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Classification for the union of the totally geodesic 3-punctured spheres in an orientable hyperbolic 3-manifold

Ken'ichi Yoshida
Graduate School of Mathematical Sciences, The University of Tokyo

1 Introduction

This note is a brief summery of my talk at RIMS workshop "Topology and Analysis of Discrete Groups and Hyperbolic Spaces". I am preparing a paper [7] for details.

We consider essential 3-punctured spheres in orientable hyperbolic 3-manifolds. In this note, a hyperbolic 3-manifold is a manifold with toroidal boundary whose interior admits a complete hyperbolic metric. Its boundary component is called a cusp. A 3-punctured sphere is the compact orientable surface of genus zero with three boundary components. We always assume that the boundary of a 3-punctured sphere is contained in the cusps.

Adams [1] showed that an essential 3-punctured sphere in a hyperbolic 3-manifold is isotopic to a totally geodesic one. Hence we naturally assume that a 3-punctured sphere is totally geodesic.

We remark on immersed 3-punctured spheres.

Theorem 1.1 (Agol [2]). *Suppose that a totally geodesic 3-punctured sphere Σ is immersed in an orientable hyperbolic 3-manifold M, and Σ is not embedded. Then M is homeomorphic to a manifold obtained by (possibly empty) Dehn filling on a cusp of the Whitehead link complement. Furthermore, Σ is contained in M as in Figure 1.*

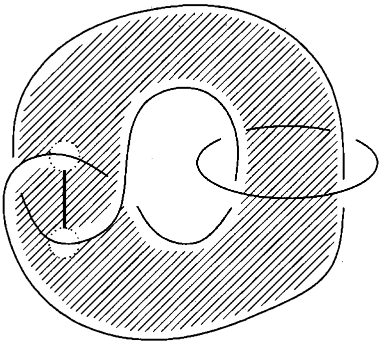

Figure 1: Non-embedded 3-punctured sphere

From now on, we consider embedded totally geodesic 3-punctured spheres in an orientable hyperbolic 3-manifold. 3-punctured spheres in a hyperbolic 3-manifold may intersect. The union of 3-punctured spheres may be complicated.

Theorem 1.2. *A connected component X of the union of all the totally geodesic 3-punctured spheres in an orientable hyperbolic 3-manifold with finitely many cusps is one of the following types:*

- *(general types)*
  $A_n (n \geq 1), B_{2n} (n \geq 1), T_3, T_4$.
- *(types determining the manifolds)*
  $Whi_{2n} (n \geq 2), Whi'_{4n} (n \geq 2), Bor_6, Mag_4, Tet_8, Pen_{10}, Oct_8$.
- *(types almost determining the manifolds)*
  $\widehat{Whi}_n (n \geq 2), \widehat{Whi'}_{2n} (n \geq 1), \widehat{Tet}_2, \widehat{Pen}_4, \widehat{Oct}_4$.

We classify the topological type of a pair $(N(X), X)$, where $N(X)$ is a regular neighborhood of X. The index indicates the number of 3-punctured spheres.

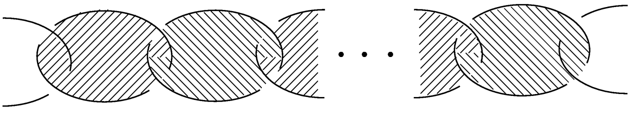

Figure 2: A_n

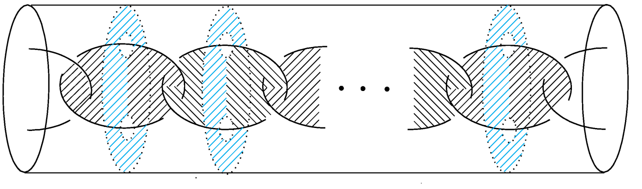

Figure 3: B_{2n}

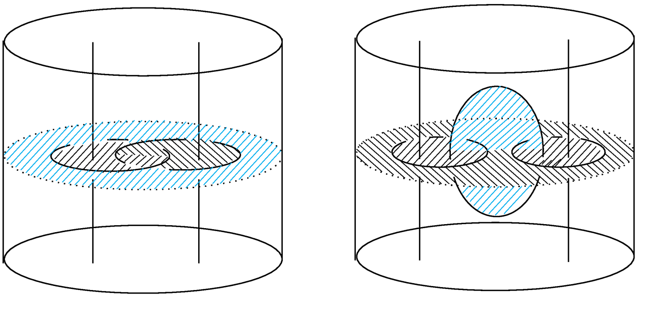

Figure 4: T_3 and T_4

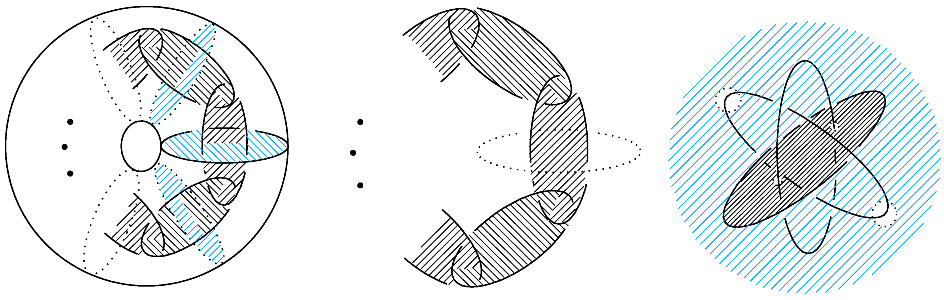

Figure 5: Whi_{2n} (Whi'_{4n}), $\widehat{Whi_n}$ ($\widehat{Whi'_{2n}}$) and Bor_6

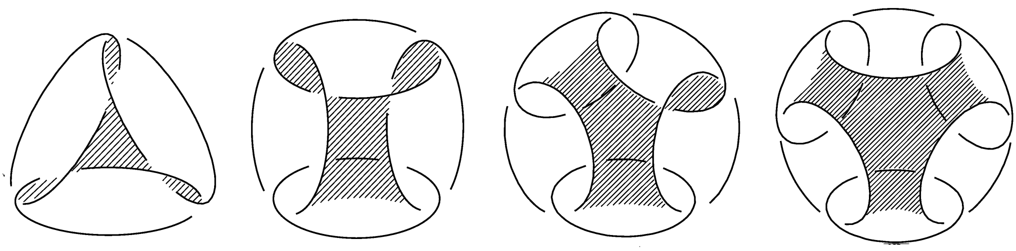

Figure 6: Mag_4, Tet_8, Pen_{10} and Oct_8

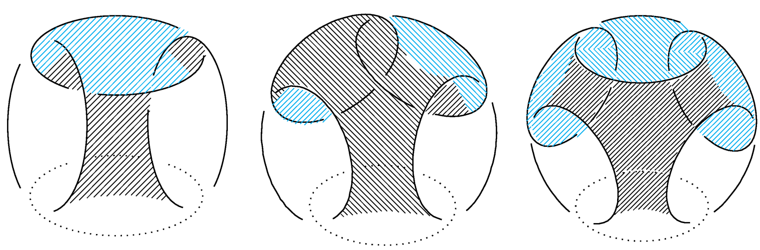

Figure 7: $\widehat{Tet_2}, \widehat{Pen_4}$ and $\widehat{Oct_4}$

The general types appear in various manifolds. For any finite multiset of general types, there is a manifold containing 3-punctured spheres of those types. When the type B_{2n}, T_3 or T_4 appears, however, there are isolated 3-punctured spheres near it. Contrastingly, each of the determining types appears only in a certain special manifold. The almost determining types appear only in manifolds obtained by Dehn fillings on a cusp of those special manifolds. For each n, the types Whi_{4n} and Whi'_{4n} have the common topology of the union, but they are distinguished by their neighborhood. The same argument holds for $\widehat{Whi_{2n}}$ and $\widehat{Whi'_{2n}}$.

We remark on non-hyperbolic 3-manifolds. If a 3-punctured sphere is contained in an irreducible 3-manifold whose boundary consisting of incompressible tori, it is cut into a 3-punctured sphere and annuli by the JSJ decomposition. Hence we are reduced to the cases for geometric pieces. A 3-punctured sphere is contained only in three Seifert 3-manifolds, which are the fiber bundles of 3-punctured sphere over a circle. The product of a 3-punctured sphere and a circle contains infinitely many (pairwise non-isotopic) 3-punctured spheres.

2 Description of the types

At first, we introduce the manifolds containing the 3-punctured spheres of special types. Let $\mathcal{W}_n$ denote the manifold in the left of Figure 5 which is an n-sheeted cyclic cover of the Whitehead link complement. Let $\mathcal{W}'_n$ denote the manifold which is obtained a half twist along a certain 3-punctured sphere of $\mathcal{W}_n$. For odd n, the manifold $\mathcal{W}'_n$ is homeomorphic to $\mathcal{W}_n$ by reversing orientation. See Kaiser-Purcell-Rollins [4] for more details. The manifold $\mathcal{W}'_2$ is the Borromean rings complement. For $3 \leq n \leq 6$, let $\mathcal{M}_n$ denote the minimally twisted hyperbolic n-chain link complement as in Figure 6. The manifold $\mathcal{M}_3$ is called the magic manifold. The manifold $\mathcal{M}_4$ has the smallest volume of the orientable hyperbolic 3-manifolds with 4 cusps [6]. The manifolds $\mathcal{M}_3, \mathcal{M}_5$ and $\mathcal{M}_6$ are conjectured to have smallest volume with respect to the number of the cusps. The 3-punctured spheres of the type $Whi_{2n}, Whi'_{4n}, Bor_6, Mag_4, Tet_8, Pen_{10}$ and Oct_8 are contained only in the manifolds $\mathcal{W}_n, \mathcal{W}'_{2n}, \mathcal{W}'_2, \mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_5$ and $\mathcal{M}_6$ respectively.

The Borromean rings complement $\mathcal{W}'_2$ has six 3-punctured spheres. We can put the Borromean rings in such a manner that each component is in a plane in $\mathbb{R}^3$. Then there are two 3-punctured spheres in the union of each plane and the infinite point as shown in the right of Figure 5.

The 3-punctured spheres of the type A_n are placed linearly, and can be considered as the most general type. If the 3-punctured spheres are placed cyclically, the union is the type $\widehat{Whi_n}$ or $\widehat{Whi'_{2n}}$. Now consider the 3-punctured spheres of the type $\widehat{Whi_n}$ or $\widehat{Whi'_{2n}}$. A regular neighborhood of the union of such 3-punctured spheres and the adjacent cusps is homeomorphic to $\mathcal{W}_n$ or $\mathcal{W}'_{2n}$. Hence if a hyperbolic 3-manifold has such 3-punctured spheres, it is obtained by Dehn filling on a cusp of $\mathcal{W}_n$ or $\mathcal{W}'_{2n}$. In fact, such a surgered hyperbolic 3-manifold except $\mathcal{M}_3, \dots, \mathcal{M}_6$ has no more 3-punctured spheres. The manifolds $\mathcal{M}_3, \dots, \mathcal{M}_6$ have more 3-punctured spheres as shown in Figure 6. Rotational symmetries give the remaining 3-punctured spheres.

The types $\widehat{Tet_2}, \widehat{Pen_4}$ and $\widehat{Oct_4}$ appear only in the manifolds obtained by Dehn fillings on a cusp of $\mathcal{M}_4, \mathcal{M}_5$ and $\mathcal{M}_6$ respectively, except $\mathcal{M}_3, \mathcal{M}_4$ and $\mathcal{M}_5$. The 3-punctured spheres of Tet_8, Pen_{10} and Oct_8 which are disjoint from the filled cusp remain in $\widehat{Tet_2}, \widehat{Pen_4}$ and $\widehat{Oct_4}$.

We describe the remaining general types. Let $\mathcal{T}_3$ and $\mathcal{T}_4$ denote the hyperbolic 3-manifolds with totally geodesic boundary obtained by cutting $\mathcal{M}_5$ and $\mathcal{M}_6$ respectively along a 3-punctured sphere. Similarly, let $\mathcal{B}_n$ denote the hyperbolic 3-manifold with totally geodesic boundary obtained by cutting $\mathcal{W}_n$ along a 3-punctured sphere which intersects only one of the other 3-punctured spheres. The types T_3, T_4 and B_{2n} appear in the manifolds containing $\mathcal{T}_3, \mathcal{T}_4$ and $\mathcal{B}_{n+1}$ respectively. Hence if there are the 3-punctured spheres of the type T_3, T_4 or B_{2n}, there are isolated 3-punctured spheres which correspond to the boundary of $\mathcal{T}_3, \mathcal{T}_4$ or $\mathcal{B}_{n+1}$.

We explain that the types Tet_8, Pen_{10} and Oct_8 are quite symmetrical in the way introduced by Dunfield-Thurston [3]. Each of $\mathcal{M}_4, \mathcal{M}_5$ and $\mathcal{M}_6$ has an involutional symmetry as shown in Figure 8, 9 and 10. The quotients by this involution is naturally decomposed by ideal polyhedra. Then the original manifolds are recovered by double branched coverings. The quotient of M_5 is the boundary of 4-dimensional simplex (a.k.a. pentachoron) made of five regular ideal tetrahedra. The quotient of M_6 is the double of a regular ideal octahedron. The 3-punctured spheres in $\mathcal{M}_4, \mathcal{M}_5$ and $\mathcal{M}_6$ correspond to the faces of ideal polyhedra in this decomposition. In particular, the manifolds $\mathcal{M}_4, \mathcal{M}_5$ and $\mathcal{M}_6$ have symmetries which can map a 3-punctured sphere to any other one.

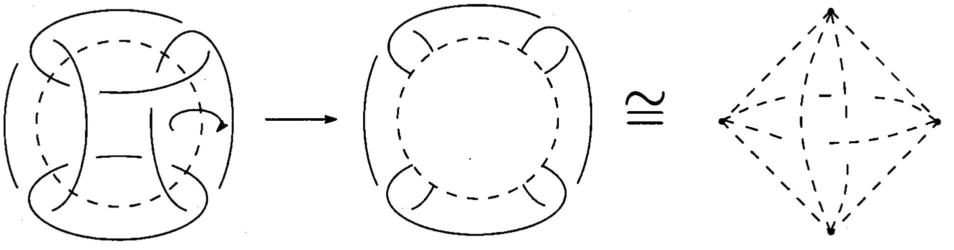

Figure 8: Quotient of $\mathcal{M}_4$

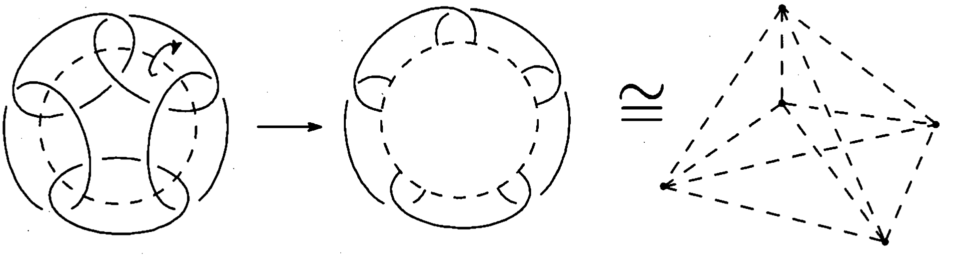

Figure 9: Quotient of $\mathcal{M}_5$

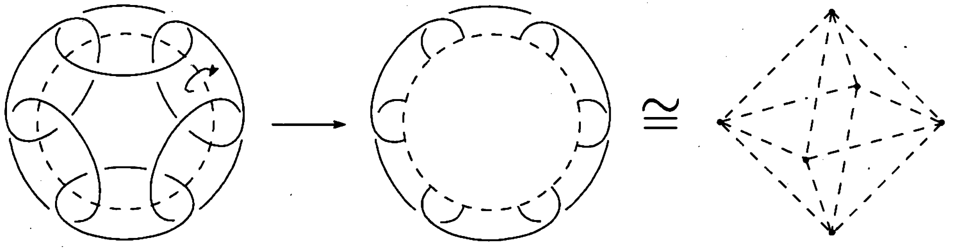

Figure 10: Quotient of $\mathcal{M}_6$

3 Number of 3-punctured spheres

For a hyperbolic 3-manifold with totally geodesic boundary, we can estimate its volume by the Euler characteristic of its boundary.

Theorem 3.1 (Miyamoto [5]). *Let N be a hyperbolic manifold with totally geodesic boundary. Then* $\mathrm{vol}(N) \geq |\chi(\partial N)|V_{oct}/2$, *where V_{oct} is the volume of a regular ideal octahedron.*

When we cut a hyperbolic 3-manifold along totally geodesic surfaces, we obtain a hyperbolic manifold with totally geodesic boundary. Hence if a hyperbolic 3-manifold M contains n disjoint 3-punctured spheres, the volume of M is at least nV_{oct}. Therefore the classification gives an estimate of the number of 3-punctured spheres by the volume. For example, 3-punctured spheres of the type A_n contains $\lfloor (n+1)/2 \rfloor$ disjoint ones.

Corollary 3.2. *Suppose that an orientable hyperbolic 3-manifold M has n 3-punctured spheres. Then $n \leq 4\mathrm{vol}(M)/V_{oct}$. The equality holds if and only if M is the manifold $\mathcal{M}_4$.*

4 Sketch of proof

There are six simple geodesics in a totally geodesic 3-punctured sphere. Each of these six geodesics is one of the two types. We indicate a geodesic whose endpoints are in different boundary components by the type α. Let the others be of the type β. At first, we consider the intersection of two 3-punctured spheres.

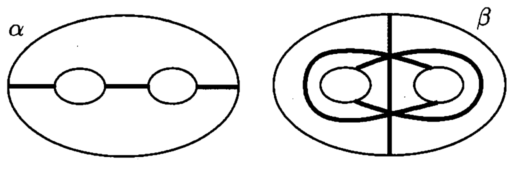

Figure 11: Simple geodesics in a 3-punctured sphere

Lemma 4.1. *Intersection of two totally geodesic 3-punctured spheres in an orientable hyperbolic 3-manifold is one of the following types:*

(o) The intersection is empty, i.e. the two 3-punctured spheres are disjoint.

(i) The intersection is one simple geodesic, and it is of the type α in both 3-punctured spheres.

(ii) The intersection is one simple geodesic, and it is of the type α in a 3-punctured sphere and of type β in the other 3-punctured sphere.

(iii) The intersection is two simple geodesics, and they are of the type α in both 3-punctured spheres.

The orientability is necessary for Lemma 4.1. For instance, a non-orientable hyperbolic 3-manifold obtained by gluing one regular ideal octahedron contains two 3-punctured spheres which intersect at three geodesics.

In the case (iii), the union of the two 3-punctured spheres is $\widehat{Whi_2}, \widehat{Whi'_2}$ or $\widehat{Tet_2}$ as in Figure 12. The boundary of a regular neighborhood of these two 3-punctured spheres and the adjacent cusps is a torus. Hence the ambient 3-manifold is obtained by (possibly empty) Dehn filling on a cusp of $\mathcal{W}_2, \mathcal{W}'_2$ or $\mathcal{M}_4$. Then the union of the 3-punctured spheres is $\widehat{Whi_2}, Whi_4, \widehat{Whi'_2}, Bor_6, \widehat{Tet_2}, Mag_4$ or Tet_8.

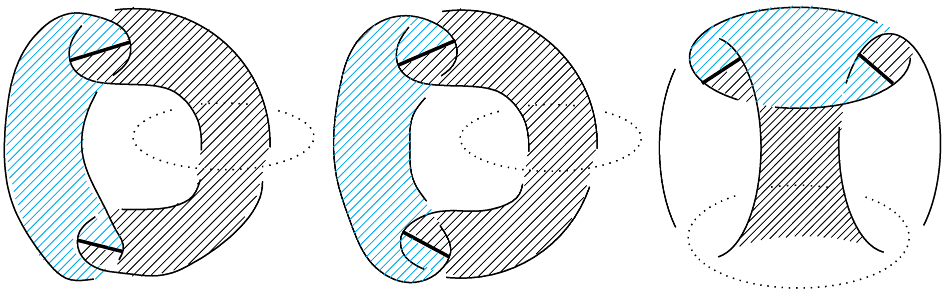

Figure 12: Two 3-punctured spheres intersecting at two geodesics

We next consider the case (ii). Suppose that 3-punctured spheres Σ_1 and Σ_2 intersect at a geodesic of the type β in Σ_1 and of the type α in Σ_2. Their union is B_2. If another 3-punctured sphere intersects Σ_2, the ambient 3-manifold is the Borromean rings complement $\mathcal{W}'_2$. Otherwise the component of the union containing Σ_1 and Σ_2 is B_{2n}, Whi_{2n} or Whi'_{4n}.

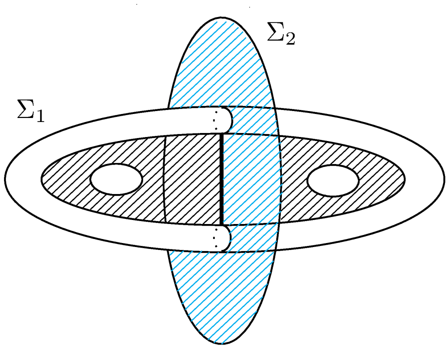

Figure 13: Two 3-punctured spheres of the type B_2

From now on, we consider a component X of the union without intersection of the type (ii) or (iii). Consider intersection C of a cusp and X. We can assume that a slope of C is $0, 1$ or ∞. Remark that there are only two geodesic of type α emanating from one component of boundary. We first consider following special types of intersection:

- If C contains three loops of slope $0, 1$ and ∞ with common intersection, X contains T_3.
- If C contains three loops of slope $0, 1$ and ∞ without common intersection, X contains $\widehat{Pen_4}$.
- If C contains four loops of slope 0 and ∞, X contains $\widehat{Oct_4}$.

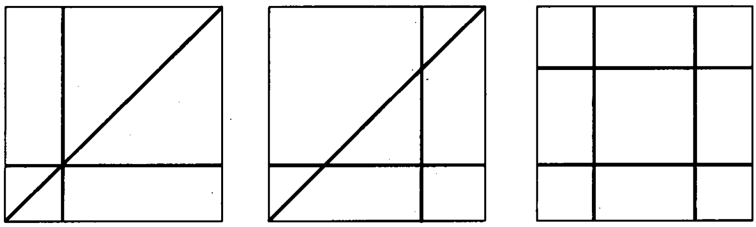

Figure 14: Special types of C (The squares are fundamental domains of cusps.)

These can be constructed explicitly. In these cases, the union X is $T_3, \widehat{Pen_4}, \widehat{Oct_4}, Oct_8$ or Pen_{10}. All the three above cases hold for Pen_{10}.

We next consider the remaining cases. Then the intersection C is one of the following general types;

- disjoint simple loops,
- two loops with one common point, or
- three loops with two common points, two of which are parallel.

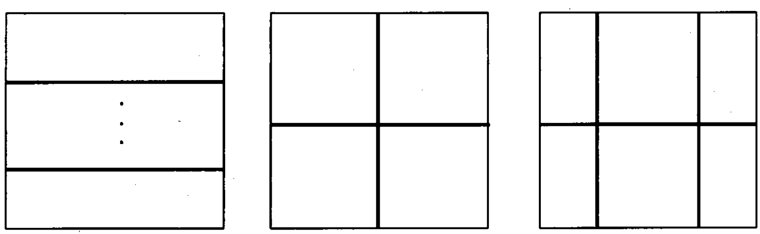

Figure 15: General types of C

If there is a 3-punctured sphere which intersects three 3-punctured spheres, the union X is T_4. Then they intersect no more 3-punctured spheres.

In the last remaining cases, the 3-punctured spheres in X form a line or a cycle. Then the union X is $A_n, \widehat{Whi_n}$ or $\widehat{Whi'_{2n}}$.

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Graduate School of Mathematical Sciences
The University of Tokyo
Tokyo 153-8914
JAPAN
E-mail address: `kyoshida@ms.u-tokyo.ac.jp`

東京大学大学院数理科学研究科 吉田 建一