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On some demonstrative embeddings into higher dimensional Thompson groups

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1 Intoduction

The Thompson group V is an infinite, simple and finitely presented group, described as a subgroup of the homeomorphism group of the Cantor set C. Brin [1] defined n-dimensional Thompson group nV for all natural number $n \geq 1$, where $1V = V$. Brin [1] showed that V and $2V$ are not isomorphic. Bleak and Lanoue [3] showed n_1V and n_2V are isomorphic if and only if $n_1 = n_2$.

V contains many groups, such as all finite groups and free groups, as its subgroups. The class of subgroups of V are closed under taking the direct product of finitely many members. However, the class is not closed under taking the free products. Bleak and Salazar-Díaz [4] proved that $\mathbb{Z}^2 * \mathbb{Z}$ does not embed in V, although there are many embeddings of $\mathbb{Z}$ and $\mathbb{Z}^2$ in V. They defined a class of well-behaved subgroups of V, demonstrative subgroups, and showed that the free product of two demonstrative subgroups can be embedded into V. It follows that any embedded $\mathbb{Z}^2$ in V is not demonstrative.

Recently, Corwin and Haymaker [5] determined which right-angled Artin groups embed into V. Belk, Bleak and Matucci [2] showed that every right-angled Artin group and its finite extensions embed into nV with sufficiently large n.

In this paper, we consider embeddings of right-angled *Coxeter* groups into higher dimensional Thompson groups. It follows from the result of [2] that every right-angled Coxeter group embeds into some nV. We explicitly construct demonstrative embeddings of each right-angled Coxeter group into nV, where n is the number of "complementary edges" in the defining graph.

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2 Right-angled Artin groups and right-angled Coxeter groups

Let Γ be a finite graph with a vertex set $V(\Gamma) = \{v_i\}_{1\leq i\leq m}$ and an edge set $E(\Gamma)$. Let

$$\bar{E}(\Gamma) = \{\{v_i, v_j\} \mid v_i \neq v_j \in V(\Gamma) \text{ are not connected by edges.}\}$$

We call the elements of $\bar{E}(\Gamma)$ *complementary edges.*

The *right-angled Artin group* corresponding to Γ, denoted by A_Γ, is a group defined by the presentation

$$A_\Gamma = \langle g_1, \ldots, g_m \mid g_i g_j = g_j g_i \text{ for all } \{v_i, v_j\} \in E(\Gamma)\rangle.$$

The *right-angled Coxeter group* corresponding to Γ, denoted by W_Γ, is a group defined by the presentation

$$W_\Gamma = \langle g_1, \ldots, g_m \mid {g_i}^2 = 1, g_i g_j = g_j g_i \text{ for all } \{v_i, v_j\} \in E(\Gamma)\rangle.$$

For example, $\mathbb{Z}^2 * \mathbb{Z}$ is a right-angled Artin group corresponding to the graph with three vertices and an edge.

To construct embeddings of free groups, the ping-pong lemma of F. Klein is known to be a useful tool. Besides the standard one, there is also the ping-pong lemma for right-angled Artin groups ([8]). It might be helpful to state a version for right-angled Coxeter groups here.

Lemma 2.1. *Let W_Γ be a right-angled Coxeter group with generators $\{g_i\}_{1\leq i\leq m}$ acting on a set X. Suppose that there exist subsets S_i $(1 \leq i \leq m)$ of X, satisfying the following conditions:*

(1) *If g_i and g_j $(i \neq j)$ commute, then $g_i(S_j) = S_j$.*

(2) *If g_i and g_j do not commute, then $g_i(S_j) \subset S_i$.*

(3) *There exists $x_0 \in X - \bigcup_{i=1}^m S_i$ such that $g_i(x_0) \in S_i$ for all i.*

Then this action is faithful.

Proof. In the following, we assume that the action is a left action. We identify words and the group elements. A prefix w_1 for a word w is a subword such that $w = w_1 w_2$ as words, for some subword w_2.

Let w be a nonempty reduced word of $\{g_1, \ldots, g_n\}$. We claim that $w(x_0) \in S_j$ for some j, and w has a prefix of the form $w_1 g_j$, where w_1 is either empty or a word of generators commuting with g_j.

We show the claim by induction on the length of w. The base case is ensured by the condition (3). We suppose that the claim holds true for reduced words with length less than l. Let $w = g_k w'$ be a reduced word of length l. By the induction hypothesis, there is some j such that $w'(x_0) \in S_j$. There is a prefix for w' of the form $w'_1 g_j$ where w'_1 is either empty or a word of generators commuting with g_j.

We first consider the case where $k \neq j$. If g_k and g_j commute, $w(x) = g_k w'(x) \in S_j$, by condition (1). There is a prefix $w_1 g_j$ for w, where $w_1 = g_k w'_1$. If g_k and g_j do not commute, $w(x) = g_k w'(x) \in S_k$, by condition (2). There is a prefix g_k of w.

Next we consider the case when $k = j$. However this case does not happen, because the reduced word w cannot have a prefix of the form $g_j w'_1 g_j$. Therefore, the claim holds true also in the case of $|w| = l$.

We have shown that $w(x_0) \neq x_0$ for any nontrivial $w \in W_\Gamma$. Therefore, the action W_Γ on X is faithful. □

3 Demonstrative embeddings into higher dimensional Thompson groups

Now we focus on the Thompson group V and its generalizations. The subgroup structure of V is not well understood. It is known that V contains free groups and many free products of its subgroups. On the other hand, there is a nonembedding result on the free product of subgroups of V.

Theorem 3.1 ([4], Theorem 1.5). *The group $\mathbb{Z}^2 * \mathbb{Z}$ does not embed in V.*

This free product is the only obstruction for right-angled Artin groups to embed into V.

Theorem 3.2 ([5]). *A right-angled Artin group A_Γ embeds into V if and only if $\mathbb{Z}^2 * \mathbb{Z}$ does not embed into A_Γ.*

In the following, we consider embeddings of right-angled Artin groups and right-angled Coxeter groups into higher dimensional Thompson groups.

We describe the definition of higher dimensional Thompson groups with notations in [1]. The symbol I denotes the half-open interval $[0, 1)$. An *n-dimensional rectangle* is an affine copy of I^n in I^n, constructed by repeating "dyadic divisions". An *n-dimensional pattern* is a finite set of n-dimensional rectangles, with pairwise disjoint, non-empty interiors and whose union is I^n. A *numbered pattern* is a pattern with a one-to-one correspondence to $\{0, 1, \dots, r-1\}$, where r is the number of rectangles in the pattern.

Let $P = \{P_i\}_{0\leq i\leq r-1}$ and $Q = \{Q_i\}_{0\leq i\leq r-1}$ be numbered patterns of the same dimension, containing the same number of rectangles in each. We define $v(P, Q)$ to be a map from I^n to itself which takes each P_i onto Q_i affinely so as to preserve the orientation.

The *n-dimensional Thompson group nV* is the group which consists of maps with the form $v(P, Q)$, where P and Q are the n-dimensional numbered patterns. The definition of $1V$ is equivalent to the definition of V.

Theorem 3.3 ([2], Theorem 1.1 and Corollary 1.3). *For every finite graph Γ, the right-angled Artin group A_Γ embeds into nV, where $n = |V(\Gamma)| + |\bar{E}(\Gamma)|$. Furthermore, every finite extension of A_Γ embeds into nV.*

By Theorem 3.3 and the fact that every right-angled Artin group is contained in some right-angled Coxeter group as a finite index subgroup [6], it follows that every right-angled Coxeter group embeds into some higher-dimensional Thompson group.

The following is the main result of this paper.

Theorem 3.4. *Let Γ be a graph with the vertex set $V(\Gamma) = \{v_i\}_{1\leq i\leq m}$. Suppose that there are nonempty subsets $\{D_i\}_{1\leq i\leq m}$ of $\{1, \dots n\}$, such that $D_i \cap D_j = \emptyset$ if and only if v_i and v_j are connected by an edge.*

(1) *The right-angled Artin group A_Γ embeds into nV.*

(2) *The right-angled Coxeter group W_Γ embeds into nV.*

Compared to Theorem 3.3, we get a better estimate for the dimension of the Thompson groups which contain A_Γ. We construct embeddings of right-angled Coxeter groups into higher-dimensional Thompson groups explicitly.

For the proof of Theorem 3.4, we borrow some notations and a lemma from [7]. For a nonempty subset D of $\{1, \dots, n\}$, a *D-slice* of I^n is an n-dimensional rectangle $S = \prod_{d=1}^{n} I_d$, where $d \in D$ if and only if I_d is properly contained in $[0, 1)$.

Lemma 3.5. *For nonempty subsets $\{D_i\}_{1\leq i\leq m}$ of $\{1, \dots, n\}$, we may take a set of n-dimensional rectangles $\{S_i\}_{1\leq i\leq m}$ satisfying*

(1) *For every i, S_i is a D_i-slice of I^n.*

(2) *$S_i \cap S_j = \emptyset$ if and only if $D_i \cap D_j \neq \emptyset$.*

(3) $\bigcup_{i=1}^{m} S_i \subsetneq I^n$.

Proof of Theorem 3.4. The proof for right-angled Artin groups is given in [7]. Here we state the proof only for right-angled Coxeter groups.

We take $\{S_i\}_{1\leq i\leq m}$ with respect to given $\{D_i\}_{1\leq i\leq m}$, according to Lemma 3.5. Let $g_i \in nV$ be a map which permute S_i and $[0,1)^n - S_i$.

We may take g_i as to change d-th coordinate of $[0,1)^n$ only if $d \in D_i$. That is, when we write $g_i(x) = g_i((x_d)_{1\leq d\leq n}) = (g_{i,d}(x))_{1\leq d\leq n}$, $g_{i,d}(x) \neq x_d$ only if $d \in D_i$. With this assumption, g_i and g_j commute when v_i and v_j are connected by an edge. Therefore, we may define a group homomorphism $\phi : W(\Gamma) \to nV$ by $\phi(v_i) = g_i$. Here, we are using the same symbols for the vertices of Γ and the corresponding generators of W_Γ.

If g_i and g_j $(i \neq j)$ commute, then $D_i \cap D_j = \emptyset$. In this case, S_j is determined only by d-th coordinates for $d \in D_j$, which are unchanged by g_i. Therefore $g_i(S_j) = S_j$, and the condition (1) in Lemma 2.1 is satisfied.

If g_i and g_j do not commute, S_i and S_j are disjoint. Therefore. $g_i(S_j) \subset g_i([0,1)^n - S_i) \subset S_i$, and the condition (2) in Lemma 2.1 is satisfied.

Condition (3) in Lemma 2.1 follows from the third assumption for $\{S_i\}_{1\leq i\leq m}$ in Lemma 3.5. □

We note that Theorem 3.4 does not give the best estimate for dimensions of the Thompson groups which contain W_Γ. For Γ with $|E(\Gamma)| \geq 1$, we need two or more dimensions to realize the conditions required in Theorem 3.4. On the other hand, many W_Γ with $|E(\Gamma)| \geq 1$ can be embedded into V. The argument of demonstrative subgroups in [4] is useful to get examples of such embeddings.

Suppose that a group G acts on a space X. A subgroup H of G is *demonstrative over* X if there is an open set $U \subset X$ so that for any two elements g_1, $g_2 \in G$, $g_1U \cap g_2U \neq \emptyset$ if and only if $g_1 = g_2$. We call U a *demonstration set.*

By definition, there is a canonical action of V on the half open interval I. Instead of this action, sometimes we consider the action of V on the Cantor set C. We identify I with the Cantor set C: the dyadic division of I corresponds to trisecting the unit interval and then taking two of them to produce open sets of C.

There are demonstrative subgroups of V over C, isomorphic to all finite groups and $\mathbb{Z}$. The class of demonstrative subgroups of V over C is closed under taking subgroups, and taking the direct product of any finite member with any member.

There is an embedding result on the free product of demonstrative subgroups.

Theorem 3.6 ([4], Theorem 1.4). *If groups K_1 and K_2 are isomorphic to some demonstrative subgroups of V over C, then $K_1 * K_2$ embeds in V.*

According to this result, for example, we may embed free products of finite groups such as a right-angled Coxeter group $(\mathbb{Z}_2 \times \mathbb{Z}_2) * \mathbb{Z}_2$ into V.

Remark 3.1. *We identify I with C, and consider the canonical action of nV on C^n. Theorem 3.4 gives demonstrative subgroups of nV over C^n. For each subgroup, any open set in C^n which corresponds to n-dimensional rectangles in $[0,1)^n - \cup_{i=1}^m S_i$ is the demonstration set.*

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