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Volume conjecture for various quantum $SO(3)$ invariants

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Introduction

After the discovery of the Jones polynomial, various quantum invariants of knots are introduced. R. Kashaev introduced knot invariants by using the quantum dilogarithm function, and observed that the hyperbolic volume of the knot complement can be obtained as a certain limit of his invariants. Kashaev's invariants turned out to be a special version of the colored Jones invariants, and we got a new path to investigate the geometric structure of a knot complement from quantum invariants. The volume conjecture is this relation between the hyperbolic volume and the colored Jones invariant. It is not proved yet, but it looks very natural because the classical dilogarithm function has a strong relation to the hyperbolic volume.

The colored Jones invariant is generalized to the Witten-Reshetikhin-Turaev invariant of 3 manifolds, but the volume conjecture did not extend to the Witten-Reshetikhin-Turaev invariant somehow. In 2015, Q. Chen and T. Yang observed that, if r is odd, the hyperbolic volume of certain closed 3 manifold is obtained by the Witten-Reshetikhin-Turaev invariant like the volume conjecture if the quantum parameter q, which is fixed as the first root of unity $\xi_r = \exp(2\pi i/r)$ for some positive integer r, is replaced by the second root of unity $\xi_r^2 = \exp(4\pi i/r)$.

On the other hand, R. Kirby and P. Melvin introduced in [8] the quantum $SO(3)$-invariant of 3 manifolds for odd r. The colored Jones invariant corresponds to a highest weight representation of $\mathcal{U}_q(sl_2)$, and the Witten-Reshetikhin-Turaev invariant is given for a positive integer r by a linear combination of some colored Jones invariants where q is specialized to the first r-th root of unity $\xi_r = \exp(2\pi i/r)$. In the construction of the Witten-Reshetikhin-Turaev invariant, by taking the linear combination of the colored Jones invariants corresponding to the odd dimensional representations of $\mathcal{U}_q(sl_2)$, which is factored by $SO(3)$ by the natural 2-fold covering from $SU(2)$ to $SO(3)$.

In this report, I would like to explain variations of the volume conjecture corresponding to various quantum $SO(3)$ invariants.

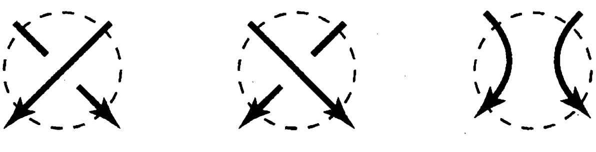
1. Quantum $SO(3)$ invariants

For a knot K, the Jones invariant $V_K(q)$ of K is determined inductively by the following skein relation. The three knots K_+, K_- and K_0

are identical outside the region where they are given in Figure 1.

$$q^{-1}\, V_{K_+}(q) - q\, V_{K_-}(q) = (q^{1/2} - q^{-1/2})\, V_{K_0}(q),$$

$$V_{\underbrace{\bigcirc\,\bigcirc\cdots\bigcirc}_{k}}(q) = (-q^{1/2} - q^{-1/2})^k.$$



K_+ $\qquad$ K_- $\qquad$ K_0

FIGURE 1. Three knots K_+, K_- and K_0 in the skein relation

The Jones polynomial is also constructed by the quantum R matrix corresponding to the vector representation of the quantum group $\mathcal{U}_q(sl_2)$. This construction is easily generalized to any irreducible representation of any quantum group obtained from a semisimple Lie algebra. The invariant coming from an irreducible representation of $\mathcal{U}_q(sl_2)$ is called **the colored Jones invariant**, which is denoted by $V_K^{(l)}(q)$, where l is the dimension of the corresponding representation. For a link L whose components are $L_1, L_2, \cdots, L_p$, we also have the colored Jones invariant $V_L^{(l_1,l_2,\cdots,l_p)}(q)$ where the component L_j is attached by the l_j dimensional representation.

Let M be a closed 3 manifold given by the surgery along a framed link $L = L_1 \cup L_2 \cup \cdots \cup L_p \subset S^3$. Then, for a positive integer r, the Witten-Reshetikhin-Turaev invariant $\tau^{(r)}(M)$ is given by the following.

$$\overline{\tau}^{(r)}(M) = \sum_{l_1=1}^{r-1} \cdots \sum_{l_p=1}^{r-1} d_q(l_1) \cdots d_q(l_p)\, V_L^{(l_1,\cdots,l_p)}(q)$$

where $q = \exp(2\pi i/r)$ and $d_q(l) = (-1)^{l-1} \frac{q^{l/2}-q^{-l/2}}{q^{1/2}-q^{-1/2}}$, and

$$\tau^{(r)}(M) = \overline{\tau}^{(r)}(U_+)^{s_+}\, \overline{\tau}^{(r)}(U_-)^{s_-}\, \overline{\tau}^{(r)}(M),$$

where $U_\pm$ is a trivial knot with ± 1 framing and s_+ (s_-) is the number of positive (negative) eigenvalues of the linking matrix of L.

Now let us introduce the quantum $SO(3)$ invariant $\tau_{SO(3)}^{(r)}$. Let r be a positive odd integer and

$$\overline{\tau}_{SO(3)}^{(r)}(M) = \sum_{l_1=1}^{\frac{r-1}{2}} \cdots \sum_{l_p=1}^{\frac{r-1}{2}} d_q(2l_1 - 1) \cdots d_q(2l_p - 1)\, V_L^{(2l_1-1,\cdots,2l_p-1)}(q),$$

and then define $\tau^{(r)}_{SO(3)}$ from $\overline{\tau}^{(r)}_{SO(3)}$ as before. Then Kirby-Melvin [8] shows that $\tau^{(r)}_{SO(3)}(M)$ is an invariant of M, and $\tau^{(r)}_{SO(3)}(M)$ is called the $SO(3)$ quantum invariant since irreducible representations of odd dimensions are factored by $\mathcal{U}_q(so_3)$. Such $SO(3)$ invariant can be defined not only for a closed 3 manifold but also for a spatial graph in any closed 3 manifold.

We also have $SO(3)$ version of the Turaev-Viro invariant of a 3 manifold, which is constructed by a tetrahedral decomposition of the manifold and quantum 6j symbols assigned to the each tetrahedron. Let $\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}^{RW}_q$ be the Racar-Wigner form of the quantum $6j$-symbol introduced by Kirillov and Reshetikhin in [9]. Then, for a positive integer r, the Turaev-Viro invariant of a 3 manifold M with tetrahedral decomposition $\mathcal{T}$ is given by

$$\mathrm{TV}^{(r)}(M) = \sum_{\text{coloring of edges}} \left(\prod_e d(e) \prod_t \{t\}^{\mathrm{RW}}_{\xi_r} \right),$$

where the coloring of edges means to assign a positive integer between 1 and $r-2$ to each edges of $\mathcal{T}$ satisfying the following condition for colors a, b, c of three edges around any face of $\mathcal{T}$,

$$a+b+c = \text{odd}, \qquad 0 \leq |a-b| < c < a+b,$$

t runs over all tetrahedra in $\mathcal{T}$, $\{t\}$ is the quantum $6j$-symbol corresponding to the six colors around T, e runs over all edges and $d(e)$ is the quantum dimension, which is the quantum integer corresponding to the color of e. For odd r, we can introduce the $SO(3)$ version of $TV^{(r)}$ by restricting the coloring of edges to odd integers as follows.

$$\mathrm{TV}^{(r)}_{SO(3)}(M) = \sum_{\text{odd coloring of edges}} \left(\prod_e d(e) \prod_t \{t\}^{\mathrm{RW}}_{\xi_r} \right).$$

2. Volume conjecture for knots and links

Kashaev introduced new knot invariant K_r for any positive integer r and found that the hyperbolic volume of the complement of some simple hyperbolic knots can be obtained from his invariants in [6]. H. Murakami and the author found in [12] that Kashaev's invariants are equal to the colored Jones invariants $V^{(r)}_K(\xi_r)$ where $\xi_r = \exp(2\pi i/r)$, the first r-th root of unity, and generalize Kashaev's observation as follows.

Volume Conjecture. For a knot K in S^3,

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| V_K^{(r)}(\xi_r) \right| = v_3 \, ||S^3 \setminus K||, \tag{1}$$

where $||M||$ is Gromov's simplicial volume of a 3 manifold M and v_3 is the hyperbolic volume of the regular ideal tetrahedron in the hyperbolic 3 space. If M is a hyperbolic manifold, then $v_3 \, ||M||$ is equal to the hyperbolic volume of M.

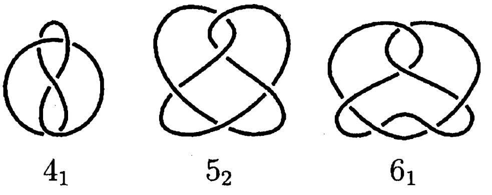


FIGURE 2. The three simplest knots 4_1, 5_2 and 6_1

Kashaev checked such relation for the three simplest hyperbolic knots 4_1, 5_2 and 6_1 as follows. The invariants of these knots are

$$V_{4_1}^{(r)}(\xi_r) = \sum_{j=0}^{r-1} |(\xi_r)_j|^2, \qquad (x)_j = \prod_{k=1}^{j} (1 - x^k),$$

$$V_{5_2}^{(r)}(\xi_r) = \sum_{0 \le j \le k \le r-1} \frac{\xi_r^{-j(j+1)/2} (\xi_r)_k^2}{(\xi_r^{-1})_k},$$

$$V_{6_1}^{(r)}(\xi_r) = \sum_{0 \le j+k \le l \le r-1} \frac{\xi_r^{(l-k-1)(l-k+1)/2} |(\xi_r)_l|^2}{(\xi_r)_j (\xi_r^{-1})_k},$$

and numerical computation shows that

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| V_K^{(r)}(\xi_r) \right| = 2.02988321\ldots \, ,$$

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| V_K^{(r)}(\xi_r) \right| = 2.82812208\ldots \, ,$$

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| V_K^{(r)}(\xi_r) \right| = 3.16396322\ldots \, .$$

The righthand side numbers coincide with the hyperbolic volumes of the complements of these knots. For the figure-eight knot 4_1, the above equality is actually proved by using the standard calculus. For other knots, to prove this conjecture is not so easy, but it is proved for some simple hypergolic knots in [14] and [16].

The colored Jones invariant $V_K^{(r)}(\xi_r)$ is a complex number and it is natural to consider about the meaning of the imaginary part or the argument. On the other hand, in the study of the hyperbolic volume, it turned out that the Chern-Simons invariant must be the imaginary part of the hyperbolic volume, and now we call $\mathrm{Vol}(M) + i\,\mathrm{CS}(M)$ the complex volume of a hyperbolic manifold M. Actually, if we don't take the modulus in (1), we get the following conjecture which is proposed in [13].

Complexified Volume Conjecture. For a hyperbolic knot K,

$$\lim_{r\to\infty} \frac{2\pi}{r} \log V_K^{(r)}(\xi_r) = \mathrm{Vol}(S^3 \setminus K) + i\,\mathrm{CS}(S^3 \setminus K) \pmod{i\,\pi^2\,\mathbf{Z}}.$$

Here we are interested in the growth rate of $V_K^{(r)}(\xi_r)$ and so the imaginary part of the limit means

$$\lim_{r\to\infty} 2\pi \arg \frac{V_K^{(r+1)}(\xi_{r+1})}{V_K^{(r)}(\xi_r)}.$$

3. Chen-Yang's observation

The colored Jones invariants of knots and links are extended to the Witten-Reshetikhin-Turaev invariant of 3 manifolds. But, for a 3 manifold M, we have

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \tau^{(r)}(M) = 0$$

and the analogy of the volume conjecture for knots and links does not hold for the Witten-Reshetikhin-Turaev invariant. However, Q. Chen and T. Yang found in [2] that the analogy of the volume conjecture seems to hold for $\tau^{(r)}(M)$ if r is odd and the parameter q is replaced by the second r-th root of unity ξ_r^2 instead of the first r-th root of unity $\xi_r = \exp(2\pi i/r)$.

Chen-Yang's Conjecture. Let M be a 3 manifold. Then

$$\lim_{n\to\infty} \frac{4\pi}{2n+1} \log \tau^{(2n+1)}(M)\Big|_{\xi_{2n+1}\to\xi_{2n+1}^2} = \mathrm{Vol}(M) + i\,\mathrm{CS}(M) \pmod{i\,\pi^2\,\mathbf{Z}}.$$

They numerically checked the above for 3 manifolds obtained by some integral surgeries along the knots 4_1 and 5_2. T. Ohtsuki announced in [15] a proof for this conjecture for 3 manifolds obtained from the integral surgeries along 4_1.

Chen and Yang also investigated the Turaev-Viro invariant [22], which is defined by using a tetrahedral decomposition of a 3 manifold. For a closed 3 manifold M, let $\mathrm{TV}^{(r)}(M)$ be the Turaev-Viro invariant of M. Then, it is known by [19] that

$$\mathrm{TV}^{(r)}(M) = \left|\tau^{(r)}(M)\right|^2.$$

This relation is proved for the case $q = \xi_r$, but the proof works well for the case $q = \xi_r^2$ if r is odd since ξ_r^2 is also a primitive r-th root of unity. Then Chen-Yang's conjecture implies that

$$\lim_{n\to\infty} \frac{2\pi}{2n+1} \log \left|\mathrm{TV}^{(2n+1)}(M)\right|_{\xi_{2n+1}\to\xi_{2n+1}^2} = \mathrm{Vol}(M).$$

The Turaev-Viro invariant is generalized by Benedetti and Petronio [1] to a 3 manifold M with cusp boundary or totally geodesic boundary decomposed into ideal tetrahedra or truncated tetrahedra which is based on the Pachner move [18] generalized for such decomposition by J. Roberts. We denote this invariant by $\mathrm{BP}^{(r)}(M)$. Detcyerry and Yang pointed out the following in [23].

Theorem (Detcyerry-Yang). Let $M = S^3 \setminus K$ for a knot K in S^3. Then

$$\mathrm{BP}^{(r)}(M) = \mu_r \sum_{n=0}^{r-2} \left|J_K^{(n)}(\xi_r)\right|^2,$$

where μ_r is a complex number which only depend on r and not on K.

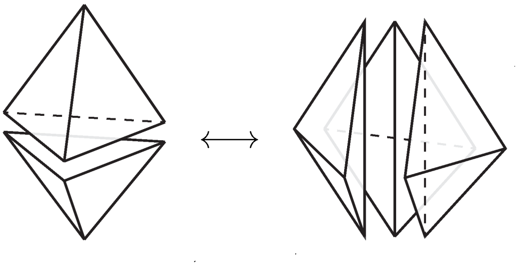


FIGURE 3. The 2$-$3 Pachner move

The first example of such decomposition is the complement of the figure-eight knot 4_1 decomposed into two regular ideal hyperbolic tetrahedra. Let M_{4_1} denote such complement. Then

$$\mathrm{BP}^{(r)}(M_{4_1}) = \sum_{a,b=0}^{(r-3)/2} d(2a+1)\,d(2b+1) \begin{Bmatrix} a & a & b \\ b & b & a \end{Bmatrix}_{\xi_r}^{RW} \begin{Bmatrix} a & a & b \\ b & b & a \end{Bmatrix}_{\xi_r}^{RW}.$$

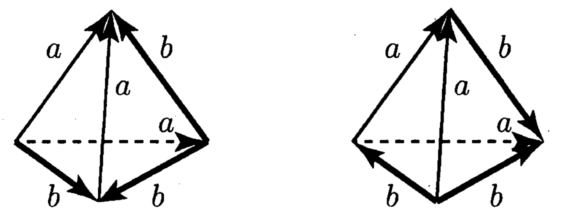


FIGURE 4. Decomposition of the figure-eight knot complement

The second example is the smallest hyperbolic 3 manifold with totally geodesic boundary $M_{\min}$, which is classified by Fujii [5]. This manifold is obtained by two truncated regular tetrahedra whose dihedral angles are all equal to $\pi/6$. All edges are glued to one edge and $\mathrm{BP}^{(r)}(M_{\min})$ is given by

$$\mathrm{BP}^{(r)}(M_{\min}) = \sum_{a=0}^{(r-3)/2} d(2a+1)\left(\begin{Bmatrix} a & a & a \\ a & a & a \end{Bmatrix}_{\xi_r}^{RW}\right)^2 .$$

For Benedetti-Petronio invariant, Chen and Yang observed its relation to the complex volume, and propose the following conjecture.

Volume Conjecture for Benedetti-Petronio invariant (Chen-Yang). Let M be a hyperbolic manifold with cusp or totally geodesic boundary and r be an odd positive integer, then

$$\lim_{n\to\infty} \frac{2\pi}{2n+1} \log \mathrm{BP}^{(2n+1)}(M)\Big|_{\xi_{2n+1}\to\xi_{2n+1}^2} = \mathrm{Vol}(M) + i\,\mathrm{CS}(M) \pmod{i\,\pi^2\,\mathbf{Z}}$$

4. Various conjectures

Chen-Yang's conjecture is generalized to various quantum $SO(3)$ invariants.

4.1. $SO(3)$-version of the Witten-Reshetikhin-Turaev invariant. Let M be a closed oriented three manifold. For a positive odd integer r greater than or equal to 3, let $\tau^{(r)}_{SO(3)}(M)$ be the $SO(3)$-version of the Witten-Reshetikhin-Turaev invariant of M introduced in [8], and let $\tilde{\tau}^{(r)}_{SO(3)}(M) = \tau^{(r)}_{SO(3)}(M)\Big|_{\xi_r\to\xi_r^2}$ be its modified $SO(3)$-version. Then the following may holds.

Conjecture 1 ($SO(3)$-version of Chen-Yang's conjecture). Let M be a closed oriented hyperbolic three manifold. Then

$$4\pi \lim_{n\to\infty} \frac{\log \widetilde{\tau}^{(2n+1)}_{SO(3)}(M)}{2n+1} = \mathrm{Vol}(M) + i\,\mathrm{CS}(M). \quad (\mathrm{mod}\ i\,\pi^2\,\mathbf{Z}).$$

Now we compute the modified Witten-Reshetikhin-Turaev invariant of a three manifolds M^f which is obtained from the Dehn surgery of the figure eight knot K with framing f. We use the following notations.

$\{n\} = \xi_r^n - \xi_r^{-n}$, $\{n,k\} = \{n\}\{n-1\}\cdots\{n-k+1\}$, $\{n\}! = \{n,n\}$.

Then the invariant $\widetilde{\tau}^{(r)}_{SO(3)}(M^f)$ is given as follows.

$$\begin{aligned}\widetilde{\tau}^{(r)}_{SO(3)}(M^f) &= \sum_{n=0}^{(r-3)/2} q^{n^2+n}\,\frac{\{2n+1\}}{\{1\}}\,V_n^{(r)}(K)\\ &= \sum_{n=0}^{(r-3)/2} q^{n^2+n}\,\frac{\{2n+1\}}{\{1\}}\sum_{k=0}^{2j}\frac{\{2n+1+k,2k+1\}}{\{1\}}.\end{aligned} \tag{2}$$

By using this formula, values of $\frac{4\pi}{r}\log\left|\widetilde{\tau}^{(r)}_{SO(3)}(M^f)\right|$ are given by the graph in Figure 5. The Chern-Simons part of the conjecture is checked by computing the values $2\pi\arg\frac{\widetilde{\tau}^{(r)}_{SO(3)}(M^f)}{\widetilde{\tau}^{(r-2)}_{SO(3)}(M^f)}$ mod π^2, which are given by the graph in Figure 6. The signature of the Chern-SImons invariant is ambiguous because the the relation of the orientation of the manifold and the choice of q or q^{-1} is not fixed commonly, and here the signature of CS and cs are opposite. The argument of such ratio seems to estimate the Chern-Simons invariant well for hyperbolic manifolds (Cf. computation in [13] also estimate the Chern-Simons invariant well for knots).

4.2. **Turaev-Viro invariant.** Let M be a closed oriented 3-manifold and $\mathrm{TV}^{(r)}_{SO(3)}(M)$ be the $SO(3)$-version of the Turaev-Viro invariant in [22] given by the quantum $6j$ symbol in the last section. Let $\widetilde{\mathrm{TV}}^{(r)}_{SO(3)}(M)$ be the modified invariant which is obtained by replacing ξ_r by ξ_r^2. Then, from the relation $\left|\widetilde{\tau}^{(r)}_{SO(3)}(M)\right|^2 = \widetilde{\mathrm{TV}}^{(r)}_{SO(3)}(M)$ and Conjecture 1, we have

Conjecture 2. For a hyperbolic closed oriented three manifold M,

$$2\pi \lim_{n\to\infty} \frac{\log\left|\widetilde{\mathrm{TV}}^{(2n+1)}_{SO(3)}(M)\right|}{2n+1} = \mathrm{Vol}(M).$$

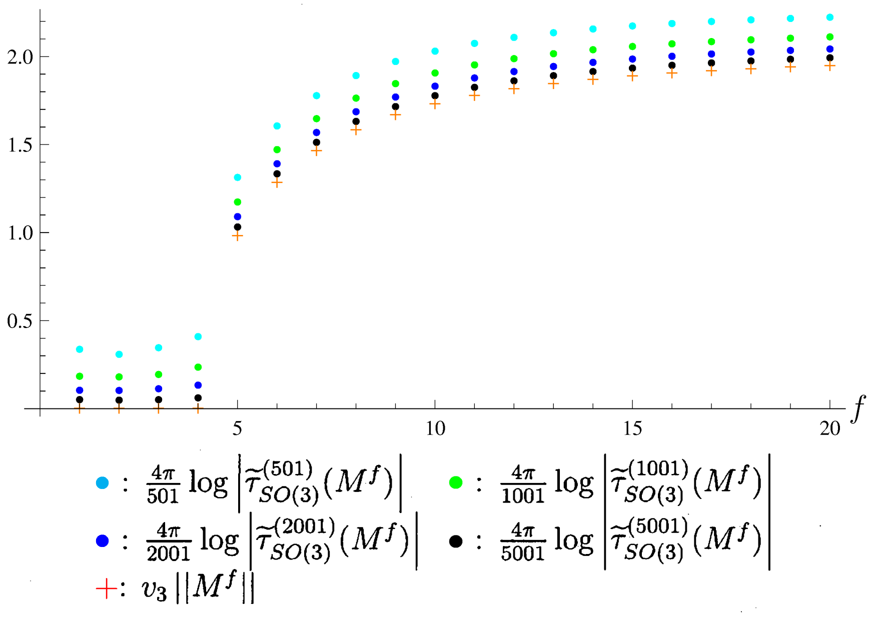
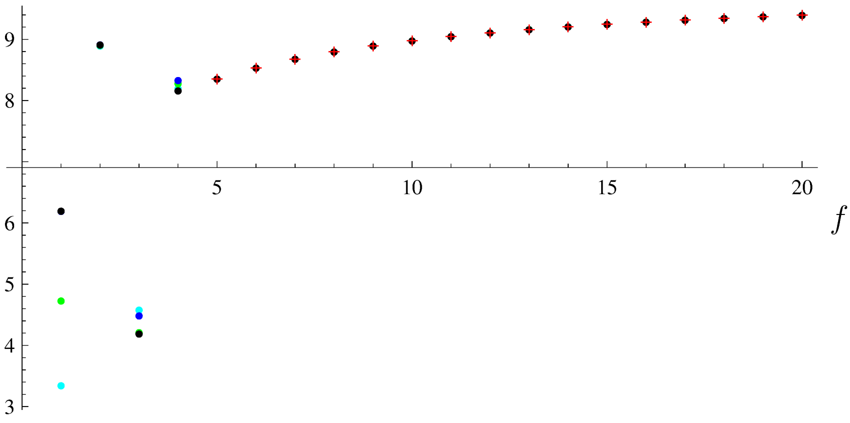


FIGURE 5. Absolute values of $\widetilde{\tau}$ of the Dehn surgery space M^f obtained from the figure eight knot with framing f. $||M^f||$ is Gromov's simplicial volume, v_3 is the hyperbolic volume of the ideal regular tetrahedron, and $v_3\,||M^f|| = \mathrm{Vol}(M^f)$ if M^f is hyperbolic.

Please note that the Turaev-VIro invariant is a real number, and here we take the absolute value of it.

4.3. **Kirillov-Reshetikhin invariant.** Kirillov and Reshetikhin extended the colored Jones invariant to knotted graphs with trivalent vertices in [9]. This is also constructed by using the Kauffman bracket and Jones-Wenzl idempotent in [7], which is called the quantum spin network. Here we consider the unitary version of the quantum spin network in [3]. Let Γ be a trivalent knotted graph with edges E_1, E_2, $\cdots$, E_e. Let j_1, j_2, $\cdots$, j_e be an $SO(3)$-admissible coloring for corresponding edges of Γ, and we denote the coloring by c. $SO(3)$-admissible means that each j_k is a positive odd integer and, for the three colors a, b, c of edges around a vertex, $a+b+c$ is an odd integer and they satisfy the triangle inequalities $0 < a < b+c$, $0 < b < c+a$ and $0 < c < a+b$. Let $\langle \Gamma, c\rangle_{SO(3)}$ denote the $SO(3)$ version of the unitary spin network in [4], which is an invariant of knotted graphs. For this invariant, following may holds.

Conjecture 3. Let M be a hyperbolic cone manifold obtained from a knotted graph Γ in S^3 with the cone angles α_1, $\cdots$, α_e at edges E_1,



$\bullet$: $2\pi \arg \dfrac{\widetilde{\tau}^{(501)}_{SO(3)}(M^f)}{\widetilde{\tau}^{(499)}_{SO(3)}(M^f)} \mod \pi^2$ $\bullet$: $2\pi \arg \dfrac{\widetilde{\tau}^{(1001)}_{SO(3)}(M^f)}{\widetilde{\tau}^{(999)}_{SO(3)}(M^f)} \mod \pi^2$

$\bullet$: $2\pi \arg \dfrac{\widetilde{\tau}^{(2001)}_{SO(3)}(M^f)}{\widetilde{\tau}^{(1999)}_{SO(3)}(M^f)} \mod \pi^2$ $\bullet$: $2\pi \arg \dfrac{\widetilde{\tau}^{(5001)}_{SO(3)}(M^f)}{\widetilde{\tau}^{(4999)}_{SO(3)}(M^f)} \mod \pi^2$

$+$: $-2\,\pi^2 \,\mathrm{cs}(M^f) \mod \pi^2$

FIGURE 6. The arguments of $\dfrac{\widetilde{\tau}^{(r)}_{SO(3)}(M^f)}{\widetilde{\tau}^{(r-2)}_{SO(3)}(M^f)}$ and the Chern-Simons invariant $\mathrm{cs}(M^f)$ of M^f which is obtained by the software *SnapPy*. Many points are hidden by the overlap. Especially, for $f \geq 5$, M^f is hyperbolic, and $2\pi \arg \dfrac{\widetilde{\tau}^{(r)}_{SO(3)}(M^f)}{\widetilde{\tau}^{(r-29)}_{SO(3)}(M^f)}$ is almost equal to $-2\pi^2\mathrm{cs}(M^f) \mod \pi^2$ for $r = 501, 1001, 2001, 5001$.

$\cdots$, E_e of Γ. Let $c^{(r)}$ be a sequence of coloring $s_k^{(r)}$ $(1 \leq k \leq e, r = 3, 5, 7, \cdots)$ for edges $E_1, \cdots, E_e$ of Γ such that

$$4\,\pi \lim_{n\to\infty} \frac{s_k^{(2n+1)}}{2n+1} = 2\,\pi - \alpha_k,$$

then

$$4\,\pi \lim_{n\to\infty} \frac{\log \left\langle \Gamma, c^{(2n+1)} \right\rangle_{SO(3)} \Big|_{\xi_{2n+1}\to\xi_{2n+1}^2}}{2n+1} = \mathrm{Vol}(M) + i\,\mathrm{CS}(M) \mod i\,\pi^2\,\mathbf{Z}$$

where $\langle \Gamma, c^{(2n+1)} \rangle_{SO(3)}$ is the $SO(3)$ version of the Kirillov-Reshetikhin invariant, which is a restriction of the original Kirillov-Reshetikhin invariant to odd colors, and $\mathrm{Vol}(M)$ and $\mathrm{CS}(M)$ are the hyperbolic volume and the Chern-Simons invariant of $M_{\alpha_1,\cdots,\alpha_\ell}$ respectively as before.

As a special case of this conjecture, we also have a conjecture for hyperbolic polyhedra.

Conjecture 3'. Let P be a hyperbolic polyhedron with edges $E_1, \cdots$, E_e whose dihedral angles at $E_1, \cdots, E_e$ are $\alpha_1, \cdots, \alpha_e$, and Γ be the planar graph obtained by the edges of Γ. Let $c^{(r)}$ be a sequence of coloring $s_k^{(r)}$ $(1 \leq k \leq e, r = 3, 5, 7, \cdots)$ for edges $E_1, \cdots, E_e$ of Γ such that $2\pi \lim_{n\to\infty} \frac{s_k^{(2n+1)}}{2n+1} = \pi - \alpha_k$, then

$$2\pi \lim_{n\to\infty} \frac{\log \langle \Gamma, c^{(r)} \rangle_{SO(3)} \Big|_{\xi_{2n+1} \to \xi_{2n+1}^2}}{2n+1} = \mathrm{Vol}(P).$$

This conjecture comes from Conjecture 3 since the corresponding cone manifold is the double of P.

Now let us compare the volume of the regular hyperbolic cube C_α with dihedral angle α and the Kirillov-Reshetikhin invariant of the graph Γ formed by the edges of a cube whose edges are all colored by the same spin k. For the admissible condition, k must be an odd integer.

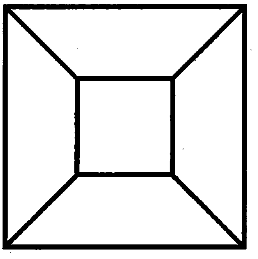
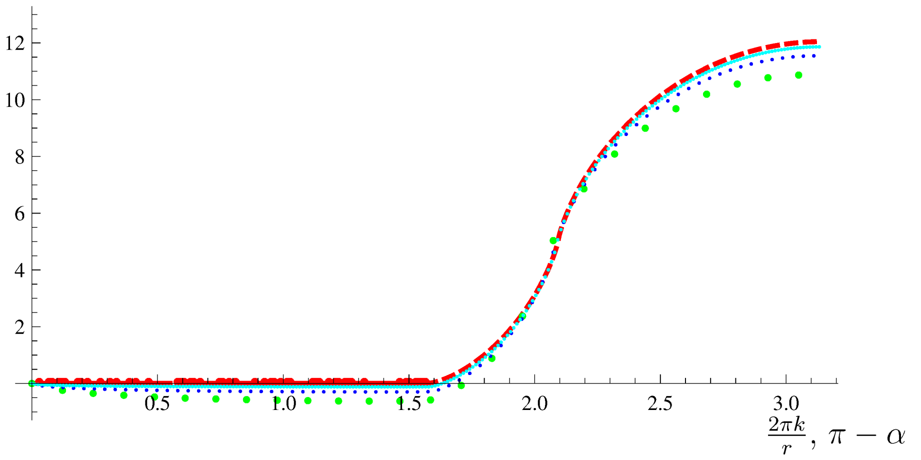


FIGURE 7. Cube graph Γ

We denote this coloring by $c(k)$. Then $\langle \Gamma, c(k) \rangle_{SO(3)}$ is evaluated as follows.

$$\langle \Gamma, c(k) \rangle_{SO(3)} = \sum_j \frac{\{2j+1\}}{\{1\}} \left(\begin{Bmatrix} k & k & k \\ k & k & 2j+1 \end{Bmatrix}_q^{RW} \right)^4.$$

The cube C_α is spherical for $\alpha > \pi/2$, Euclidean for $\alpha = \pi/2$, hyperbolic for $\pi/3 < \alpha < \pi/2$, ideal for $\alpha = \pi/3$, truncated (ultra-ideal) for



$\bullet$: $\frac{2\pi}{103} \operatorname{Re} \log \langle \Gamma, c(k) \rangle_{SO(3)} \Big|_{q=\exp \frac{4\pi i}{103}}$ $\bullet$: $\frac{2\pi}{303} \operatorname{Re} \log \langle \Gamma, c(k) \rangle_{SO(3)} \Big|_{q=\exp \frac{4\pi i}{303}}$

$\bullet$: $\frac{2\pi}{1003} \operatorname{Re} \log \langle \Gamma, c(k) \rangle_{SO(3)} \Big|_{q=\exp \frac{4\pi i}{1003}}$ - - - : $\operatorname{Vol}(C_{\pi-\alpha})$

FIGURE 8. Values of the Kirillov-Reshetikhin invariants of cubic graphs and the volumes of C_α.

$0 \leq \alpha \leq \pi/3$ and it becomes the right-angled ideal regular cuboctahedron for $\alpha = 0$ whose volume is $12.046\ldots$.

Now we show the second example. Let $K_{3,3}$ be the complete bipartite graph with six vertices and Ξ be the knotted graph which is the embedding of $K_{3,3}$ as in Figure 9. We give the coloring k for all edges and denote this coloring by $c(k)$. Then $\langle \Xi, c(k) \rangle_{SO(3)}$ is given by

$$\langle \Xi, c(k) \rangle_{SO(3)} = \sum_j (-1)^k q^{(k^2+k-j^2-j)/2} \frac{\{2j+1\}}{\{1\}} \left(\begin{Bmatrix} k & k & k \\ k & k & 2j+1 \end{Bmatrix}_q^{RW} \right)^3 .$$

Some values of $| \langle \Xi, c(k) \rangle_{SO(3)} |$ and some volumes of the corresponding cone manifolds are given in Figure 10. The arguments of $\langle \Xi, c(k) \rangle_{SO(3)}$ are investigated in Figure 11. The author don't know the definition of the Chern-Simons invariant of the corresponding cone manifold and relation to the Chern-Simons invariant is not checked yet.

4.4. **Yokota invariant.** The Kirillov-Reshetikhin invariant is defined for spatial graphs with trivalent vertices, while the Yokota invariant in [24] is defined for spatial graphs with multivalent vertices. Let Γ be a colored spatial graph with odd colors and v be a vertex of Γ whose valency is k. We expand the vertex v as in Figure 12, and let $\widetilde{\Gamma}$ be the

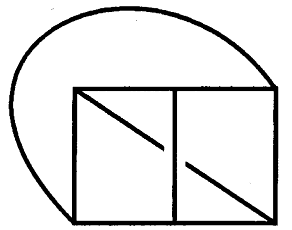


FIGURE 9. The knotted graph Ξ which is a embedding of $K_{3,3}$ in S^3.

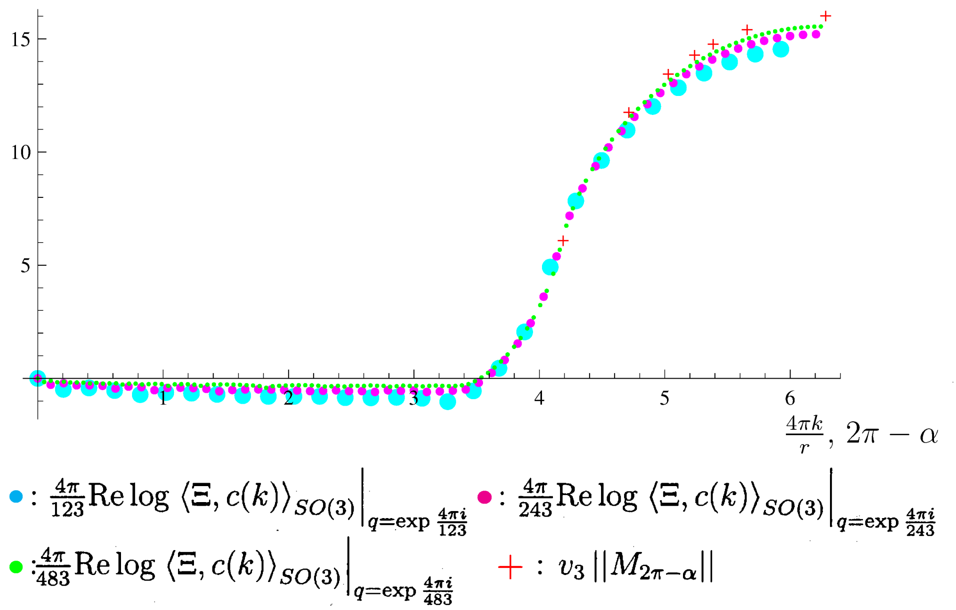
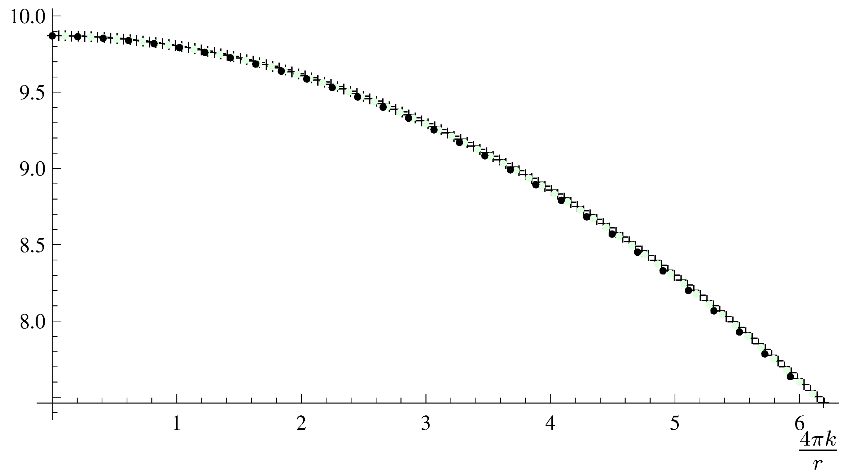


FIGURE 10. Values of the Kirillov-Reshetikhin invariants of Ξ and the volumes of corresponding come manifold M_α with the cone angle α for all edges where $\alpha = 0$, $\pi/5$, $2\pi/7$, $\pi/3$, $2\pi/5$, $\pi/2$, $2\pi/3$, which are computed by the software *Orb* [17].

resulting graph. Now we define the Yokota invariant $\langle\langle \Gamma, c\rangle\rangle_{SO(3)}$ by using the Kirillov-Reshetikhin invariant. Here c denote the coloring of Γ.

$$\langle\langle \Gamma, c\rangle\rangle_{SO(3)} = \sum_{i_1,\cdots,i_p} \prod_{r=1}^{p} \frac{\{2\,i_r+1\}}{\{1\}} < \widetilde{\Gamma}, \tilde{c} >_{SO(3)} < \overline{\widetilde{\Gamma}}, \tilde{c} >_{SO(3)} .$$

$\overline{\widetilde{\Gamma}}$ is the mirror image of $\widetilde{\Gamma}$ and $\tilde{c}$ is the coloring of $\widetilde{\Gamma}$ which is a extension of c by adding the coloring $i_1, \cdots, i_p$ for the newly added edges by the



$+ : \frac{2\pi}{123}\,\mathrm{Im}\log\dfrac{\langle\Xi,c(k)\rangle_{SO(3)}\big|_{q=\xi_{123}^2}}{\langle\Xi,c(k)\rangle_{SO(3)}\big|_{q=\xi_{125}^2}}$ $\quad \bullet : \frac{2\pi}{243}\,\mathrm{Re}\log\dfrac{\langle\Xi,c(k)\rangle_{SO(3)}\big|_{q=\xi_{243}^2}}{\langle\Xi,c(k)\rangle_{SO(3)}\big|_{q=\xi_{245}^2}}$

$\bullet : \frac{2\pi}{483}\,\mathrm{Im}\log\dfrac{\langle\Xi,c(k)\rangle_{SO(3)}\big|_{q=\xi_{483}^2}}{\langle\Xi,c(k)\rangle_{SO(3)}\big|_{q=\xi_{485}^2}}$

FIGURE 11. Arguments of the Kirillov-Reshetikhin invariants of Ξ

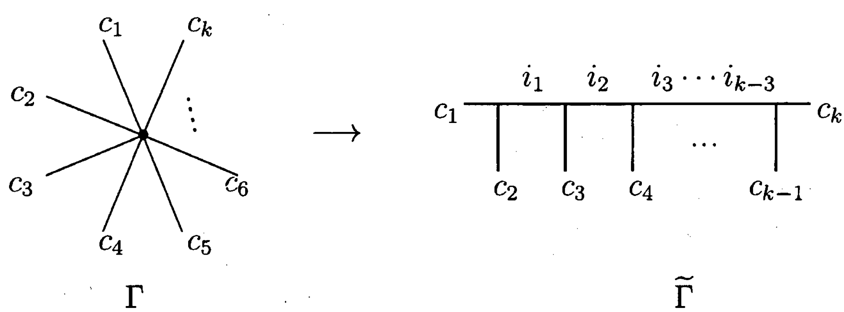


FIGURE 12. Expansion at k-valent vertex

expansion of the multivalent vertices. This does not depend on the expansion of the multivalent vertices and this is actually an invariant of the spatial graph. For this invariant, following may holds.

Conjecture 4. Let M be a hyperbolic cone manifold obtained from a knotted graph Γ in S^3 with the cone angles $\alpha_1, \cdots, \alpha_e$ at edges $E_1, \cdots, E_e$ of Γ. Let $c^{(r)}$ be a sequence of coloring $s_k^{(r)}$ ($1 \leq k \leq e, r = 3, 5, 7, \cdots$) for edges $E_1, \cdots, E_e$ of Γ such that $4\pi \lim_{n\to\infty} \frac{s_k^{(2n+1)}}{2n+1} =$

$2\pi - \alpha_k$, then

$$2\pi \lim_{n\to\infty} \frac{\log\langle\langle\Gamma, c^{(2n+1)}\rangle\rangle_{SO(3)}\big|_{\xi_{2n+1}\to\xi_{2n+1}^2}}{2n+1} = \mathrm{Vol}(M).$$

Conjecture 4'. Let P be a hyperbolic polyhedron with edges $E_1, \cdots, E_e$ whose dihedral angles at $E_1, \cdots, E_e$ are $\alpha_1, \cdots, \alpha_e$, and Γ be the planar graph obtained by the edges of Γ. Let $c^{(r)}$ be a sequence of coloring $s_k^{(r)}$ $(1 \leq k \leq e, r = 3, 5, 7, \cdots)$ for edges $E_1, \cdots, E_e$ of Γ such that $2\pi \lim_{n\to\infty} \frac{s_k^{(2n+1)}}{2n+1} = \pi - \alpha_k$, then

$$\pi \lim_{n\to\infty} \frac{\log\langle\langle\Gamma, c^{(2n+1)}\rangle\rangle_{SO(3)}\big|_{\xi_{2n+1}\to\xi_{2n+1}^2}}{2n+1} = \mathrm{Vol}(P).$$

For example, let Π be a square pyramid with all dihedral angles α. If $\alpha < \pi/2$, then Π is realized as a truncated hyperbolic pyramid which is a half of the hyperbolic cube. The Yokota invariant of Π with coloring $c(k)$ which assigns a positive odd integer k to all edges is given by

$$\langle\langle\Gamma, c(k)\rangle\rangle_{SO(3)} = \sum_j \frac{\{2j+1\}}{\{1\}} \left(\begin{Bmatrix} k & k & k \\ k & k & 2j+1 \end{Bmatrix}_q^{RW} \right)^4.$$

This is equal to the Kirillov-Reshetikhin invariant of the cube graph, and the Conjecture 4 seems to holds for this graph according to the computation in Figure 8.

5. Conclusion

By replacing the first primitive root of unity $\xi_r = \exp(2\pi i/r)$ with the second primitive root of unity ξ_r^2, the volume conjecture for the quantum knot invariant is extended to various cases. I hope that the observations in this note will help to prove the conjecture generally.

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