma F_{23}\\ \partial_1^\Gamma F_{31} + \partial_2^\Gamma F_{32} + \partial_3^\Gamma F_{33}\end{pmatrix},$$

$$\mathrm{curl} f := \nabla\times f = \begin{pmatrix}\partial_2 f_3 - \partial_3 f_2\\ \partial_3 f_1 - \partial_1 f_3\\ \partial_1 f_2 - \partial_2 f_1\end{pmatrix}, \quad \mathrm{curl}_\Gamma f := \nabla_\Gamma \times f = \begin{pmatrix}\partial_2^\Gamma f_3 - \partial_3^\Gamma f_2\\ \partial_3^\Gamma f_1 - \partial_1^\Gamma f_3\\ \partial_1^\Gamma f_2 - \partial_2^\Gamma f_1\end{pmatrix}.$$

We call grad_Γ a *surface gradient*, div_Γ a *surface divergent*, and curl_Γ a *surface curl* operators. Note that in general $\mathrm{div}_\Gamma F\cdot n \neq 0$ and $\mathrm{curl} f\cdot n \neq 0$. It is easy to check that

$$\begin{aligned}&\mathrm{div}(\mathrm{grad} g) = \Delta g, \quad \mathrm{div}(\mathrm{grad} f) = \Delta f, \quad \mathrm{div}({}^t(\mathrm{grad} f)) = \mathrm{grad}(\mathrm{div} f),\\ &\mathrm{curl}(\mathrm{curl} f) = -\Delta f + \mathrm{div}({}^t(\mathrm{grad} f)) = -\Delta f + \mathrm{grad}(\mathrm{div} f),\\ &\mathrm{div}_\Gamma(\mathrm{grad}_\Gamma g) = \Delta_\Gamma g, \quad \mathrm{div}_\Gamma(\mathrm{grad}_\Gamma f) = \Delta_\Gamma f,\\ &\mathrm{curl}_\Gamma(\mathrm{curl}_\Gamma f) = -\Delta_\Gamma f + \mathrm{div}_\Gamma({}^t(\mathrm{grad}_\Gamma f)).\end{aligned}$$

Remark that, in general, if $i \neq j$ then

$$\partial_i^\Gamma \partial_j^\Gamma g \neq \partial_j^\Gamma \partial_i^\Gamma g.$$

Remark that the mean curvature $H_\Gamma = H_\Gamma(x,t)$ is defined by

$$H_\Gamma = -\mathrm{div}_\Gamma n = -(\partial_1^\Gamma n_1 + \partial_2^\Gamma n_2 + \partial_3^\Gamma n_3) = -(\partial_1 n_1 + \partial_2 n_2 + \partial_3 n_3).$$

Now we define some strain rate tensors. For smooth functions $f = f(x,t) = {}^t(f_1, f_2, f_3)$,

$$D(f) \equiv D^+(f) := \frac{1}{2}\{(\nabla f) + {}^t(\nabla f)\} = \frac{1}{2}\begin{pmatrix} 2\partial_1 f_1 & \partial_2 f_1 + \partial_1 f_2 & \partial_3 f_1 + \partial_1 f_3 \\ \partial_1 f_2 + \partial_2 f_1 & 2\partial_2 f_2 & \partial_3 f_2 + \partial_2 f_3 \\ \partial_1 f_3 + \partial_3 f_1 & \partial_2 f_3 + \partial_3 f_2 & 2\partial_3 f_3 \end{pmatrix},$$

$$\mathbb{D}_\Gamma(f) \equiv \mathbb{D}_\Gamma^+ := \frac{1}{2}\{(\nabla_\Gamma f) + {}^t(\nabla_\Gamma f)\} = \frac{1}{2}\begin{pmatrix} 2\partial_1^\Gamma f_1 & \partial_2^\Gamma f_1 + \partial_1^\Gamma f_2 & \partial_3^\Gamma f_1 + \partial_1^\Gamma f_3 \\ \partial_1^\Gamma f_2 + \partial_2^\Gamma f_1 & 2\partial_2^\Gamma f_2 & \partial_3^\Gamma f_2 + \partial_2^\Gamma f_3 \\ \partial_1^\Gamma f_3 + \partial_3^\Gamma f_1 & \partial_2^\Gamma f_3 + \partial_3^\Gamma f_2 & 2\partial_3^\Gamma f_3 \end{pmatrix},$$

$$D_\Gamma(f) \equiv D_\Gamma^+(f) := \frac{1}{2}\{(P_\Gamma \nabla_\Gamma f) + {}^t(P_\Gamma \nabla_\Gamma f)\}.$$

We call $D(f)$ a *strain rate tensor*, $\mathbb{D}_\Gamma(f)$ a *tangential strain rate tensor*, and $D_\Gamma(f)$ a *projected strain rate tensor*. See [12] for surface strain rate tensors. Note that

$$P_\Gamma \nabla_\Gamma f = \begin{pmatrix} \partial_1^\Gamma f_1 - n_1(n\cdot\partial_1^\Gamma f) & \partial_2^\Gamma f_1 - n_1(n\cdot\partial_2^\Gamma f) & \partial_3^\Gamma f_1 - n_1(n\cdot\partial_3^\Gamma f) \\ \partial_1^\Gamma f_2 - n_2(n\cdot\partial_1^\Gamma f) & \partial_2^\Gamma f_2 - n_2(n\cdot\partial_2^\Gamma f) & \partial_3^\Gamma f_2 - n_2(n\cdot\partial_3^\Gamma f) \\ \partial_1^\Gamma f_3 - n_3(n\cdot\partial_1^\Gamma f) & \partial_2^\Gamma f_3 - n_3(n\cdot\partial_2^\Gamma f) & \partial_3^\Gamma f_3 - n_3(n\cdot\partial_3^\Gamma f) \end{pmatrix}.$$

Moreover,

$$D^-(f) := \frac{1}{2}\{(\nabla f) - {}^t(\nabla f)\} = \frac{1}{2}\begin{pmatrix} 0 & \partial_2 f_1 - \partial_1 f_2 & \partial_3 f_1 - \partial_1 f_3 \\ \partial_1 f_2 - \partial_2 f_1 & 0 & \partial_3 f_2 - \partial_2 f_3 \\ \partial_1 f_3 - \partial_3 f_1 & \partial_2 f_3 - \partial_3 f_2 & 0 \end{pmatrix},$$

$$\mathbb{D}_\Gamma^- := \frac{1}{2}\{(\nabla_\Gamma f) - {}^t(\nabla_\Gamma f)\} = \frac{1}{2}\begin{pmatrix} 0 & \partial_2^\Gamma f_1 - \partial_1^\Gamma f_2 & \partial_3^\Gamma f_1 - \partial_1^\Gamma f_3 \\ \partial_1^\Gamma f_2 - \partial_2^\Gamma f_1 & 0 & \partial_3^\Gamma f_2 - \partial_2^\Gamma f_3 \\ \partial_1^\Gamma f_3 - \partial_3^\Gamma f_1 & \partial_2^\Gamma f_3 - \partial_3^\Gamma f_2 & 0 \end{pmatrix},$$

$$D_\Gamma^-(f) := \frac{1}{2}\{(P_\Gamma \nabla_\Gamma f) - {}^t(P_\Gamma \nabla_\Gamma f)\}.$$

It is clear that

$$\mathrm{curl} f \cdot \mathrm{curl} f = 2D^-(f) : D^-(f),$$
$$\mathrm{curl}_\Gamma f \cdot \mathrm{curl}_\Gamma f = 2\mathbb{D}_\Gamma^-(f) : \mathbb{D}_\Gamma^-(f).$$

Here $f \cdot f = f_1 f_1 + f_2 f_2 + f_3 f_3$ and $M : M = \sum_{i,j=1}^3 M_{ij}^2$, where $M = (M_{ij})_{3\times 3}$.

Let us now study some operators on surfaces.

Lemma 5.1 (Fundamental properties of P_Γ, ∇_Γ, $\mathbb{D}_\Gamma$, and D_Γ).
Let $f = f(x,t) = {}^t(f_1, f_2, f_3)$, $g = g(x,t)$, and $F = F(x,t) = (F_{ij})_{3\times 3}$ be smooth functions. Then

$$P_\Gamma n = {}^t(0,0,0), \tag{5.1}$$
$$P_\Gamma^2 = P_\Gamma, \tag{5.2}$$
$$P_\Gamma(\nabla g) = \nabla_\Gamma g, \tag{5.3}$$
$$n \cdot (\nabla_\Gamma g) = 0, \tag{5.4}$$
$${}^t(P_\Gamma F)n = {}^t(0,0,0), \tag{5.5}$$
$$(P_\Gamma f) \cdot n = 0, \tag{5.6}$$
$$f = P_\Gamma f + (f \cdot n)n, \tag{5.7}$$
$$D_\Gamma(f) = P_\Gamma D(f) P_\Gamma = P_\Gamma \mathbb{D}_\Gamma(f) P_\Gamma, \tag{5.8}$$
$$D_\Gamma^-(f) = P_\Gamma D^-(f) P_\Gamma = P_\Gamma \mathbb{D}_\Gamma^-(f) P_\Gamma. \tag{5.9}$$

Proof of Lemma 5.1.

We first prove (5.1) and (5.2). Since $n_1^2 + n_2^2 + n_3^2 = 1$, we observe that

$$\begin{aligned} P_\Gamma n &= \begin{pmatrix} 1 - n_1 n_1 & -n_1 n_2 & -n_1 n_3 \\ -n_2 n_1 & 1 - n_2 n_2 & -n_2 n_3 \\ -n_3 n_1 & -n_3 n_2 & 1 - n_3 n_3 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \\ &= \begin{pmatrix} n_1 - n_1(n_1^2 + n_2^2 + n_3^2) \\ n_2 - n_2(n_1^2 + n_2^2 + n_3^2) \\ n_3 - n_3(n_1^2 + n_2^2 + n_3^2) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

We also see that

$$P_\Gamma^2 = \begin{pmatrix} 1-n_1n_1 & -n_1n_2 & -n_1n_3 \\ -n_2n_1 & 1-n_2n_2 & -n_2n_3 \\ -n_3n_1 & -n_3n_2 & 1-n_3n_3 \end{pmatrix} \begin{pmatrix} 1-n_1n_1 & -n_1n_2 & -n_1n_3 \\ -n_2n_1 & 1-n_2n_2 & -n_2n_3 \\ -n_3n_1 & -n_3n_2 & 1-n_3n_3 \end{pmatrix}$$

$$= \begin{pmatrix} (1-n_1^2)^2 + n_1^2n_2^2 + n_1^2n_3^2 & n_1n_2(n_1^2+n_2^2+n_3^2-2) & n_1n_3(n_1^2+n_2^2+n_3^2-2) \\ n_2n_1(n_1^2+n_2^2+n_3^2-2) & n_2^2n_1^2 + (1-n_2^2)^2 + n_2^2n_3^2 & n_2n_3(n_1^2+n_2^2+n_3^2-2) \\ n_3n_1(n_1^2+n_2^2+n_3^2-2) & n_3n_2(n_1^2+n_2^2+n_3^2-2) & n_3^2n_1^2 + n_3^2n_2^2 + (1-n_3^2)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1-n_1n_1 & -n_1n_2 & -n_1n_3 \\ -n_2n_1 & 1-n_2n_2 & -n_2n_3 \\ -n_3n_1 & -n_3n_2 & 1-n_3n_3 \end{pmatrix} = P_\Gamma.$$

Secondly, we show (5.3) and (5.4). Let $g = g(x,t)$ be a smooth function. By definition, we check that

$$P_\Gamma(\nabla g) = \begin{pmatrix} 1-n_1n_1 & -n_1n_2 & -n_1n_3 \\ -n_2n_1 & 1-n_2n_2 & -n_2n_3 \\ -n_3n_1 & -n_3n_2 & 1-n_3n_3 \end{pmatrix} \begin{pmatrix} \partial_1 g \\ \partial_2 g \\ \partial_3 g \end{pmatrix}$$

$$= \begin{pmatrix} \partial_1 g - n_1(n_1\partial_1 g + n_2\partial_2 g + n_3\partial_3 g) \\ \partial_2 g - n_2(n_1\partial_1 g + n_2\partial_2 g + n_3\partial_3 g) \\ \partial_3 g - n_3(n_1\partial_1 g + n_2\partial_2 g + n_3\partial_3 g) \end{pmatrix}$$

$$= \begin{pmatrix} \partial_1^\Gamma g \\ \partial_2^\Gamma g \\ \partial_3^\Gamma g \end{pmatrix} = \nabla_\Gamma g.$$

We also check that

$$n \cdot (\nabla_\Gamma g) = n_1\partial_1 g + n_1\partial_2 g + n_3\partial_3 g - \sum_{j=1}^{3}(n_1^2+n_2^2+n_3^2)n_j\partial_j g = 0.$$

Thirdly, we derive (5.5). Let $F = F(x,t) = (F_{ij})_{3\times 3}$ be a smooth function. Since

$$P_\Gamma F = \begin{pmatrix} F_{11} - \sum_{j=1}^3 n_1n_jF_{j1} & F_{12} - \sum_{j=1}^3 n_1n_jF_{j2} & F_{13} - \sum_{j=1}^3 n_1n_jF_{j3} \\ F_{21} - \sum_{j=1}^3 n_2n_jF_{j1} & F_{22} - \sum_{j=1}^3 n_2n_jF_{j2} & F_{22} - \sum_{j=1}^3 n_2n_jF_{j3} \\ F_{31} - \sum_{j=1}^3 n_3n_jF_{j1} & F_{23} - \sum_{j=1}^3 n_3n_jF_{j2} & F_{33} - \sum_{j=1}^3 n_3n_jF_{j3} \end{pmatrix},$$

if follows from the fact $n_1^2 + n_2^2 + n_3^2 = 1$ that

$$^t(P_\Gamma F)n = \begin{pmatrix} n_1F_{11} + n_2F_{21} + n_3F_{31} - \sum_{j=1}^3(n_1^2+n_2^2+n_3^2)n_jF_{j1} \\ n_1F_{12} + n_2F_{22} + n_3F_{32} - \sum_{j=1}^3(n_1^2+n_2^2+n_3^2)n_jF_{j2} \\ n_1F_{13} + n_2F_{23} + n_3F_{33} - \sum_{j=1}^3(n_1^2+n_2^2+n_3^2)n_jF_{j3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Fourthly, we deduce (5.6) and (5.7). Let $f = f(x,t) = {}^t(f_1, f_2, f_3)$ be a smooth function. By the argument similar to derive (5.4) and (5.5), we see that

$$\begin{aligned}(P_\Gamma f)\cdot n &= \begin{pmatrix} f_1 - \sum_{j=1}^3 n_1 n_j f_j \\ f_2 - \sum_{j=1}^3 n_2 n_j f_j \\ f_3 - \sum_{j=1}^3 n_3 n_j f_j \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \\ &= n_1 f_1 + n_2 f_2 + n_3 f_3 - \sum_{j=1}^3 (n_1^2 + n_2^2 + n_3^2) n_j f_j = 0.\end{aligned}$$

By definition, we check that

$$\begin{aligned}P_\Gamma f + (f\cdot n)n &= \begin{pmatrix} 1-n_1n_1 & -n_1n_2 & -n_1n_3 \\ -n_2n_1 & 1-n_2n_2 & -n_2n_3 \\ -n_3n_1 & -n_3n_2 & 1-n_3n_3 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} + (f_1n_1 + f_2n_2 + f_3n_3)\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \\ &= \begin{pmatrix} 1-n_1n_1 & -n_1n_2 & -n_1n_3 \\ -n_2n_1 & 1-n_2n_2 & -n_2n_3 \\ -n_3n_1 & -n_3n_2 & 1-n_3n_3 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} + \begin{pmatrix} n_1n_1 & n_1n_2 & n_1n_3 \\ n_2n_1 & n_2n_2 & n_2n_3 \\ n_3n_1 & n_3n_2 & n_3n_3 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \\ &= \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = f.\end{aligned}$$

Finally, we attack (5.8) and (5.9). Let $f = f(x,t) = {}^t(f_1, f_2, f_3)$ be a smooth function. A direct calculation gives

$$P_\Gamma \nabla f = \begin{pmatrix} \partial_1 f_1 - n_1(n\cdot\partial_1 f) & \partial_2 f_1 - n_1(n\cdot\partial_2 f) & \partial_3 f_1 - n_1(n\cdot\partial_3 f) \\ \partial_1 f_2 - n_2(n\cdot\partial_1 f) & \partial_2 f_2 - n_2(n\cdot\partial_2 f) & \partial_3 f_2 - n_2(n\cdot\partial_3 f) \\ \partial_1 f_3 - n_3(n\cdot\partial_1 f) & \partial_2 f_3 - n_3(n\cdot\partial_2 f) & \partial_3 f_3 - n_3(n\cdot\partial_3 f) \end{pmatrix},$$

$$P_\Gamma \nabla_\Gamma f = \begin{pmatrix} \partial_1^\Gamma f_1 - n_1(n\cdot\partial_1^\Gamma f) & \partial_2^\Gamma f_1 - n_1(n\cdot\partial_2^\Gamma f) & \partial_3^\Gamma f_1 - n_1(n\cdot\partial_3^\Gamma f) \\ \partial_1^\Gamma f_2 - n_2(n\cdot\partial_1^\Gamma f) & \partial_2^\Gamma f_2 - n_2(n\cdot\partial_2^\Gamma f) & \partial_3^\Gamma f_2 - n_2(n\cdot\partial_3^\Gamma f) \\ \partial_1^\Gamma f_3 - n_3(n\cdot\partial_1^\Gamma f) & \partial_2^\Gamma f_3 - n_3(n\cdot\partial_2^\Gamma f) & \partial_3^\Gamma f_3 - n_3(n\cdot\partial_3^\Gamma f) \end{pmatrix},$$

$$P_\Gamma (\nabla f) P_\Gamma = \begin{pmatrix} \partial_1^\Gamma f_1 - n_1(n\cdot\partial_1^\Gamma f) & \partial_2^\Gamma f_1 - n_1(n\cdot\partial_2^\Gamma f) & \partial_3^\Gamma f_1 - n_1(n\cdot\partial_3^\Gamma f) \\ \partial_1^\Gamma f_2 - n_2(n\cdot\partial_1^\Gamma f) & \partial_2^\Gamma f_2 - n_2(n\cdot\partial_2^\Gamma f) & \partial_3^\Gamma f_2 - n_2(n\cdot\partial_3^\Gamma f) \\ \partial_1^\Gamma f_3 - n_3(n\cdot\partial_1^\Gamma f) & \partial_2^\Gamma f_3 - n_3(n\cdot\partial_2^\Gamma f) & \partial_3^\Gamma f_3 - n_3(n\cdot\partial_3^\Gamma f) \end{pmatrix},$$

and

$$P_\Gamma (\nabla_\Gamma f) P_\Gamma = \begin{pmatrix} \partial_1^\Gamma f_1 - n_1(n\cdot\partial_1^\Gamma f) & \partial_2^\Gamma f_1 - n_1(n\cdot\partial_2^\Gamma f) & \partial_3^\Gamma f_1 - n_1(n\cdot\partial_3^\Gamma f) \\ \partial_1^\Gamma f_2 - n_2(n\cdot\partial_1^\Gamma f) & \partial_2^\Gamma f_2 - n_2(n\cdot\partial_2^\Gamma f) & \partial_3^\Gamma f_2 - n_2(n\cdot\partial_3^\Gamma f) \\ \partial_1^\Gamma f_3 - n_3(n\cdot\partial_1^\Gamma f) & \partial_2^\Gamma f_3 - n_3(n\cdot\partial_2^\Gamma f) & \partial_3^\Gamma f_3 - n_3(n\cdot\partial_3^\Gamma f) \end{pmatrix}.$$

Here we used the fact that $n_1\partial_1^\Gamma g + n_2\partial_2^\Gamma g + n_3\partial_3^\Gamma g = 0$. Therefore we find that

$$P_\Gamma(\nabla f)P_\Gamma = P_\Gamma(\nabla_\Gamma f)P_\Gamma = P_\Gamma\nabla_\Gamma f.$$

Since P_Γ is a symmetric matrix, we observe that

$$\begin{aligned} 2P_\Gamma D(v)P_\Gamma = P_\Gamma(\nabla f)P_\Gamma + P_\Gamma({}^t(\nabla f))P_\Gamma &= P_\Gamma(\nabla f)P_\Gamma + {}^t(P_\Gamma(\nabla f)P_\Gamma) \\ &= P_\Gamma(\nabla_\Gamma f)P_\Gamma + {}^t(P_\Gamma(\nabla_\Gamma f)P_\Gamma) \\ &= 2P_\Gamma\mathbb{D}_\Gamma(f)P_\Gamma. \end{aligned}$$

Similarly, we check that

$$\begin{aligned} 2P_\Gamma\mathbb{D}_\Gamma(f)P_\Gamma &= P_\Gamma(\nabla_\Gamma f)P_\Gamma + {}^t(P_\Gamma(\nabla_\Gamma f)P_\Gamma) \\ &= P_\Gamma\nabla_\Gamma f + {}^t(P_\Gamma\nabla_\Gamma f) = 2D_\Gamma(f). \end{aligned}$$

In the same manner, we see that

$$P_\Gamma D^-(f)P_\Gamma = P_\Gamma\mathbb{D}_\Gamma^-(f)P_\Gamma = D_\Gamma^-(f).$$

Therefore the lemma follows. □

6 Laplace Operators on Surfaces

We introduce several Laplace operators on surfaces. Let $\Gamma(t)(=\{\Gamma(t)\}_{0<t<T})$ be a smooth evolving 2-dimensional surface in $\mathbb{R}^3$.

Lemma 6.1. *Suppose that $\Gamma(t)$ is a closed surface. Assume that for every smooth functions $f = f(x,t) = {}^t(f_1, f_2, f_3)$ and $\varphi = \varphi(x,t) = {}^t(\varphi_1, \varphi_2, \varphi_3)$,*

$$\begin{aligned} \int_{\Gamma(t)} (\nabla_\Gamma f) : (\nabla_\Gamma \varphi)\, d\mathcal{H}_x^2 &= \int_{\Gamma(t)} A_1 f \cdot \varphi\, d\mathcal{H}_x^2, \\ \int_{\Gamma(t)} (P_\Gamma\nabla_\Gamma f) : (P_\Gamma\nabla_\Gamma \varphi)\, d\mathcal{H}_x^2 &= \int_{\Gamma(t)} A_2 f \cdot \varphi\, d\mathcal{H}_x^2, \\ 2\int_{\Gamma(t)} \mathbb{D}_\Gamma(f) : \mathbb{D}_\Gamma(\varphi)\, d\mathcal{H}_x^2 &= \int_{\Gamma(t)} A_3 f \cdot \varphi\, d\mathcal{H}_x^2, \\ 2\int_{\Gamma(t)} D_\Gamma(f) : D_\Gamma(\varphi)\, d\mathcal{H}_x^2 &= \int_{\Gamma(t)} A_4 f \cdot \varphi\, d\mathcal{H}_x^2, \\ 2\int_{\Gamma(t)} \mathbb{D}_\Gamma^-(f) : \mathbb{D}_\Gamma^-(\varphi)\, d\mathcal{H}_x^2 &= \int_{\Gamma(t)} A_5 f \cdot \varphi\, d\mathcal{H}_x^2. \end{aligned}$$

Then

$$\begin{aligned} A_1 f &= -\mathrm{div}_\Gamma\{\nabla_\Gamma f\} = -\Delta_\Gamma f,\\ A_2 f &= -\mathrm{div}_\Gamma\{P_\Gamma \nabla_\Gamma f\},\\ A_3 f &= -\mathrm{div}_\Gamma\{2\mathbb{D}_\Gamma(f)\} = -\mathrm{div}_\Gamma\{(\nabla_\Gamma f) + {}^t(\nabla_\Gamma f)\},\\ A_4 f &= -\mathrm{div}_\Gamma\{2D_\Gamma(f)\} = -\mathrm{div}_\Gamma\{(P_\Gamma\nabla_\Gamma f) + {}^t(P_\Gamma \nabla_\Gamma f)\},\\ A_5 f &= -\mathrm{div}_\Gamma\{2\mathbb{D}_\Gamma^-(f)\} = -\mathrm{div}_\Gamma\{(\nabla_\Gamma f) - {}^t(P_\Gamma\nabla_\Gamma f)\}. \end{aligned}$$

Assume in addition that $\varphi \cdot n = 0$. *Then*

$$\begin{aligned} A_1 f &= -P_\Gamma \mathrm{div}_\Gamma\{\nabla_\Gamma f\},\\ A_2 f &= -P_\Gamma \mathrm{div}_\Gamma\{P\nabla_\Gamma f\},\\ A_3 f &= -P_\Gamma \mathrm{div}_\Gamma\{2\mathbb{D}_\Gamma(f)\},\\ A_4 f &= -P_\Gamma \mathrm{div}_\Gamma\{2D_\Gamma(f)\},\\ A_5 f &= -P_\Gamma \mathrm{div}_\Gamma\{(\nabla_\Gamma f) - {}^t(P_\Gamma\nabla_\Gamma f)\}. \end{aligned}$$

Proof of Lemma 6.1. Using integration by parts on the surface and the fact that $\sum_{j=1}^3 n_j \partial_j^\Gamma g = 0$, we prove Lemma 6.1. See [7] for details. □

7 On the viscous terms of our systems

We consider the viscous term of incompressible fluid systems (3.1) and (3.2). Let $\Gamma(t)(=\{\Gamma(t)\}_{0<t<T})$ be a smooth evolving 2-dimensional closed surface in $\mathbb{R}^3$. Applying Lemma 6.1, we can derive the viscous term of our systems. In fact, the following proposition hold:

Proposition 7.1. *Let* $\mu > 0$ *and* $0 < t < T$. *For smooth function* $f = {}^t(f_1, f_2, f_3)$, *Set*

$$E_D[f](t) = \int_{\Gamma(t)} \mu |D_\Gamma(f)|^2 \, d\mathcal{H}_x^2.$$

Then for every smooth function $\varphi = {}^t(\varphi_1, \varphi_2, \varphi_3) \in C_0^\infty(\Gamma(t))$,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E_D[v + \varepsilon\varphi](t) = -\int_{\Gamma(t)} \mathrm{div}_\Gamma\{2\mu D_\Gamma(v)\}\cdot\varphi \, d\mathcal{H}_x^2.$$

Moreover, the two assertions hold:
(i) *Assume that for every* $\varphi = {}^t(\varphi_1, \varphi_2, \varphi_3) \in C_0^\infty(\Gamma(t))$ *satisfying* $\mathrm{div}_\Gamma \varphi = 0$,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E_D[v + \varepsilon\varphi](t) = 0.$$

Then there is a function σ *such that*

$$\mathrm{div}_\Gamma\{2\mu D_\Gamma(v)\} = \mathrm{grad}_\Gamma \sigma + \sigma H n.$$

(ii) *Assume that for every* $\varphi = {}^t(\varphi_1, \varphi_2, \varphi_3) \in C_0^\infty(\Gamma(t))$ *satisfying* $\mathrm{div}_\Gamma \varphi = 0$ *and* $\varphi \cdot n = 0$,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E_D[v + \varepsilon\varphi](t) = 0.$$

Then there is a function σ *such that*

$$P_\Gamma \mathrm{div}_\Gamma \{2\mu D_\Gamma(v)\} = \mathrm{grad}_\Gamma \sigma.$$

See [8] for the proof of Proposition 7.1.

From Proposition 7.1, we have

$$\frac{\delta E_D}{\delta v}\bigg|_{\mathrm{div}_\Gamma v = 0} = \mathrm{div}_\Gamma\{2\mu D_\Gamma(v)\} - \mathrm{grad}_\Gamma \sigma - \sigma H n,$$

$$\frac{\delta E_D}{\delta v}\bigg|_{\mathrm{div}_\Gamma v = 0,\ v\cdot n = 0} = P_\Gamma \mathrm{div}_\Gamma\{2\mu D_\Gamma(v)\} - \mathrm{grad}_\Gamma \sigma.$$

Therefore we obtain the viscous term and pressure term of our systems. Note that we do not use the Boussinesq-Scriven law directly in order to derive the viscous term of our fluid systems on an evolving surface.

The Appendix (I) in [8] showed that $P_\Gamma \mathrm{div}_\Gamma\{2D_\Gamma(v)\} \neq P_\Gamma \Delta_\Gamma v + P_\Gamma K_\Gamma v$, where K_Γ is the Gaussian curvature. Now we study the case when we can use another Laplace operator $\widehat{\Delta}_\Gamma$ to show that $P_\Gamma \mathrm{div}_\Gamma\{2D_\Gamma(v)\} = P_\Gamma \widehat{\Delta}_\Gamma v + P_\Gamma K_\Gamma v$. Now we explain the argument.

From the previous sections, we find that

$$\begin{aligned}
2D_\Gamma(v) = P_\Gamma((\nabla v) + {}^t(\nabla v))P_\Gamma &= L_1 + L_2 + L_3 + L_4,\\
2D_\Gamma^-(v) = P_\Gamma((\nabla v) - {}^t(\nabla v))P_\Gamma &= L_1 - L_2 + L_3 - L_4,\\
P_\Gamma \nabla_\Gamma v &= L_1 + L_3,\\
{}^t(P_\Gamma \nabla_\Gamma v) &= L_2 + L_4,
\end{aligned}$$

where

$$L_1 = \nabla_\Gamma v = \begin{pmatrix} \partial_1^\Gamma v_1 & \partial_2^\Gamma v_1 & \partial_3^\Gamma v_1 \\ \partial_1^\Gamma v_2 & \partial_2^\Gamma v_2 & \partial_3^\Gamma v_2 \\ \partial_1^\Gamma v_3 & \partial_2^\Gamma v_3 & \partial_3^\Gamma v_3 \end{pmatrix},$$

$$L_2 = {}^t(L_1) = \begin{pmatrix} \partial_1^\Gamma v_1 & \partial_1^\Gamma v_2 & \partial_1^\Gamma v_3 \\ \partial_2^\Gamma v_1 & \partial_2^\Gamma v_2 & \partial_2^\Gamma v_3 \\ \partial_3^\Gamma v_1 & \partial_3^\Gamma v_2 & \partial_3^\Gamma v_3 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} -n_1(n\cdot\partial_1^\Gamma v) & -n_1(n\cdot\partial_2^\Gamma v) & -n_1(n\cdot\partial_3^\Gamma v) \\ -n_2(n\cdot\partial_1^\Gamma v) & -n_2(n\cdot\partial_2^\Gamma v) & -n_2(n\cdot\partial_3^\Gamma v) \\ -n_3(n\cdot\partial_1^\Gamma v) & -n_3(n\cdot\partial_2^\Gamma v) & -n_3(n\cdot\partial_3^\Gamma v) \end{pmatrix},$$

$$L_4 = {}^t(L_3) = \begin{pmatrix} -n_1(n\cdot\partial_1^\Gamma v) & -n_2(n\cdot\partial_1^\Gamma v) & -n_3(n\cdot\partial_1^\Gamma v) \\ -n_1(n\cdot\partial_2^\Gamma v) & -n_2(n\cdot\partial_2^\Gamma v) & -n_3(n\cdot\partial_2^\Gamma v) \\ -n_1(n\cdot\partial_3^\Gamma v) & -n_2(n\cdot\partial_3^\Gamma v) & -n_3(n\cdot\partial_3^\Gamma v) \end{pmatrix}.$$

A direct calculation gives

$$\mathrm{div}_\Gamma L_1 = \begin{pmatrix} \Delta_\Gamma v_1 \\ \Delta_\Gamma v_2 \\ \Delta_\Gamma v_3 \end{pmatrix} = \Delta_\Gamma v,$$

$$\mathrm{div}_\Gamma L_2 = \begin{pmatrix} (\partial_1^\Gamma)^2 v_1 + \partial_2^\Gamma\partial_1^\Gamma v_2 + \partial_3^\Gamma\partial_1^\Gamma v_3 \\ \partial_1^\Gamma\partial_2^\Gamma v_1 + (\partial_2^\Gamma)^2 v_2 + \partial_3^\Gamma\partial_2^\Gamma v_3 \\ \partial_1^\Gamma\partial_3^\Gamma v_1 + \partial_2^\Gamma\partial_3^\Gamma v_2 + (\partial_3^\Gamma)^2 v_3 \end{pmatrix},$$

$$\mathrm{div}_\Gamma L_3 = \begin{pmatrix} -\sum_{j=1}^3 \partial_j^\Gamma n_1(n\cdot\partial_j^\Gamma v) - n_1\sum_{j=1}^3(\partial_j^\Gamma n\cdot\partial_j^\Gamma v) - n_1(n\cdot\Delta_\Gamma v) \\ -\sum_{j=1}^3 \partial_j^\Gamma n_2(n\cdot\partial_j^\Gamma v) - n_2\sum_{j=1}^3(\partial_j^\Gamma n\cdot\partial_j^\Gamma v) - n_2(n\cdot\Delta_\Gamma v) \\ -\sum_{j=1}^3 \partial_j^\Gamma n_3(n\cdot\partial_j^\Gamma v) - n_3\sum_{j=1}^3(\partial_j^\Gamma n\cdot\partial_j^\Gamma v) - n_3(n\cdot\Delta_\Gamma v) \end{pmatrix},$$

$$\mathrm{div}_\Gamma L_4 = \begin{pmatrix} -(\partial_1^\Gamma n_1 + \partial_2^\Gamma n_2 + \partial_3^\Gamma n_3)(n\cdot\partial_1^\Gamma v) \\ -(\partial_1^\Gamma n_1 + \partial_2^\Gamma n_2 + \partial_3^\Gamma n_3)(n\cdot\partial_2^\Gamma v) \\ -(\partial_1^\Gamma n_1 + \partial_2^\Gamma n_2 + \partial_3^\Gamma n_3)(n\cdot\partial_3^\Gamma v) \end{pmatrix}.$$

It is easy to check that

$$\mathrm{div}_\Gamma L_2 = \begin{pmatrix} \partial_1^\Gamma(\mathrm{div}_\Gamma v) + (\partial_2^\Gamma\partial_1^\Gamma v_2 - \partial_1^\Gamma\partial_2^\Gamma v_2) + (\partial_3^\Gamma\partial_1^\Gamma v_3 - \partial_1^\Gamma\partial_3^\Gamma v_3) \\ \partial_2^\Gamma(\mathrm{div}_\Gamma v) + (\partial_1^\Gamma\partial_2^\Gamma v_1 - \partial_2^\Gamma\partial_1^\Gamma v_1) + (\partial_3^\Gamma\partial_2^\Gamma v_3 - \partial_2^\Gamma\partial_3^\Gamma v_3) \\ \partial_3^\Gamma(\mathrm{div}_\Gamma v) + (\partial_1^\Gamma\partial_3^\Gamma v_1 - \partial_3^\Gamma\partial_1^\Gamma v_1) + (\partial_2^\Gamma\partial_3^\Gamma v_2 - \partial_3^\Gamma\partial_2^\Gamma v_2) \end{pmatrix}.$$

Assume that $v\cdot n = 0$. Since $\partial_j^\Gamma(v\cdot n) = 0$, we have

$$-v\cdot\partial_j^\Gamma n = \partial_j^\Gamma v\cdot n. \tag{7.1}$$

Using (7.1), we have

$$\mathrm{div}_\Gamma L_3 = \begin{pmatrix} \sum_{j=1}^3 \partial_j^\Gamma n_1(\partial_j^\Gamma n\cdot v) - n_1\sum_{j=1}^3(\partial_j^\Gamma n\cdot\partial_j^\Gamma v) - n_1(n\cdot\Delta_\Gamma v) \\ \sum_{j=1}^3 \partial_j^\Gamma n_2(\partial_j^\Gamma n\cdot v) - n_2\sum_{j=1}^3(\partial_j^\Gamma n\cdot\partial_j^\Gamma v) - n_2(n\cdot\Delta_\Gamma v) \\ \sum_{j=1}^3 \partial_j^\Gamma n_3(\partial_j^\Gamma n\cdot v) - n_3\sum_{j=1}^3(\partial_j^\Gamma n\cdot\partial_j^\Gamma v) - n_3(n\cdot\Delta_\Gamma v) \end{pmatrix},$$

$$\mathrm{div}_\Gamma L_4 = \begin{pmatrix} (\partial_1^\Gamma n_1 + \partial_2^\Gamma n_2 + \partial_3^\Gamma n_3)(\partial_1^\Gamma n\cdot v) \\ (\partial_1^\Gamma n_1 + \partial_2^\Gamma n_2 + \partial_3^\Gamma n_3)(\partial_2^\Gamma n\cdot v) \\ (\partial_1^\Gamma n_1 + \partial_2^\Gamma n_2 + \partial_3^\Gamma n_3)(\partial_3^\Gamma n\cdot v) \end{pmatrix}.$$

Assume that $\mathrm{div}_\Gamma v = 0$. Using the following principal coordinates in [6]:

$$\begin{aligned}
&\partial_1^\Gamma n_1 = \partial_1 n_1 = -\kappa_1,\\
&\partial_1^\Gamma n_2 = \partial_1 n_2 = 0,\\
&\partial_2^\Gamma n_1 = \partial_2 n_1 = 0,\\
&\partial_2^\Gamma n_2 = \partial_2 n_2 = -\kappa_2,\\
&n = {}^t(0,0,1),\\
&\kappa_1 + \kappa_2 = H_\Gamma,\\
&\kappa_1 \kappa_2 = K_\Gamma,
\end{aligned}$$

where H_Γ denotes the mean curvature and K_Γ the Gaussian curvature, we see that

$$P_\Gamma \mathrm{div}_\Gamma L_1 = \begin{pmatrix} (\partial_1^2 + \partial_2^2) v_1 \\ (\partial_1^2 + \partial_2^2) v_2 \\ 0 \end{pmatrix} = P_\Gamma \Delta_\Gamma v,$$

$$P_\Gamma \mathrm{div}_\Gamma L_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$P_\Gamma \mathrm{div}_\Gamma L_3 = \begin{pmatrix} \kappa_1^2 v_1 \\ \kappa_2^2 v_2 \\ 0 \end{pmatrix},$$

$$P_\Gamma \mathrm{div}_\Gamma L_4 = \begin{pmatrix} (\kappa_1 + \kappa_2)\kappa_1 v_1 \\ (\kappa_1 + \kappa_2)\kappa_2 v_2 \\ 0 \end{pmatrix}.$$

This implies that

$$P_\Gamma \mathrm{div}_\Gamma L_3 - P_\Gamma \mathrm{div}_\Gamma L_4 = \begin{pmatrix} -K_\Gamma v_1 \\ -K_\Gamma v_2 \\ 0 \end{pmatrix}$$

Note that

$$P_\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \partial_3^\Gamma = 0.$$

Therefore we see that

$$P_\Gamma \mathrm{div}_\Gamma \{2 D_\Gamma^-(v)\} = P_\Gamma \Delta_\Gamma v - P_\Gamma K_\Gamma v$$

if $v \cdot n = 0$ and $\mathrm{div}_\Gamma v = 0$. However, we do not have the information about $P_\Gamma \mathrm{div}_\Gamma D_\Gamma(v)$.

Next we introduce another type type of differential operators on surface to study the viscous term of our systems.

Definition 7.2 (Another type of differential operators on surfaces). For smooth function $f = f(t,x) = {}^t(f_1, f_2, f_3)$ and $i, j, k = 1, 2, 3$,

$$\begin{aligned}\widehat{\partial}_j f :=& P_\Gamma \partial_j^\Gamma (P_\Gamma f) = P_\Gamma(\partial_j^\Gamma P_\Gamma) f + P_\Gamma \partial_j^\Gamma f,\\ \widehat{\partial}_j f_k :=& e_k \cdot (\widehat{\partial}_j f),\\ \widehat{\partial}_i \widehat{\partial}_j f :=& \widehat{\partial}_i(\widehat{\partial}_j f)\\ \equiv& P_\Gamma(\partial_i^\Gamma P_\Gamma)(\partial_j^\Gamma P_\Gamma) f + P_\Gamma(\partial_i^\Gamma \partial_j^\Gamma P_\Gamma) f + P_\Gamma(\partial_j^\Gamma P_\Gamma)\partial_i^\Gamma f + P_\Gamma(\partial_i^\Gamma P_\Gamma)\partial_j^\Gamma f + P_\Gamma \partial_i^\Gamma \partial_j^\Gamma f,\end{aligned}$$

and

$$\widehat{\partial}_i \widehat{\partial}_j f_k := e_k \cdot (\widehat{\partial}_i \widehat{\partial}_j f).$$

Here $e_1 = {}^t(1,0,0)$, $e_2 = {}^t(0,1,0)$, and $e_3 = {}^t(0,0,1)$.

Note that if $v \cdot n = 0$ then

$$\widehat{\partial}_1 f_1 + \widehat{\partial}_2 f_2 + \widehat{\partial}_3 f_3 = \partial_1^\Gamma f_1 + \partial_2^\Gamma f_2 + \partial_3^\Gamma f_3.$$

Lemma 7.3. *Let $f = f(t,x) = {}^t(f_1, f_2, f_3)$ be a smooth function. Assume that $f \cdot n = 0$. Then*

$$\begin{pmatrix}(\widehat{\partial}_2\widehat{\partial}_1 - \widehat{\partial}_1\widehat{\partial}_2) f_2 + (\widehat{\partial}_3\widehat{\partial}_1 - \widehat{\partial}_1\widehat{\partial}_3) f_3\\ (\widehat{\partial}_1\widehat{\partial}_2 - \widehat{\partial}_2\widehat{\partial}_1) f_1 + (\widehat{\partial}_3\widehat{\partial}_1 - \widehat{\partial}_1\widehat{\partial}_3) f_3\\ (\widehat{\partial}_1\widehat{\partial}_3 - \widehat{\partial}_3\widehat{\partial}_1) f_1 + (\widehat{\partial}_3\widehat{\partial}_2 - \widehat{\partial}_2\widehat{\partial}_3) f_2\end{pmatrix} = K_\Gamma P_\Gamma f = K_\Gamma f.$$

Proof. Let $f = f(t,x) = {}^t(f_1, f_2, f_3)$ be a smooth function with $P_\Gamma f = f$. Since

$$\begin{aligned}\widehat{\partial}_2\widehat{\partial}_1 f &= P_\Gamma(\partial_2^\Gamma P_\Gamma)(\partial_1^\Gamma P_\Gamma) f + P_\Gamma(\partial_2^\Gamma \partial_1^\Gamma P_\Gamma) f + P_\Gamma(\partial_1^\Gamma P_\Gamma)\partial_2^\Gamma f + P_\Gamma(\partial_2^\Gamma P_\Gamma)\partial_1^\Gamma f + P_\Gamma \partial_2^\Gamma \partial_1^\Gamma f,\\ \widehat{\partial}_1\widehat{\partial}_2 f &= P_\Gamma(\partial_1^\Gamma P_\Gamma)(\partial_2^\Gamma P_\Gamma) f + P_\Gamma(\partial_1^\Gamma \partial_2^\Gamma P_\Gamma) f + P_\Gamma(\partial_2^\Gamma P_\Gamma)\partial_1^\Gamma f + P_\Gamma(\partial_1^\Gamma P_\Gamma)\partial_2^\Gamma f + P_\Gamma \partial_1^\Gamma \partial_2^\Gamma f,\end{aligned}$$

we have

$$\begin{aligned}&\widehat{\partial}_2\widehat{\partial}_1 f - \widehat{\partial}_1\widehat{\partial}_2 f =\\ &P_\Gamma\{(\partial_2^\Gamma P_\Gamma)(\partial_1^\Gamma P_\Gamma) - (\partial_1^\Gamma P_\Gamma)(\partial_2^\Gamma P_\Gamma)\} f + P_\Gamma\{(\partial_2^\Gamma \partial_1^\Gamma P_\Gamma) - (\partial_1^\Gamma \partial_2^\Gamma P_\Gamma)\} f + P_\Gamma(\partial_2^\Gamma \partial_1^\Gamma f - \partial_1^\Gamma \partial_2^\Gamma f).\end{aligned}$$

Using the principal coordinates at the origin(see previous page), we observe that

$$\begin{aligned}\widehat{\partial}_2\widehat{\partial}_1 f - \widehat{\partial}_1\widehat{\partial}_2 f &= P_\Gamma\{(\partial_2^\Gamma P_\Gamma)(\partial_1^\Gamma P_\Gamma) - (\partial_1^\Gamma P_\Gamma)(\partial_2^\Gamma P_\Gamma)\} f\\ &= \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\left[\begin{pmatrix}0&0&0\\0&0&\kappa_2\\0&\kappa_2&0\end{pmatrix}\begin{pmatrix}0&0&\kappa_1\\0&0&0\\\kappa_1&0&0\end{pmatrix} - \begin{pmatrix}0&0&\kappa_1\\0&0&0\\\kappa_1&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&\kappa_2\\0&\kappa_2&0\end{pmatrix}\right]\begin{pmatrix}f_1\\f_2\\f_3\end{pmatrix}\\ &= \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\left[\begin{pmatrix}0&0&0\\\kappa_1\kappa_2&0&0\\0&0&0\end{pmatrix} - \begin{pmatrix}0&\kappa_1\kappa_2&0\\0&0&0\\0&0&0\end{pmatrix}\right]\begin{pmatrix}f_1\\f_2\\f_3\end{pmatrix} = \begin{pmatrix}-\kappa_1\kappa_2 f_2\\\kappa_1\kappa_2 f_1\\0\end{pmatrix}.\end{aligned}$$

We also use the principal coordinates to see that

$$\widehat{\partial}_3\widehat{\partial}_1 f - \widehat{\partial}_1\widehat{\partial}_3 f = \widehat{\partial}_2\widehat{\partial}_3 f - \widehat{\partial}_3\widehat{\partial}_2 f = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

By definition, we see that

$$\begin{pmatrix} (\widehat{\partial}_2\widehat{\partial}_1 - \widehat{\partial}_1\widehat{\partial}_2)f_2 + (\widehat{\partial}_3\widehat{\partial}_1 - \widehat{\partial}_1\widehat{\partial}_3)f_3 \\ (\widehat{\partial}_1\widehat{\partial}_2 - \widehat{\partial}_2\widehat{\partial}_1)f_1 + (\widehat{\partial}_3\widehat{\partial}_1 - \widehat{\partial}_1\widehat{\partial}_3)f_3 \\ (\widehat{\partial}_1\widehat{\partial}_3 - \widehat{\partial}_3\widehat{\partial}_1)f_1 + (\widehat{\partial}_3\widehat{\partial}_2 - \widehat{\partial}_2\widehat{\partial}_3)f_2 \end{pmatrix} = \begin{pmatrix} \kappa_1\kappa_2 f_1 \\ \kappa_1\kappa_2 f_2 \\ 0 \end{pmatrix} = K_\Gamma f = K_\Gamma P_\Gamma f.$$

□

Note that in general for smooth function $f = f(t,x) = {}^t(f_1, f_2, f_3)$,

$$\begin{pmatrix} (\widehat{\partial}_2\widehat{\partial}_1 - \widehat{\partial}_1\widehat{\partial}_2)f_2 + (\widehat{\partial}_3\widehat{\partial}_1 - \widehat{\partial}_1\widehat{\partial}_3)f_3 \\ (\widehat{\partial}_1\widehat{\partial}_2 - \widehat{\partial}_2\widehat{\partial}_1)f_1 + (\widehat{\partial}_3\widehat{\partial}_2 - \widehat{\partial}_2\widehat{\partial}_3)f_3 \\ (\widehat{\partial}_1\widehat{\partial}_3 - \widehat{\partial}_3\widehat{\partial}_1)f_1 + (\widehat{\partial}_2\widehat{\partial}_3 - \widehat{\partial}_3\widehat{\partial}_2)f_2 \end{pmatrix} \neq \begin{pmatrix} (\partial_2^\Gamma\partial_1^\Gamma - \partial_1^\Gamma\partial_2^\Gamma)f_2 + (\partial_3^\Gamma\partial_1^\Gamma - \partial_1^\Gamma\partial_3^\Gamma)f_3 \\ (\partial_1^\Gamma\partial_2^\Gamma - \partial_2^\Gamma\partial_1^\Gamma)f_1 + (\partial_3^\Gamma\partial_2^\Gamma - \partial_2^\Gamma\partial_3^\Gamma)f_3 \\ (\partial_1^\Gamma\partial_3^\Gamma - \partial_3^\Gamma\partial_1^\Gamma)f_1 + (\partial_2^\Gamma\partial_3^\Gamma - \partial_3^\Gamma\partial_2^\Gamma)f_2 \end{pmatrix}.$$

Here we set for $f = f(x,t) = {}^t(f_1, f_2, f_3)$,

$$\widehat{\mathrm{div}_\Gamma} f := \widehat{\partial}_1 f_1 + \widehat{\partial}_2 f_2 + \widehat{\partial}_3 f_3,$$
$$\widehat{\Delta}_\Gamma f := ((\widehat{\partial}_1)^2 + (\widehat{\partial}_2)^2 + (\widehat{\partial}_3)^2) f.$$

Now we assume that $v \cdot n = 0$. By definition, we see that

$$\widehat{\partial}_i v_j = \partial_i^\Gamma v_j - n_j(n \cdot \partial_i^\Gamma v).$$

Therefore we can write

$$2D_\Gamma(v) = (L_1 + L_3) + (L_2 + L_4) = \begin{pmatrix} \widehat{\partial}_1 v_1 & \widehat{\partial}_2 v_1 & \widehat{\partial}_3 v_1 \\ \widehat{\partial}_1 v_2 & \widehat{\partial}_2 v_2 & \widehat{\partial}_3 v_2 \\ \widehat{\partial}_1 v_3 & \widehat{\partial}_2 v_3 & \widehat{\partial}_3 v_3 \end{pmatrix} + \begin{pmatrix} \widehat{\partial}_1 v_1 & \widehat{\partial}_1 v_2 & \widehat{\partial}_1 v_3 \\ \widehat{\partial}_2 v_1 & \widehat{\partial}_2 v_2 & \widehat{\partial}_2 v_3 \\ \widehat{\partial}_3 v_1 & \widehat{\partial}_3 v_2 & \widehat{\partial}_3 v_3 \end{pmatrix}.$$

Since

$$\mathrm{div}_\Gamma\{L_1 + L_3\} = \widehat{\mathrm{div}_\Gamma}\{L_1 + L_3\},$$
$$\mathrm{div}_\Gamma\{L_2 + L_4\} = \widehat{\mathrm{div}_\Gamma}\{L_2 + L_4\},$$

we have

$$P_\Gamma \mathrm{div}_\Gamma\{2D_\Gamma(v)\} = P_\Gamma\widehat{\Delta}_\Gamma v + P_\Gamma \begin{pmatrix} \widehat{\partial}_1(\widehat{\mathrm{div}_\Gamma} v) + (\widehat{\partial}_2\widehat{\partial}_1 - \widehat{\partial}_1\widehat{\partial}_2)v_2 + (\widehat{\partial}_3\widehat{\partial}_1 - \widehat{\partial}_1\widehat{\partial}_3)v_3 \\ \widehat{\partial}_2(\widehat{\mathrm{div}_\Gamma} v) + (\widehat{\partial}_1\widehat{\partial}_2 - \widehat{\partial}_2\widehat{\partial}_1)v_1 + (\widehat{\partial}_3\widehat{\partial}_2 - \widehat{\partial}_2\widehat{\partial}_3)v_3 \\ \widehat{\partial}_3(\widehat{\mathrm{div}_\Gamma} v) + (\widehat{\partial}_1\widehat{\partial}_3 - \widehat{\partial}_3\widehat{\partial}_1)v_1 + (\widehat{\partial}_2\widehat{\partial}_3 - \widehat{\partial}_3\widehat{\partial}_2)v_2 \end{pmatrix}.$$

Now we assume that $\widehat{\mathrm{div}_\Gamma} v = 0$. Note that $\mathrm{div}_\Gamma v = \widehat{\mathrm{div}_\Gamma} v$ if $v \cdot n = 0$. Applying Lemma 7.3, we find that

$$P_\Gamma \mathrm{div}_\Gamma \{2D_\Gamma(v)\} = P_\Gamma \widehat{\Delta}_\Gamma v + K_\Gamma P_\Gamma v.$$

Therefore we conclude that if $v \cdot n = 0$ and $\mathrm{div}_\Gamma v = 0$ then we can write the viscous term of our system as Laplace operator + Gaussian term.

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