数理解析研究所講究録
第2062巻 2018年 46-59

Ford and Dirichlet domains for certain cone manifolds*

Hirotaka Akiyoshi †

1 Introduction

The main purpose of this note is a progress report on the ongoing project whose goal is to achieve the variations of the Jorgensen theory for certain cone manifolds. The Jorgensen theory studies the punctured torus groups, and characterizes the combinatorial structures of the Ford domains (see [3] and [2]). Here a *punctured torus group* is a Kleinian group freely generated by two elements with parabolic commutator; the space of punctured torus groups contains the quasifuchsian space for the once-punctured torus T_0 (see Figure 1). As a corollary to this, we can obtain a practical algorithm for a subgroup of $\mathrm{PSL}_2(\mathbb{C})$ generated by two elements with parabolic commutator to be discrete and free as well as it has infinite covolume.

There are several variations of the Jorgensen theory, though some of them is still conjectural. They are introduced in Section 3. Briefly, one of them is to replace the cusp of T_0 with a cone point (Variation I), another one is to produce accidental

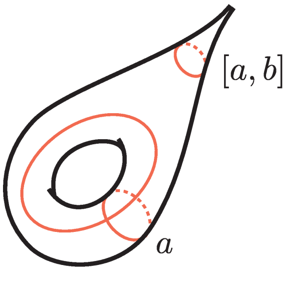

Figure 1: Once-punctured torus T_0

*This work was supported by JSPS KAKENHI Grant Numbers 15H03620, 16K05153.
†Department of Mathematics, Graduate School of Science, Osaka City University, 3-3-138, Sugimoto, Sumiyoshi-ku Osaka, 558-8585 Japan

cusps and further changing it to cone singularities (Variation II), and the other one is the mixture of the two variations I and II (Variation III).

Variation II was introduced in [2] whose goal is to construct hyperbolic structures on the two-bridge knot and link complements, concretely. It gives a path from punctured torus groups to two-bridge knot groups via cone hyperbolic structures. On the way, we can characterize the combinatorial structures of Ford domains for groups in the Riley slice of Schottky slice. Our project concerning Variation III also tries to establish a variation of the result for "cone Riley slice" (see Section 4). We believe that it will shed a new light on Martin's project of searching for small hyperbolic quotients.

In Section 5, we give a new proposal which is expected to connect coned torus in Variation I and cone Riley slice.

2 Dirichlet and Ford domains

Throughout the paper we employ the upper half space model for the hyperbolic 3-space $\mathbb{H}^3$, namely, $\mathbb{H}^3 = \{(z,t) \,|\, z \in \mathbb{C},\, t > 0\} \subset \mathbb{C} \times \mathbb{R}$. For $\theta \in (0, 2\pi)$, let $\mathrm{Sec}_\theta = \{(z,t) \,|\, 0 \leq \arg z \leq \theta,\, t > 0\} \subset \mathbb{H}^3$ and $\mathbb{H}^3_\theta$ the quotient space obtained from Sec_θ by identifying its vertical boundary half planes by the θ-rotation about the vertical geodesic $\langle \infty, 0 \rangle$. We call the image of $\langle \infty, 0 \rangle$ the *cone singularity* of $\mathbb{H}^3_\theta$. Let $p : \mathrm{Sec}_\theta \to \mathbb{H}^3_\theta$ be the projection. For a horoball H centered at ∞, we call $p(H \cap \mathrm{Sec}_\theta)$ a *horoball with cone singularity* (of cone angle θ).

A *3-dimensional cone manifold* is a 3-manifold M with a collection of arcs Σ, called the *singular locus*, and an assignment of a positive number to each arc, called the *cone angle*. A *cone hyperbolic structure* on a 3-dimensional cone manifold M is a metric such that for any point x in $M - \Sigma$ (resp. Σ) there is a neighborhood U of x such that the pair (U, x) is isometric to a pair (V, y) of an open set V in $\mathbb{H}^3$ and $y \in V$ (resp. an open set V in $\mathbb{H}^3_\theta$ and a singular point in V, where θ is the cone angle assigned to the component of Σ containing x). In this paper, we call a cone manifold equipped with a cone hyperbolic structure a *cone hyperbolic manifold*.

Let M be a complete, connected 3-dimensional cone hyperbolic manifold such that the cone angle assigned to each component of the cone singularity Σ is less than 2π.

First, we recall the notion of Dirichlet domains. Fix a point b in M. The *Dirichlet domain with respect to* b is defined to be the completion (with respect to the path metric) of the complement in M of the *cut locus* $\mathrm{Cut}(b)$ *with respect to* b, where $\mathrm{Cut}(b)$ is the closure of the set of points $x \in M$ such that there are at least two shortest paths from x to b. For example, Dirichlet domains are effectively used in the former study on cone hyperbolic structures in the proof of the "orbifold theorem". One of the most important property of Dirichlet domains would be that a Dirichlet domain is a contractible set whose interior embeds in M as a star shaped (resp. convex) dense subset whenever all the cone angles are less than 2π (resp. π).

Our interest is in the Ford domains and its relation with Dirichlet domains for cone manifolds. Before giving our definition of Ford domains for cone hyperbolic manifolds, we give a quick review on it for the case without cone singularity. In that case the manifold is obtained as the quotient space of $\mathbb{H}^3$ by a certain Kleinian group, say Γ. By fixing an identification of $\mathbb{H}^3$ with the upper half space model, the Ford domain for $M = \mathbb{H}^3/\Gamma$ is obtained as the common exterior to the isometric hemispheres for the elements of Γ which does not fix ∞.

The Ford domain is not a fundamental domain for Γ in general, however, its intersection with a fundamental domain for the stabilizer of ∞ is. Compared with Dirichlet domains, sometimes it is easier to let Ford domains to be canonical: For instance, if the manifold has precisely one cusp, then the Ford domain with respect to the fixed point of a parabolic element corresponding to the cusp. Furthermore, if the manifold is moreover of finite volume, then the "geometric dual" of the Ford domain is the canonical decomposition for the manifold in the sense of Epstein-Penner and Weeks.

In the case with cone singularity, we do not have a canonical universal covering space; in fact, the topological universal covering space does not seem to be useful enough, because possibly there are more than one shortest paths connecting two points in the covering space. In order to avoid the problems, we employ the equivalent intrinsic characterization of Ford domains. To this end, we need an additional condition that M contains an "admissible subset":

Definition 2.1. A subset C of M is said to be *admissible* if it is isometric to one of the following spaces:

1. A horoball with cone singularity in $\mathbb{H}^3_\theta$ for $\theta \in (0, 2\pi)$.

2. The quotient of a horoboll H in $\mathbb{H}^3$ by a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ consisting of parabolic transformations stabilizing H.

Suppose that M contains an admissible subset C. The *cut locus* $\mathrm{Cut}(C)$ *with respect to* C is defined to be the closure of the set of points $x \in M$ such that there are at least two shortest paths from x to C. As a corollary to the Hopf-Rinow theorem, we see that the cut locus $\mathrm{Cut}(C)$ is a locally finite union of totally geodesic pieces.

Definition 2.2. For a cone hyperbolic manifold M which contains an admissible subset C, the *Ford domain for M with respect to C* is defined to be the completion of $M - \mathrm{Cut}(C)$ with respect to the path metric.

We remark that the intrinsic characterization is based on the property of isometric spheres: The isometric hemisphere of an element γ of $\mathrm{PSL}_2(\mathbb{C})$ with $\gamma(\infty) \neq \infty$

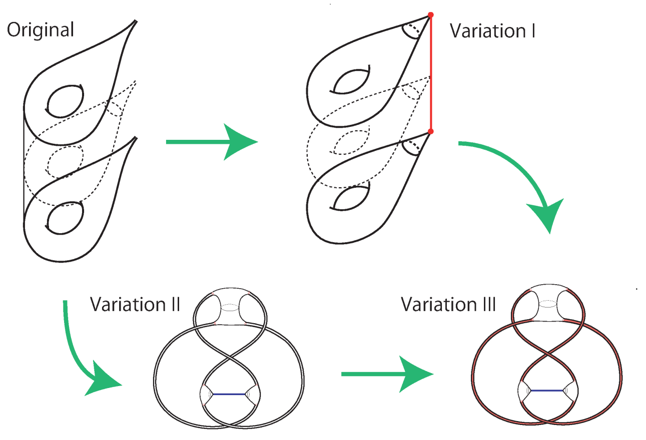

Figure 2: Variations of Jorgensen theory

is equal to the equidistant locus from H and $\gamma^{-1}(H)$ for any horoball H with center ∞.

Let S be a complete cone hyperbolic surface with a single cone point q of angle at most 2π. The hyperbolic structure is canonically extended to a complete cone hyperbolic structure on the product $M = S \times I$. We can see that the cone hyperbolic manifold M contains an admissible subset C isometric to a horoball with cone singularity.

Proposition 2.3. *Let D be the Dirichlet domain for S with respect to q, and F the Ford domain for M with respect to C. Then C is isometric to the product $D \times I$ which is endowed with the canonical hyperbolic structure induced from D.*

3 Several variations of the Jorgensen theory

There are several variations of the Jorgensen theory. The original theory deals with punctured torus groups, which correspond to hyperbolic structures on the manifold, M_0, obtained as the product of T_0 and the open interval I. As Variation I, which is expected to be obtained by an author's ongoing project, the cusp of M_0 is changed to cone singularity of cone angle $\theta \in (0, 2\pi)$. The cone manifold is denoted by M_θ. One can apply the original Jorgensen theory for four-times punctured sphere Kleinian groups by using the commensurability illustrated in Figure 3. As Variation II, we create an additional cusp at one of the ends, then pass through the boundary of the deformation space of complete hyperbolic structures via cone hyperbolic structures. Variation II is obtained by a joint project with Sakuma, Wada and Yamashita, which is outlined in [2]. The expected Variation III is the combination of Variations

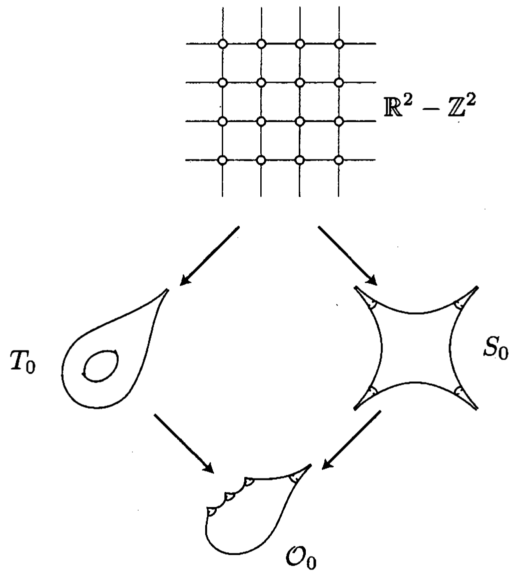

Figure 3: Fricke surfaces

I and II, where the main cusp is changed to cone singularity and pass through the boundary of the deformation space of cone hyperbolic structures via cone hyperbolic structures with additional cone singularity.

3.1 Original theory

The original Jorgensen theory characterizes the combinatorial structures of Ford domains for punctured torus groups. For the once-punctured torus T_0, fix a pair of generators $a, b \in \pi_1(T_0)$ as depicted in Figure 1. A representation $\rho : \pi_1(T_0) \to \mathrm{PSL}_2(\mathbb{C})$ with parabolic $\rho([a,b])$ is always normalized by a conjugation so that $\rho([a,b]) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

The simplest example of the Ford domain is depicted in the left hand side of Figure 4. The visible isometric hemispheres are those of $\rho(a^{\pm 1})$, $\rho(b^{\pm 1})$, $\rho((ab)^{\pm 1})$ and their conjugates by $\rho([a,b]^k)$ $(k \in \mathbb{Z})$. By gluing the pairs of faces of Ford domain, we can obtain a 2-dimensional complex embedded in M_0, which is called the *Ford complex*. For the simplest example, the Ford complex is equal to the product of a spine of T_0 and the interval. A generic Ford domain is depicted in the left hand side of Figure 5. In general, two *terminal* spines for T_0 are determined corresponding to the upper and lower most sequences of faces, and the intermediate faces are described by using the shortest path in the complex of spine of T_0 which

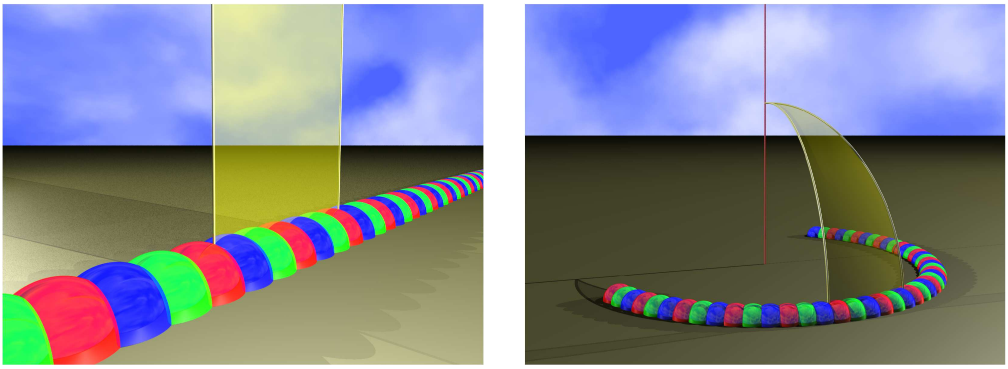

Figure 4: Simplest examples of Ford domains for original and coned case

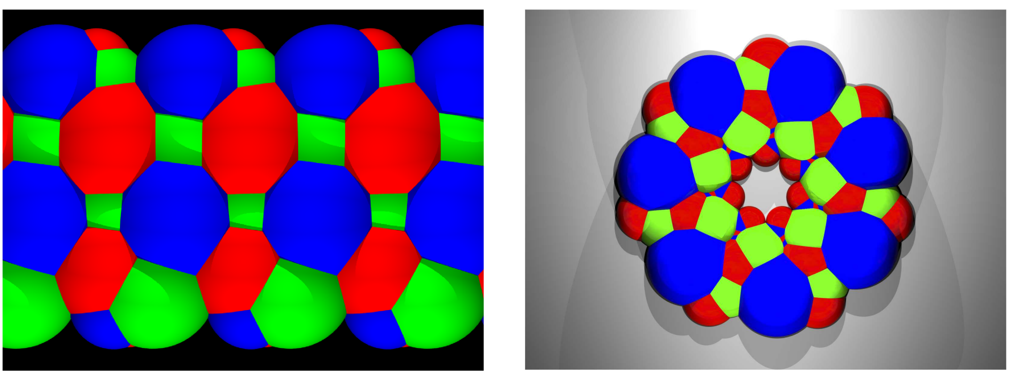

Figure 5: Generic Ford domains for original case and the case for $\theta = 4\pi/7$

connects the terminal spines.

The following theorem is stated in the unfinished manuscript due to Jorgensen [3] (see [2] for the complete proof).

Theorem 3.1. *For any quasifuchsian punctured torus group, the combinatorial structure of its Ford domain is characterized as above.*

The theory gives a practical algorithm for a given representation $\rho : \pi_1(T_0) \to \mathrm{PSL}_2(\mathbb{C})$ with parabolic $\rho([a,b])$ to be discrete or not.

3.2 Variation I

There is a variation corresponding to cone hyperbolic structures on the cone manifold, M_θ, which is obtained by replacing the cusp of M_0 with cone singularity. We remark that it is still conjectural, though there are many numerical results supporting the variation (cf. [1]). We also remark that a certain family of cone hyperbolic structures are concretely constructed by Jorgensen [4]: Each of them has discrete holonomy representation and a fundamental domain is determined. It is used to construct a hyperbolic structure on a closed surface bundle.

The (conjectural) characterization for the Ford domains for cone hyperbolic structures on M_θ under an appropriate finiteness condition is parallel to that for the original case. We only need to modify the description corresponding to the replacement of the cusp with cone singularity (see the right hand sides of Figures 4 and 5). The fundamental polyhedron for M_θ with such combinatorial structure is said to be *good.* There are several conjectures for cone hyperbolic structures on M_θ to have a good fundamental polyhedron (see [1]).

Proposition 3.2. *Suppose that a cone hyperbolic manifold M_θ has a good fundamental polyhedron F. Then F is equal to the Ford domain for M_θ.*

3.3 Variation II

Another variation is introduced in the author's joint work with Sakuma, Wada and Yamashita, whose goal is to construct the hyperbolic structures on hyperbolic two-bridge knot and link complements. For that purpose, we started with the quasifuchsian hyperbolic structures on the product N_0 of four-times punctured sphere S_0, and the interval I. Then observed the characterization is valid even for the *cusp groups* for N_0, which have accidental parabolics at one or two ends. The middle figure in the first row of Figure 6 is a schematic picture of a "double cusp" manifold, the left figure in the second row is a schematic picture of a "single cusp" manifold. A slight modification near the cusp, the Ford domain can be deformed to a polyhedron which produces a complete cone hyperbolic structure on the cone manifold N_θ obtained

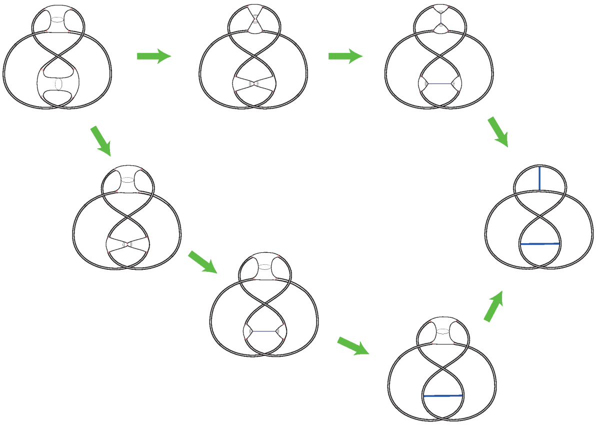

Figure 6: Paths to a two-bridge knot complement

by changing the accidental cusps to cone singularities as depicted in the adjacent pictures. It is proved that the deformation is valid until the cone angles become 2π. The bottom picture illustrates the terminal cone manifold of the deformation from a single cusp manifold: the hyperbolic structure is not singular anymore, as the cone angle becomes 2π. The space of hyperbolic structures obtained in such a way is equal to the Riley slice of Schottky space. We omit a detailed explanation of the deformation of Ford domains, for which please see [2].

As for the fundamental polyhedron introduced in [2], we can show the following.

Proposition 3.3. *The fundamental polyhedron introduced in [2] is the Ford domain for the cone hyperbolic manifolds.*

3.4 Variation III

We expect that there is a variation of the theory for cone manifolds obtained as the mixture of Variations I and II, namely, the cone manifolds obtained by changing the main cusp with cone singularity of cone angle θ and creating accidental cusps then modifying it to additional cone singularity of cone angle α. We denote the cone manifold by $N_\theta(\alpha)$. As a naive expectation, there will be a variation for any $\theta \in (0, 2\pi)$ and $\alpha \in (0, \alpha_\theta)$ for $\alpha_\theta = 2\pi - \theta$. The maximum angle α_θ is crucial for our approach because of the following observation: Figure 7 illustrates the developed image of lifts in Sec_θ to $\mathbb{H}^3$ if a generic Ford domain for M_θ introduced in Variation I (cf. Figure 5). The picture in the right hand side illustrates the dual cell complex to the Ford domain, whose vertices are centers of isometric hemispheres supporting the Ford domain, and two vertices are connected by an edge if the corresponding faces of Ford domain share an edge. Figures 8 and 9 illustrates deformation of

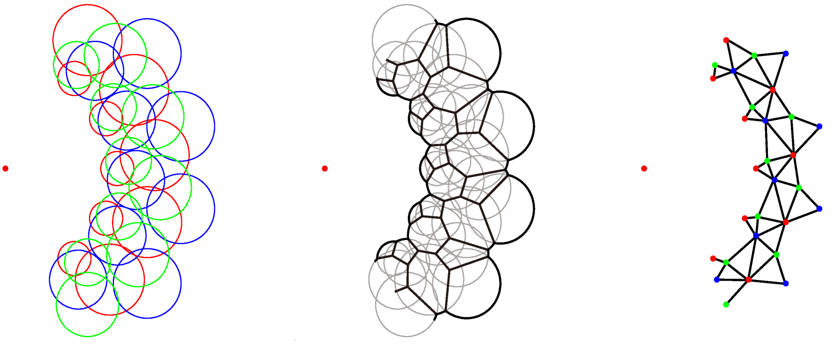

Figure 7: Ford domain for M_θ and N_θ; the developed image of lifts in Sec_θ to $\mathbb{H}^3$

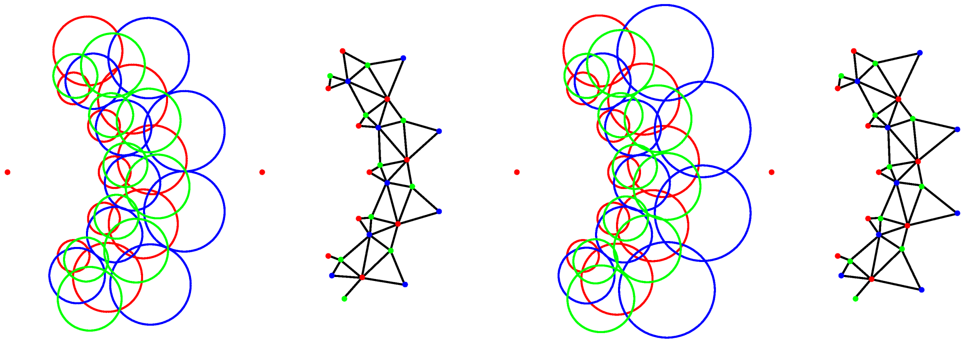

Figure 8: Creation of a cusp and cone singularity; from a limit of N_θ to $N_\theta(\alpha)$

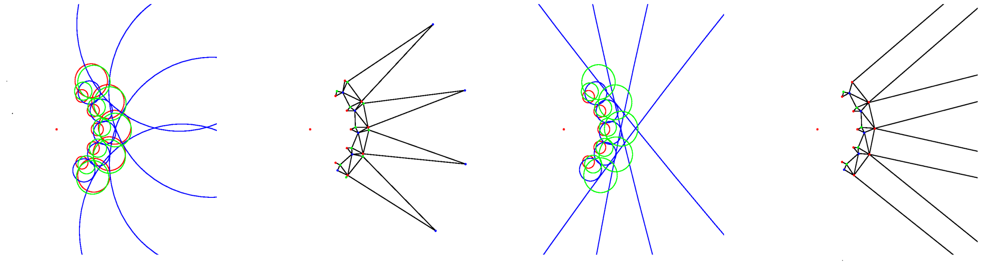

Figure 9: Deformation of $N_\theta(\alpha)$ with $\alpha \to \alpha_\theta$

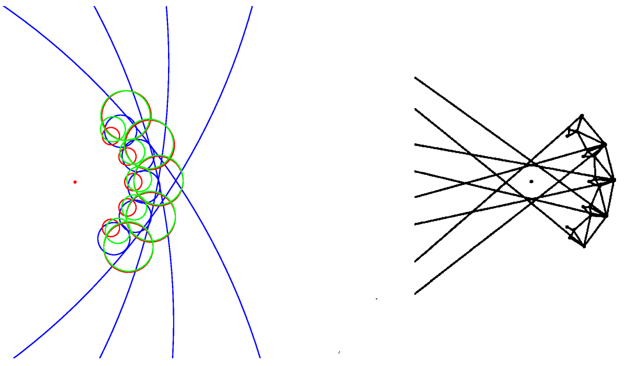

Figure 10: Outermost isometric hemispheres become inside out when $\alpha > \alpha_\theta$

Ford domain where an accidental cusp is created, changed to cone singularity, and then the cone singularity touches ∞ at $\alpha = \alpha_\theta$. In the figures, we omit the detailed information as illustrated in the middle picture of Figure 7. By a deformation, we can produce a cusp as a limit of Variation I as depicted in the two pictures in the left hand side of Figure 8: Pairs of circles contained in the family of outermost circles are tangent to each other, and the points of tangency are fixed points of parabolic isometries corresponding to the new cusp. In the two pictures in the right hand side of Figure 8, each pair of circles become intersect transversely at two points, which are the endpoints of the axis of an elliptic isometry which projects onto the cone singularity. As illustrated in Figure 9, the combinatorial structure of the Ford domain does not change until the radii of outermost circles diverge. The change in combinatorial structure at the limit can be described in a parallel way that is given in [2]. We can show that the cone angle at the limit is equal to α_θ ($< 2\pi$). The rack of angle is related to the cone angle θ, and so it seems that we cannot arrive at the "cone Riley slice" (see the next section 4) by using Ford domains only; in fact, the outermost isometric hemispheres become inside out when the cone angle $\alpha > \alpha_\theta$ (see Figure 10). An approach to overcome this problem will be proposed in Section 5.

4 Cone Riley slice

In [6], Series, Tan and Yamashita study $\mathrm{PSL}_2(\mathbb{C})$-representations of the group $\langle P, Q, R \mid P^2 = Q^2 = R^2 = (PQ)^2 = (RQP)^3 \rangle$. In particular, they describe the shape of Dirichlet domains for the faithful discrete image of such a representation ρ under a specific normalization: The fixed points in $\mathbb{C}$ of $\rho(RQP)$ (resp. $\rho(P)$ and $\rho(Q)$) are $\pm\sqrt{3}$ (resp. contained in the unit circle centered at the origin), and the basepoint is chosen to be the point of intersection of the axes of $\rho(PQ)$ and

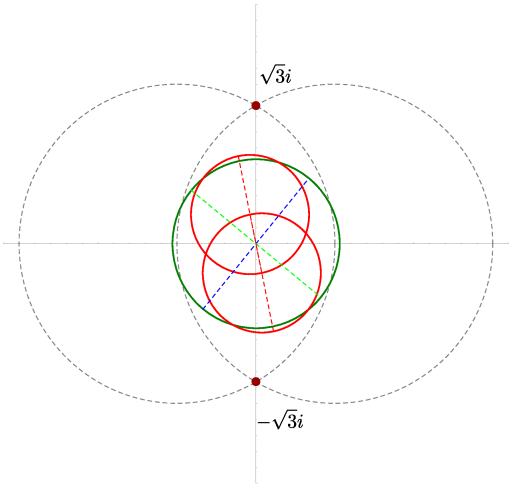

Figure 11: An example of Dirichlet domain studied in [6]; the picture is produced following the algorithm of "PQR.py" by using Mathematica

$\rho(RQP)$. Moreover, Yamashita produced a python code "PQR.py" [7] for viewing many examples; the software works as a simple version of "OPTi" [8] produced by Wada for the original Jorgensen theory. Figure 11 illustrates an example of Dirichlet domain which is displayed by "PQR.py".

In 2014, as a comment to Martin's talk (cf. [5] and Yamashita's picture of the cover page of the volume containing the paper) about searching for small hyperbolic quotients at the seminar in Hiroshima University, Yamashita pointed out that their project concerning [6] is closely related to Martin's ongoing project. In particular, he said that the combinatorial structure of their Dirichlet domain seems (almost) the same as that of the Ford domain for a group in the Riley slice of Schottky space.

For $n \geq 3$, let $\mathcal{R}_n$ be the space of faithful discrete $\mathrm{PSL}_2(\mathbb{C})$-representations of the group $\langle P, Q, R \mid P^2 = Q^2 = R^2 = (PQ)^2 = (RQP)^n \rangle$. We call $\mathcal{R}_n$ a *cone Riley slice.* We can check that a variation of the "chain rule" (cf. Chapter 4 of [2], which is a fundamental property of the original theory) for the bisecting planes. The author is studying $\mathcal{R}_n$ collaborating with Yamashita for solving the following conjecture. We believe that it provides a practical sufficient condition for a given representation being discrete, faithful and with infinite covolume.

Conjecture 4.1. Any element ρ in the interior of $\mathcal{R}_n$ is good, in the sense that the combinatorial structure of the Dirichlet domain for the image of ρ with basepoint at the intersection of the axes of $\rho(PQ)$ and $\rho(RQP)$ is the same after a natural modification as that of a group in the Riley slice of Schottky slice.

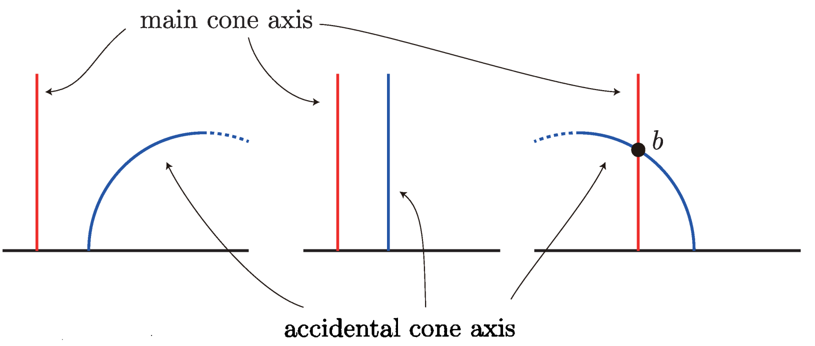

Figure 12: Schematic pictures near $\alpha = \alpha_\theta$

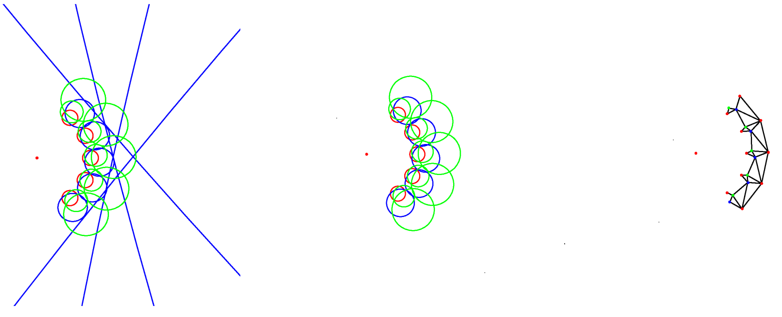

Figure 13: Modification by ignoring vertical planes

5 Path from coned torus to cone Riley slice

As mentioned in Section 3.4, the outermost isometric hemispheres become inside out when the cone angle of accidental cone singularity exceeds α_θ (recall Figures 9 and 10). Figure 12 contains three pictures, which schematically illustrate the configuration of the main and accidental cone singularity in a lift in Sec_θ of the Ford domain; from left to right they correspond to the cases $\alpha < \alpha_\theta$, $\alpha = \alpha_\theta$, $\alpha > \alpha_\theta$, respectively. When $\alpha > \alpha_\theta$, the two axes intersect at a point, b_θ.

We propose to employ the Dirichlet domain with respect to b_θ, for the case $\alpha > \alpha_\theta$. To be precise, the limit of the Ford domains for $N_\theta(\alpha)$ as $\alpha \to \alpha_\theta$ is not the Ford domain for the geometric limit $N_\theta(\alpha_\theta)$, in the sense of our definition, because the limit does not contain an admissible subset. We must extend the definition of Ford domain so that it is well defined even for cone hyperbolic manifolds whose cone singularity contains components that "share an endpoint". For now, we simply ignore the vertical planes which appear in the limit (see Figure 13). Then the polyhedra changes as in Figure 14; from left to right, the first picture is the "modified Ford domain" for $N_\theta(\alpha_\theta)$, the remaining pictures are the Dirichlet domains with

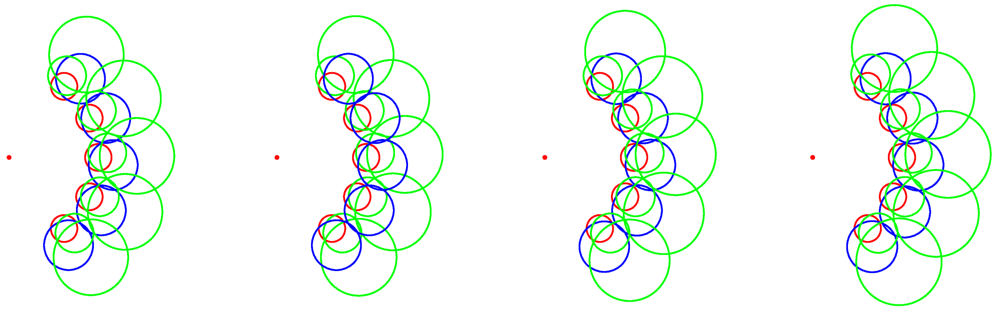

Figure 14: Modified Ford domain and deformation of Dirichlet domain

respect to b_θ for $\alpha > \alpha_\theta$, and the fourth picture is for $\alpha = 2\pi$. We remark that the final picture coincides with the Dirichlet domain for a group in the cone Riley slice introduced in [6]. The pictures in Figures 11 and 14 seem quite different, because the normalizations are different. We can check we can obtain one picture from the other by changing the normalization properly.

We close the note with a question.

Question 5.1. Is there a natural path which connects cone Riley slices and cone manifolds whose singular locus is the two-bridge knots and links?

References

[1] Towards the variation of Jorgensen's theory for the torus with a single cone point, RIMS Kôkyûroku No.1936 (2015), 120–131.

[2] H. Akiyoshi, M. Sakuma, M. Wada, Y. Yamashita, Punctured torus groups and 2-bridge knot groups I, Lecture Notes in Mathematics, **1909**. Springer, Berlin, 2007.

[3] T. Jørgensen, On pairs of once-punctured tori, Kleinian groups and hyperbolic 3-manifolds (Warwick, 2001), 183–207, London Math. Soc. Lecture Note Ser., **299**, Cambridge Univ. Press, Cambridge, 2003.

[4] T. Jørgensen, Compact 3-manifolds of constant negative curvature fibering over the circle, Ann. of Math. (2) **106** (1977), no. 1, 61–72.

[5] G. J. Martin, Siegel's Problem in Three Dimensions, Notices of the AMS, **63** (2016), no. 11, 1244–1247.

[6] C. Series, S. P. Tan, Y. Yamashita, The diagonal slice of Schottky space, arXiv:1409.6863v1.

[7] Y. Yamashita, "PQR.py", a python code, available at
`http://vivaldi.ics.nara-wu.ac.jp/~yamasita/DiagonalSlice/`

[8] M. Wada, "OPTi", a software for MacOS, available from the Mac App Store.