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ON DECOMPOSITION SPACES, ALEXANDROFF SPACES AND RELATED TOPICS

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1. INTRODUCTION

This note is an extended version of the author's talk with the same title at the workshop "Research Trends on Set-theoretic and Geometric Topology and their cooperation with various branches" held at RIMS Kyoto University, June 12 - 14 , 2017.

2. A VERY NAIVE MOTIVATION

Let us consider the "natural" decomposition of the real line $\mathbb{R}$:

$$\mathbb{R} = (-\infty, 0) \sqcup \{0\} \sqcup (0, \infty).$$

In fact, the origin is not important and one can consider the decomposition $\mathbb{R} = (-\infty, p) \sqcup \{p\} \sqcup (p, \infty)$ "at any point p".

At "the International Conference on Set-Theoretic Topology and Its Applications, Yokohama 2015", held at Kanagawa University (Yokohama Campus), August 24 - 26, 2015, I gave a talk entitled " "Counting" spaces and related topics", in which using this decomposition I explained why *the Euler–Poincaré characteristic has to be* $V - E + F - \cdots$, *the alternating sum of the number* V *of vertices, the number* E *of edges, the number* F *of faces and so on.* The reason is roughly speaking as follows: The usual counting c of a finite set, i.e., the cardinality, satisfies the following four properties for finite sets:

- $A \cong A'$ (set-theoretic isomorphism) $\Rightarrow c(A) = c(A')$,
- $c(A) = c(B) + c(A \setminus B)$ for $B \subset A$
- $c(A \times B) = c(A) \cdot c(B)$
- $c(\text{one element}) = 1$

Similarly we let c be a "counting" of "nice" topological spaces such that

(1) $X \cong X'$ (*homeomorphism = topological isomorphism*) $\Rightarrow c(X) = c(X')$,
(2) $c(X) = c(Y) + c(X \setminus Y)$ for a *closed* subset $Y \subset X$
(3) $c(X \times Y) = c(X)c(Y)$
(4) $c(\text{one-point space}) = 1$

We apply this "counting" c to the above decomposition. Then we get

$$c(\mathbb{R}) = c((-\infty, 0)) + c(\{0\}) + c((0, \infty)).$$

Since $(-\infty, 0) \cong \mathbb{R} \cong (0, \infty)$, the conditions (1), (2) and (4) imply that $c(\mathbb{R}) = -1$ and (3) implies that $c(\mathbb{R}^n) = (-1)^n$. As further generalizations I gave a quick survey about motivic Hirzebruch class [11] (cf. [43]) as a related topic. For the extended version of this talk, see [44].

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The "Euler–Poincaré characteristic" story about the decomposition $\mathbb{R} = (-\infty, 0) \sqcup \{0\} \sqcup (0, \infty)$ is kind of *"algebraic topological" (= algebra + topology)*. In the present work, we consider a *"general topological"* aspect of the decomposition $\mathbb{R} = (-\infty, 0) \sqcup \{0\} \sqcup (0, \infty)$.

Let $\mathcal{D} = \{(-\infty, 0), \{0\}, (0, \infty)\}$ be the decomposition set and let $N = (-\infty, 0), O = \{0\}, P = (0, \infty)$ (where N stands for "negative", O "origin", P "positive"). Considering each decomposition piece as a point, or by introducing the equivalence relation $x \sim_{\mathcal{D}} y \Leftrightarrow$ both x and y belong to one decomposition piece, and then taking the set of the equivalence classes, we get the quotient map

$$\pi_{\mathcal{D}} : \mathbb{R} \to \mathbb{R}/\mathcal{D} = \{N, O, P\}.$$

The quotient topology $\tau_{\pi_{\mathcal{D}}}$ of the quotient map, i.e., the strongest or largest topology on the target $\mathbb{R}/\mathcal{D}$ such that the quotient map $\pi_{\mathcal{D}} : \mathbb{R} \to \mathbb{R}/\mathcal{D}$ becomes continuous is

$$\tau_{\pi_{\mathcal{D}}} = \{\emptyset, \{N, O, P\}, \{N\}, \{P\}, \{N, P\}\}.$$

Consider another decomposition $\mathcal{D}'$ of $\mathbb{R}$, similar to the above $\mathcal{D}$:

$$\mathcal{D}' = \{(-\infty, -1), [-1, 1], (1, \infty)\},$$

and the quotient map $\pi_{\mathcal{D}'} : \mathbb{R} \to \mathbb{R}/\mathcal{D}'$. For the target $\mathbb{R}/\mathcal{D}'$ we use the same symbols $N = \pi_{\mathcal{D}'}((-\infty, -1)), O = \pi_{\mathcal{D}'}([-1, 1])$ and $P = \pi_{\mathcal{D}'}((1, \infty))$. The quotient topology for $\mathbb{R}/\mathcal{D}'$ is the same as above: $\Big\{\emptyset, \{N, O, P\}, \{N\}, \{P\}, \{N, P\}\Big\}$.

It is easy to see that the quotient map $\pi_{\mathcal{D}'} : \mathbb{R} \to \mathbb{R}/\mathcal{D}'$ is *not* an open map, since the image $\pi_{\mathcal{D}'}((0, 1)) = \{O\}$ is a closed set, not an open set, although $(0, 1)$ is an open set of $\mathbb{R}$. However the quotient map $\pi_{\mathcal{D}} : \mathbb{R} \to \mathbb{R}/\mathcal{D}$ *is* an open map. One needs to prove it; for example the proof goes as follows. Let U be an open set of $\mathbb{R}$. Then either $0 \in U$ or $0 \notin U$. If $0 \notin U$, then $\pi_{\mathcal{D}}(U)$ is $\{N\}$, $\{P\}$ or $\{N, P\}$, which is an open set. If $0 \in U$, then by the definition of an open set of $\mathbb{R}$, there exists an open interval $(-\epsilon, \epsilon)$ such that $0 \in (-\epsilon, \epsilon) \subset U$ where $\epsilon > 0$, hence $U \cap (-\infty, 0) \neq \emptyset$ and $U \cap (0, \infty) \neq \emptyset$. Therefore, if U is an open set and $0 \in U$, then $\pi_{\mathcal{D}}(U) = \{N, O, P\}$, which is an open set. Thus the quotient map $\pi_{\mathcal{D}} : \mathbb{R} \to \mathbb{R}/\mathcal{D}$ is an open map.

We can imagine that if a given decomposition $\mathcal{D}$ of a topological space X has lots of pieces, say 100 pieces, then it would be not easy or quite tedious to check whether the quotient map $\pi_{\mathcal{D}} : X \to X/\mathcal{D}$ is open or not. In fact we can see the openness of the quotient map via the proset-structure of the quotient space $X/\mathcal{D}$, which we will discuss below.

3. Decompositions and Decomposition spaces

Before going on to prosets, in this section we recall some basic facts of decompositions and decomposition spaces, for later use.

Let $\mathcal{D} = \{D_\lambda | \lambda \in \Lambda\}$ be a decomposition of a topological space X, i.e.,

(1) $D_\lambda \cap \mu = \emptyset$ if $\lambda \neq \mu$,
(2) $X = \bigcup_{\lambda \in \Lambda} D_\lambda$.

Let $\pi_{\mathcal{D}} : X \to X/\mathcal{D}$ be the quotient map. Let $\tau_{\pi_{\mathcal{D}}}$ be the quotient topology on the target $X/\mathcal{D}$. Then the topological space $(X/\mathcal{D}, \tau_\pi)$ is called *the decomposition space* and the continuous map $\pi_{\mathcal{D}} : X \to (X/\mathcal{D}, \tau_\pi)$ is called *the decomposition map.* If the content is clear, we sometimes delete the topology $\tau_{\pi_{\mathcal{D}}}$. The decomposition map $\pi_{\mathcal{D}}$ is also sometimes denoted simply by π for the sake of simplicity.

That the decomposition map $\pi : X \to X/\mathcal{D}$ is open (closed) means that for any open (closed) set G of X the image $\pi(G)$ is open (closed) in $X/\mathcal{D}$, which implies by the definition of the quotient topology on the decomposition space $X/\mathcal{D}$ that $\pi^{-1}(\pi(G))$ is open (closed). Here we note that

$$\pi^{-1}(\pi(G)) = \bigcup\{D_\lambda \mid D_\lambda \cap G \neq \emptyset\}.$$

Thus *the decomposition map* $\pi : X \to X/\mathcal{D}$ *is open (closed) if and only if* $\bigcup\{D_\lambda \mid D_\lambda \cap G \neq \emptyset\}$ *is open (closed) for any open (closed) set* G. Here we remark that given a subset G of X each D_λ either intersects G or does not intersect G, i.e., either $D_\lambda \cap G \neq \emptyset$ or $D_\lambda \cap G = \emptyset$, namely $D_\lambda \subset G^c = X \setminus G$. Thus we can split the decomposition $\mathcal{D}$ into two disjoint parts:

$$X = \bigcup_{\lambda \in \Lambda} D_\lambda = \left(\bigcup\{D_\lambda \mid D_\lambda \cap G \neq \emptyset\}\right) \cup \left(\bigcup\{D_\lambda \mid D_\lambda \subset X \setminus G\}\right).$$

$\bigcup\{D_\lambda \mid D_\lambda \cap G \neq \emptyset\}$ is open (closed) if and only if $\bigcup\{D_\lambda \mid D_\lambda \subset X \setminus G\}$ is closed (open). If G is open (closed), then $G^c = X \setminus G$ is closed (open). Therefore we can also say that *the decomposition map* $\pi : X \to X/\mathcal{D}$ *is open (closed) if and only if* $\bigcup\{D_\lambda \mid D_\lambda \subset F\}$ *is closed (open) for any closed (open) set* F. Therefore we can see the following:

Proposition 3.1. *Let* $\mathcal{D}$ *be a decomposition of a topological space* X *and let* $\pi : X \to X/\mathcal{D}$ *be the decomposition map.*

(1) *The decomposition map* $\pi : X \to X/\mathcal{D}$ *is open if and only if* $\bigcup\{D_\lambda \mid D_\lambda \cap U \neq \emptyset\}$ *is open for any open set* U *in* X*, or* $\bigcup\{D_\lambda \mid D_\lambda \subset F\}$ *is closed for any closed set* F *of* X*.*
(2) *The decomposition map* $\pi : X \to X/\mathcal{D}$ *is closed if and only if* $\bigcup\{D_\lambda \mid D_\lambda \cap F \neq \emptyset\}$ *is closed for any closed set* F *in* X*, or* $\bigcup\{D_\lambda \mid D_\lambda \subset U\}$ *is open for any open set* U *of* X*.*

Remark 3.2. This proposition is just a paraphrasing of the definition and it does not help much to see if the decomposition map is an open map or a closed map or not.

Here we recall the following classical definitions:

Definition 3.3 (R. L. Moore [32], cf. [33]). Let $\mathcal{D}$ be a decomposition of a topological space X.

(1) $\mathcal{D}$ is called *an upper semicontinuous decomposition* if $\bigcup\{D_\lambda \mid D_\lambda \subset U\}$ is open for any open set U of X (thus, the decomposition map $\pi : X \to X/\mathcal{D}$ is closed).
(2) $\mathcal{D}$ is called *a lower semicontinuous decomposition* if $\bigcup\{D_\lambda \mid D_\lambda \subset F\}$ is closed for any closed set F of X (thus, the decomposition map $\pi : X \to X/\mathcal{D}$ is open).
(3) $\mathcal{D}$ is called *a continuous decomposition* if it is both upper semicontinuous and lower semicontinuous.

Proposition 3.4. *Let* $f : X \to Y$ *be a surjective continuous map. If* f *is open or closed, then the topology of* Y *is equal to the quotient topology induced by the map* f*.*

Remark 3.5. The converse statement does not hold, as seen below.

Using Propostion 3.4 we can show the following:

Proposition 3.6. *Let* $\mathcal{D}_i$ $(1 \leqq i \leqq n)$ *be a lower semicontinuous decomposition of a topological space* X_i $((1 \leqq i \leqq n))$*. Then the product* $\mathcal{D}_1 \times \cdots \mathcal{D}_n$ *is a lower semicontinuous decomposition of the product* $X_1 \times \cdots \times X_n$ *and we have the homeomorphism*

$$(X_1 \times \cdots \times X_n)/(\mathcal{D}_1 \times \cdots \times \mathcal{D}_n) \cong (X_1/\mathcal{D}_1) \times \cdots \times (X_n/\mathcal{D}_n).$$

Remark 3.7. The decomposition theory of decomposing a (metric) space into continuum (i.e., compact connected space) was developed by R. L. Moore in 1920s and later by R. H. Bing in 1950s (e.g., see [13]). Moore's famous theorem [32] is that *if* $\mathcal{D}$ *is an upper semicontinuous decomposition of the 2-dimensional Euclidean space* $\mathbb{R}^2$ *into continua, none of which separates* $\mathbb{R}^2$*, then* *the decomposition space* $\mathbb{R}^2$ *is homeomorphic to the Euclidean space* $\mathbb{R}^2$*:* $\mathbb{R}^2/\mathcal{D} \cong \mathbb{R}^2$. However, as to the 3-dimensional Euclidean space $\mathbb{R}^3$, it is not the case, which was proved by R. H. Bing [5]: *There exists an upper semicontinuous decomposition* $\mathcal{D}$ *of the 3-dimensional Euclidean space* $\mathbb{R}^3$ *into continua, none of which separates* $\mathbb{R}^3$*, such that the decomposition space* $\mathbb{R}^2/\mathcal{D}$ *is neither homeomorphic to the Euclidean space* $\mathbb{R}^3$ *nor a manifold.* But this decomposition space $\mathbb{R}^2/\mathcal{D}$ satisfies that $(\mathbb{R}^3/\mathcal{D}) \times \mathbb{R}^1 \cong \mathbb{R}^4$ (see [6]). This decomposition space $\mathbb{R}^2/\mathcal{D}$ is the famous *Bing's dogbone space.* It should be noted that this kind of topology is called *wild topology* and a similar wild topology and decomposition theory was used in M. Freedman's famous proof of the 4-dimensional Poincaré conjecture.

4. Prosets and Alexandroff spaces

A preorder on a set P is a relation $\leqq$ which is reflexive ($a \leqq a$) and transitive ($a \leqq b, b \leqq c \Rightarrow a \leqq c$). A set $(P, \leqq)$ equipped with a preorder $\leqq$ is called a *proset* (preordered set). If a preorder is anti-symmetric ($a \leqq b, b \leqq a \Rightarrow a = b$), then it is called a partial order and a set with a partial order is called a *poset* (partial ordered set). $a \leqq b$ is also denoted by $a \to b$ using arrow.

Definition 4.1 (Alexandroff topology [1])**.** Let X be a topological space. If *the intersection of any family* of open sets is open (or equivalently, *the union of any family of closed sets is closed*), then the topology is called an *Alexandroff topology* and the space is called an *Alexandroff space* (cf. [35]).

We note that any finite topological space, i.e., a finite set with a topology, is an Alexandroff space. For finite topological spaces, e.g., see [4, 26, 27, 28, 29, 30, 36].

Given a proset $(X, \leqq)$, we define $U \subset X$ *to be an open set by* $x \in U, x \leqq y \Rightarrow y \in U$. In other words, if we let $U_x := \{y \in X \mid x \leqq y\}$, then $\{U_x \mid x \in X\}$ is the base for the topology. This topology is denoted by $\tau_{\leqq}$.

Lemma 4.2. *For a proset* $(X, \leqq)$*, the topological space* $(X, \tau_{\leqq})$ *is an Alexandroff space. The topology* $\tau_{\leqq}$ *is called the Alexandroff topology of the preorder.*

Conversely, for a topological space (X, τ), we define the order $x \leqq_\tau y \Leftrightarrow x \in \overline{\{y\}}$, which is called *specialization order*. Certainly this is a preorder, but not necessarily a partial order.

Lemma 4.3. *If* $f : (X, \leqq_1) \to (Y, \leqq_2)$ *is a monotone (order-preserving) function, i.e.,* $x \leqq_1 y$ *implies* $f(x) \leqq_2 f(y)$*, then* $f : (X, \tau_{\leqq_1}) \to (Y, \tau_{\leqq_2})$ *is a continuous map.*

Lemma 4.4. *If* $f : (X, \tau_1) \to (Y, \tau_2)$ *is a continuous map, then* $f : (X, \leqq_{\tau_1}) \to (Y, \leqq_{\tau_2})$ *is a monotone function.*

Let $\mathcal{P}roset$ be the category of prosets and monotone functions of prosets, $\mathcal{A}lex$ the category of Alexandroff spaces and continuous maps and $\mathcal{T}op$ the category of topological spaces and continuous maps. Then we have covariant functors $\mathcal{T} : \mathcal{P}roset \to \mathcal{A}lex$ and $\mathcal{P} : \mathcal{T}op \to \mathcal{P}roset$. For any proset $(X, \leqq)$ we have $(\mathcal{P} \circ \mathcal{T})((X, \leqq)) = (X, \leqq)$, i.e., $\mathcal{P} \circ \mathcal{T} = Id_{\mathcal{P}roset}$ However, in general, for a topological space (X, τ) we have $(\mathcal{T} \circ \mathcal{P})((X, \tau)) \neq (X, \tau)$, i.e., $\mathcal{T} \circ \mathcal{P} \neq Id_{\mathcal{T}op}$. The reason is that $(\mathcal{T} \circ \mathcal{P})((X, \tau))$ is always an Alexandroff space, even if the original space (X, τ) is not an Alexandroff space, i.e., the topology of $(\mathcal{T} \circ \mathcal{P})((X, \tau))$ is stronger that the original topology τ.

However, if we restrict the functor $\mathcal{P} : \mathcal{T}op \to \mathcal{P}roset$ to the subcategory $\mathcal{A}lex$ of Alexandroff spaces, then we have $(\mathcal{T} \circ \mathcal{P})((X, \tau)) = (X, \tau)$, i.e., $\mathcal{T} \circ \mathcal{P} = Id_{\mathcal{A}lex}$. Therefore we have $\mathcal{P} \circ \mathcal{T} = Id_{\mathcal{P}roset}$, $\mathcal{T} \circ \mathcal{P} = Id_{\mathcal{A}lex}$. Thus Alexandroff spaces and prosets are equivalent.

Proposition 4.5. *If we define an open set in a proset by "up-set", then* F *is a closed set if and only if* F *is a "down-set", i.e.,* $x \in F, y \leqq x \Rightarrow y \in F$.

Remark 4.6. For a proset, in the above an open set is defined using "up-set". However, an open set can be also defined using "down-set", as in [4]. In this case, a closed is defined by "up-set". Namely the roles of open set and closed set are exchanged to each other.

Corollary 4.7. *Let Λ be a poset. In the Alexandroff topological space $\mathcal{T}(\Lambda)$, any singleton $\{\lambda\}$ is locally closed.*

Proposition 4.8. *Let $(P_i, \leqq_i)$ be a proset $(1 \leqq i \leqq n)$. Then the preorder $\leqq$ of the proset $\mathcal{P}\left((P_1, \tau_{\leqq_1}) \times \cdots \times (P_n, \tau_{\leqq_n})\right)$ of the product of the Alexandroff spaces $\mathcal{T}((P_i, \leqq_i)) = (P_i, \tau_{\leqq_i})$ is given by $(x_1, \cdots, x_n) \leqq (y_1, \cdots, y_n) \Leftrightarrow x_1 \leqq_1 y_1, \cdots, x_n \leqq_n y_n$.*

In fact, we get the following commutative diagram for the category product:

$$\begin{array}{ccc} \mathcal{P}roset \times \cdots \times \mathcal{P}roset & \underset{\mathcal{P}\times\cdots\times\mathcal{P}}{\overset{\mathcal{T}\times\cdots\times\mathcal{T}}{\rightleftarrows}} & \mathcal{A}lex \times \cdots \times \mathcal{A}lex \\ \times \downarrow & & \downarrow \times \\ \mathcal{P}roset & \underset{\mathcal{P}}{\overset{\mathcal{T}}{\rightleftarrows}} & \mathcal{A}lex. \end{array}$$

Definition 4.9. A decomposition $\mathcal{D}$ of a topological space X such that the decomposition space $X/\mathcal{D}$ becomes an Alexandroff space is called *an Alexandroff decomposition.*

Corollary 4.10. *Let $\mathcal{D}_i$ $(1 \leqq i \leqq n)$ be a lower semicontinuous Alexandroff decomposition of a topological space X_i $((1 \leqq i \leqq n))$. Then the product $\mathcal{D}_1 \times \cdots \mathcal{D}_n$ is a lower semicontinuous Alexandroff decomposition of the product $X_1 \times \cdots \times X_n$ and we have the homeomorphism*

$$(X_1 \times \cdots \times X_n)/(\mathcal{D}_1 \times \cdots \times \mathcal{D}_n) \cong \mathcal{T}\Big((X_1/\mathcal{D}_1, \leqq_{\tau_{\pi_1}}) \times \cdots \times (X_n/\mathcal{D}_n, \leqq_{\tau_{\pi_n}})\Big).$$

Remark 4.11. In the above proof it is crucial that the finite product of Alexandroff spaces is again an Alexandroff space. As a matter of fact, it is not true in the case of an infinite product of Alexandroff spaces, as it is known that the Cantor set is an infinite product of $\{0, 1\}$ but is not an Alexandroff space (e.g., see [2]), although $\{0, 1\}$ is an Alexandroff space because it is a finite topological space.

Remark 4.12. It follows from the above Corollary 4.10 that we determine the topology of the decomposition space $(X_1 \times \cdots \times X_n)/(\mathcal{D}_1 \times \cdots \times \mathcal{D}_n)$ by looking at the proset structure of the product $(X_1/\mathcal{D}_1, \leqq_{\tau_{\pi_1}}) \times \cdots \times (X_n/\mathcal{D}_n, \leqq_{\tau_{\pi_n}})$, where the preorder is $\leqq_{\tau_{\pi_1}} \times \cdots \times \leqq_{\tau_{\pi_n}}$, i.e.,

$$(a_1, \cdots, a_n) \leqq_{\tau_{\pi_1}} \times \cdots \times \leqq_{\tau_{\pi_n}} (b_1, \cdots, b_n) \Longleftrightarrow a_i \leqq_{\tau_{\pi_i}} b_i \ (\forall i).$$

Here we give some examples of decomposition spaces and their associated prosets. The first two examples are given above, but we repeat them.

Example 4.13. Consider the above-mentioned $\mathcal{D} = \{(-\infty, 0), \{0\}, (0, \infty)\}$ of the real line $\mathbb{R}$. For the quotient map $\pi : \mathbb{R} \to \mathbb{R}/\mathcal{D} = \{N, O, P\}$, the quotient topology for $\mathbb{R}/\mathcal{D}$ is $\tau_\pi = \Big\{\emptyset, \{N\}, \{P\}, \{N, P\}, \{N, O, P\}\Big\}$ and the proset (in fact poset) $\mathcal{P}((\mathbb{R}/\mathcal{D}, \tau_\pi))$ is (we do not write the reflexivity):

$$O \leqq N, O \leqq P, \quad N \longleftarrow O \longrightarrow P\,.$$

The decomposition map $\pi : \mathbb{R} \to \mathbb{R}/\mathcal{D}$ is open.

Example 4.14. $\mathcal{D}' = \{(-\infty, -1), [-1, 1], (1, \infty)\}$ is another decomposition of $\mathbb{R}$. For the quotient map $\pi : \mathbb{R} \to \mathbb{R}/\mathcal{D}' = \{N, O, P\}$, using the same symbols, the quotient topology for $\mathbb{R}/\mathcal{D}'$ is the same as above: $\tau_\pi = \Big\{\emptyset, \{N\}, \{P\}, \{N,P\}, \{N,O,P\}\Big\}$. The decomposition map $\pi : \mathbb{R} \to \mathbb{R}/\mathcal{D}'$ is not open.

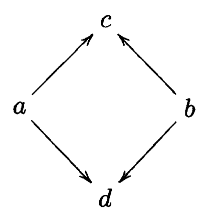
Example 4.15. $\mathcal{D} = \{\mathbb{Q}, \mathbb{R}\setminus\mathbb{Q}\}$ is a decomposition of the real line $\mathbb{R}$ into the rational part $\mathbb{Q}$ and the irrational part $\mathbb{R}\setminus\mathbb{Q}$ and for the quotient map $\pi : \mathbb{R} \to \mathbb{R}/\mathcal{D}$ we let $q = \pi(\mathbb{Q}), p = \pi(\mathbb{R}\setminus\mathbb{Q})$. Then the quotient topology for $R/\mathcal{D}$ is the indiscrete topology: $\tau_\pi = \Big\{\emptyset, \{p,q\}\Big\}$ and the proset $\mathcal{P}((\mathbb{R}/\mathcal{D}, \tau_\pi))$ is:

$$p \leqq q, q \leqq p, \ p \rightleftarrows q\,.$$

The decomposition map $\pi : \mathbb{R}^1 \to \mathbb{R}^1/\mathcal{D}$ is open.

Example 4.16. For the circle $S^1 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 = 1\}$, consider the decomposition $\mathcal{D} = \Big\{\{(-1,0)\}, \{(1,0)\}, H^+ = \{(x,y) \in S^1 \,|\, y > 0\}, H_- = \{(x,y) \in S^1 \,|\, y < 0\}\Big\}$ and the quotient map $\pi : S^1 \to S^1/\mathcal{D}$. Let $a = \pi((-1,0)), b = \pi((1,0)), c = \pi(H^+), d = \pi(H_-)$. Then the quotient topology for $S^1/\mathcal{D}$ is $\tau_\pi = \Big\{\emptyset, \{c\}, \{d\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}\Big\}$ and the proset (in fact poset) $\mathcal{P}((S^1/\mathcal{D}, \tau_\pi))$ is

$$a \leqq c, \quad b \leqq c, \quad a \leqq d, \quad b \leqq d,$$



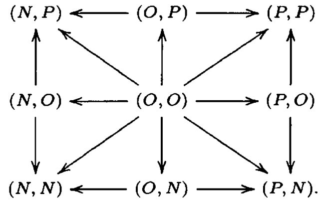
The decomposition map $\pi : S^1 \to S^1/\mathcal{D}$ is open. This four-point poset is well-known as *the pseudo-circle*, denoted $\mathbb{S}^1$, which is weakly homotopic to the standard circle S^1, i.e., $\pi_n(\mathbb{S}^1) \cong \pi_n(S^1)$ for any $n \geqq 1$.

Example 4.17. For $S^1 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 = 1\}$, consider another decomposition $\mathcal{D}' = \Big\{\{(-1,0)\}, B = \Big\{(x,y) \in S^1 \,\Big|\, \frac{2}{3} \leqq x \leqq 1\Big\}, C = H^+ \setminus B, D = H_- \setminus B\Big\}$ and the quotient map $\pi : S^1 \to S^1/\mathcal{D}$. Here H^+, H_- are as the above example. Let $a = \pi((-1,0)), b = \pi(B), c = \pi(C), d = \pi(D)$. Then the quotient topology for $S^1/\mathcal{D}'$ is the same as above: $\tau_\pi = \Big\{\emptyset, \{c\}, \{d\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}\Big\}$. However the decomposition map $\pi : S^1 \to S^1/\mathcal{D}$ is not open.

Example 4.18. (see also Example 4.24 below) Take the product of n-copies of the decomposition $\mathcal{D} = \{(-\infty,0), \{0\}, (0,\infty)\}$ of $\mathbb{R}$ in Example 4.13; $\mathcal{D}^n = \mathcal{D}\times\cdots\times\mathcal{D}$ is a decomposition of $\mathbb{R}^n$. It follows from Corollary 4.10 that we have

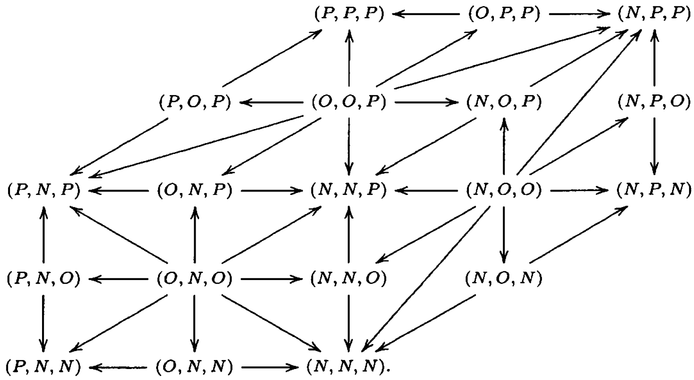
$$\begin{aligned}\mathbb{R}^n/\mathcal{D}^n &= \mathcal{T}\Big((\mathbb{R}/\mathcal{D}, \leqq_{\tau_\pi}) \times\cdots\times (\mathbb{R}/\mathcal{D}, \leqq_{\tau_\pi})\Big)\\ &= \mathcal{T}\Big((\{N,O,P\}, \leqq) \times\cdots\times (\{N,O,P\}, \leqq)\Big)\\ &= \mathcal{T}\Big((\{N,O,P\}\times\cdots\times\{N,O,P\}, \leqq\times\cdots\times\leqq)\Big)\end{aligned}$$

In the case when $n = 2$, the proset (in fact poset) $(\{N,O,P\}\times\{N,O,P\}, \leqq \times \leqq)$ is the following:



From this poset structure we can determine all the open sets of $\mathbb{R}^2/\mathcal{D}^2$.

In the case when n is 3, we just write down the following poset, which is a part of the whole poset $(\{N,O,P\} \times \{N,O,P\} \times \{N,O,P\}, \leqq \times \leqq \times \leqq)$:



Now we discuss a criteria for the decomposition map being open via order $\leqq$. In [37] Dai Tamaki proves the following *"preorder versus frontier-condition" criterion for being open* :

Theorem 4.19. *Let Λ be a poset and let $\pi : X \to \Lambda$ be a surjective continuous map for the Alexandroff topology on Λ. Let $D_\lambda := \pi^{-1}(\lambda)$. π is open if and only if $\lambda \leqq \mu \iff D_\lambda \subset \overline{D_\mu}$.*

In his theorem the target Λ is a poset, however it can be a proset.

Corollary 4.20. *Let $\mathcal{D} = \{D_\lambda\}_{\lambda \in \Lambda}$ be a decomposition of a topological space X such that the decomposition space $X/\mathcal{D}$ becomes an Alexandroff space and let $\leqq_{\tau_\pi}$ be the preorder of the proset $\mathcal{P}((X/\mathcal{D}, \tau_\pi))$ associated to the Alexandroff space. Then the decomposition map $\pi : X \to X/\mathcal{D} = \Lambda$ is open if and only if $\lambda \leqq \tau_\pi \mu \iff D_\lambda \subset \overline{D_\mu}$.*

Then the above proposition implies that an Alexandroff decomposition is lower semicontinuous if and only if $\lambda \leqq \tau_\pi \mu \iff D_\lambda \subset \overline{D_\mu}$ for the decomposition space $\Lambda = X/\mathcal{D}$.

Remark 4.21. For the examples above where the decomposition maps are not open, certainly $\lambda \leqq \mu \Leftrightarrow D_\lambda \subset \overline{D_\mu}$ does not hold.

We can define the preorder on the quotient set $X/\mathcal{D} = \Lambda$ by

$$\lambda \leqq^* \mu \iff D_\lambda \subset \overline{D_\mu}.$$

The above proposition means that the decomposition map $\pi : X \to X/\mathcal{D}$ is open if and only if the proset $\mathcal{P}((X/\mathcal{D}, \tau_\pi))$ is the same as the proset $(X/\mathcal{D}, \leqq^*)$.

Now, when $(X/\mathcal{D}, \leqq^*)$ is defined as above, we have the Alexandroff space $\mathcal{T}((X/\mathcal{D}, \leqq^*)) = (X/\mathcal{D}, \tau_{\leqq^*})$ and a natural question is:

(4.22) *Is the quotient map $\pi : X \to (X/\mathcal{D}, \tau_{\leqq^*})$ continuous?*

The following is an answer to this question:

Theorem 4.23. *Let $\mathcal{D} = \{D_\lambda\}_{\lambda \in \Lambda}$ be a decomposition of a topological space X. The quotient map $\pi : X \to \mathcal{T}((X/\mathcal{D}, \leqq^*)) = (X/\mathcal{D}, \tau_{\leqq^*})$ is continuous if and only if $\mathcal{T}((X/\mathcal{D}, \leqq^*)) = (X/\mathcal{D}, \tau_{\leqq^*})$ is the decomposition space $(X/\mathcal{D}, \tau_\pi)$ (thus $\mathcal{D}$ is an Alexandroff decomposition).*

Theorem 4.23 can be applied to real hyperplane arrangements, as follows:

Example 4.24. Let $\mathcal{A} = \{H_1, H_2, \cdots, H_k\}$ be a real hyperplane arrangement of $\mathbb{R}^n$. Here H_i is a hyperplane defined by an affine form or a linear polynomial $\ell_i = a_{i0} + a_{i1}x_1 + \cdots + a_{ir}x_r + \cdots + a_{in}x_n$: $H_i = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid \ell_i(x_1, x_2, \cdots, x_n) = 0\}$. The hyperplane arrangement defines the decomposition $\mathcal{D}(\mathcal{A})$, which is obtained as follows: We let

$$H_i^- := \{(x_1, \cdots, x_n) \mid \ell_i(x_1, \cdots, x_n) < 0\}, H_i^+ := \{(x_1, \cdots, x_n) \mid \ell_i(x_1, \cdots, x_n) > 0\}.$$

We also let $H_i^0 := H_i$. Then $\mathcal{D}(\mathcal{A}) := \left\{\cap_{i=1}^k A_i \mid A_i \in \{H_i^-, H_i^0, H_i^+\}\right\}$. Here we note that some $\bigcap_{i=1}^k A_i$ can be empty, which is then deleted. The poset $\mathcal{P}((\mathbb{R}^n/\mathcal{D}(\mathcal{A}), \tau_\pi))$ of the decomposition space $(\mathbb{R}^n/\mathcal{D}(\mathcal{A}), \tau_\pi)$ is nothing but the so-called *face poset* $F(\mathcal{A})$, which is the oriented matroid (see [34])[1].

Remark 4.25. In fact, Example 4.18 above is a special case of the above hyperplane arrangement, namely it is the case of the so-called *coordinate hyperplane arrangement*: $\mathcal{A} = \{\{x_1 = 0\}, \{x_2 = 0\}, \cdots, \{x_n = 0\}\}$.

As to the above question (4.22), the frontier condition is "basically" a sufficient condition for the continuity of $\pi : X \to \mathcal{T}((X/\mathcal{D}, \leqq^*)) = (X/\mathcal{D}, \tau_{\leqq^*})$. The following was proved by Hiro Lee Tanaka ([38]):

Proposition 4.26. *Let X be a topological space and let $\pi : X \to \Lambda$ be a surjective map to a set Λ, and let $D_\lambda := \pi^{-1}(\lambda)$ and we define the preorder by $\lambda \leqq \mu \Leftrightarrow D_\lambda \subset \overline{D_\mu}$. If the following two conditions hold, then the map $\pi : X \to \Lambda$ is continuous for the Alexandroff topology for Λ:*

1. *(frontier condition) if $D_\lambda \cap \overline{D_\mu} \neq \emptyset$, then $D_\lambda \subset \overline{D_\mu}$.*
2. *For any closed subset $C \subset \Lambda$, $\bigcup_{\lambda \in C} \overline{D_\lambda}$ is closed. (Note that if Λ is a finite set, then this condition is automatic.)*

5. Poset-stratified spaces

So far, we have not discussed a poset-structure of the proset $\mathcal{P}((X/\mathcal{D}, \tau_\pi))$ of the decomposition space $(X/\mathcal{D}, \tau_\pi)$. As Example 4.15 shows, the proset $\mathcal{P}((X/\mathcal{D}, \tau_\pi))$ is not necessarily a poset. A necessary condition is the following proposition, which follows from Corollary 4.7 above:

Proposition 5.1. *Let $\mathcal{D} = \{D_\lambda | \lambda \in \Lambda\}$ be an Alexandroff decomposition of a topological space X. If the proset $\mathcal{P}((X/\mathcal{D}, \tau_\pi))$ of the decomposition space $(X/\mathcal{D}, \tau_\pi)$ is a poset, then each piece D_λ is locally closed.*

At the moment we do not know if the converse statement holds or not, although we have not found a counterexample such that each piece D_λ is locally closed, but the proset $\mathcal{P}((X/\mathcal{D}, \tau_\pi))$ is not a poset. For the converse statement, we can show the following

Theorem 5.2. *Let $\mathcal{D} = \{D_\lambda | \lambda \in \Lambda\}$ be a decomposition of a topological space X such that the decomposition map $\pi : X \to X/\mathcal{D}$ is open. Then, if each piece D_λ is locally closed, then the proset $\mathcal{P}((X/\mathcal{D}, \tau_\pi))$ is a poset.*

---

[1]In [34] the partial order $\leqq$ is the reversed one. To get the same situation as in [34] we just define the Alexandroff topology via "down-set" instead of "up-set".

Corollary 5.3. *Let Let $\mathcal{D} = \{D_\lambda | \lambda \in \Lambda\}$ be an Alexandroff decomposition of a topological space X and suppose that the decomposition map $\pi : X \to X/\mathcal{D}$ is open. Then the proset $\mathcal{P}((X/\mathcal{D}, \tau_\pi))$ is a poset if and only each piece D_λ is locally closed.*

Remark 5.4. Indeed, in the case of Example 4.15, one can show that the rational part $\mathbb{Q}$ and the irrational part $\mathbb{R} \setminus \mathbb{Q}$ are both not locally closed.

Theorem 5.2 follows from Corollary 4.20 above and the following proposition (cf. [24]):

Proposition 5.5. *Let $\mathcal{D} = \{D_\lambda | \lambda \in \Lambda\}$ be an Alexandroff decomposition of a topological space X such that each piece D_λ is locally closed. If we define the preorder $\lambda \leqq^* \mu \iff D_\lambda \subset \overline{D_\mu}$, then this preorder is a partial order, i.e. $(X/\mathcal{D}, \leqq^*)$ is a poset.*

Corollary 5.6. *Let $\mathcal{D} = \{D_\lambda | \lambda \in \Lambda\}$ be a finite stratification (i.e., $|\Lambda| < \infty$), namely such that*

(1) $D_\lambda \cap D_\mu = \emptyset$ *if* $\lambda \neq \mu$.
(2) $X = \bigcup_\lambda D_\lambda$.
(3) *(locally closed) Each D_λ is a locally closed set*
(4) *(frontier condition)* $D_\lambda \cap \overline{D_\mu} \neq \emptyset \Longrightarrow D_\lambda \subset \overline{D_\mu}$.

Then the decomposition map $\pi : X \to X/\mathcal{D}$ is a continuous map to a poset with the Alexandroff topology.

Such a continuous map from a topological space to a poset considered as a topological space with the Alexandroff topology have been studied in recent papers (e.g., [3], [12], [25], [37], etc.)

In Example 4.14, the decomposition is not such a finite stratification in the above sense and the decomposition map is not an open map, but it is a continuous map to a poset with the Alexandroff topology.

Definition 5.7. A *poset-stratified space* X is a pair $(X, X \xrightarrow{s} P)$ of a topological space X and a continuous map $s : X \to P$ where P is a poset considered as the associated Alexandroff space.

Remark 5.8. The notion of poset-stratified space seems to be due to Jacob Lurie [25]. For a poset-stratified space $(X, X \xrightarrow{s} P)$, X is the underlying topological space and $s : X \to P$ is considered as *a structure of poset-stratification.* If the context is clear, then we just write a poset-stratified space X, just like writing a topological space X without referring to which topology to be considered on it.

The category of poset-stratified spaces is denoted by $\mathcal{S}trat$. The objects are pairs $(X, X \xrightarrow{s} P)$ of a topological space X and a continuous map $s : X \to P$ from the space X to a poset P with the Alexandroff topology associated to the poset P. Given two poset-stratified spaces $(X, X \xrightarrow{s} P)$ and $(X', X' \xrightarrow{s'} P)$, a morphism from $(X, X \xrightarrow{s} P)$ to $(X', X' \xrightarrow{s'} P')$ is a pair of a continuous map $f : X \to X'$ and a monotone map $q : P \to P'$ (thus it is a continuous map for the associated Alexandroff spaces) such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{s} & P \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ X' & \xrightarrow[s']{} & P'. \end{array}$$

6. A POSET-STRATIFIED-SPACE STRUCTURE OF THE SET HOMOTOPY SET $[X, Y]$

In this section we show that for topological spaces X, Y the (unbased) homotopy set $[X, Y]$ can be considered as a poset-stratified space in a natural way ([42]).

First we observe the following:

Lemma 6.1. *Given a proset $(P, \leqq)$, we define the following relation on P:*

$$a, b \in P, a \sim b \iff a \leqq b \text{ and } b \leqq a.$$

(1) *The relation $\sim$ is an equivalence relation and we denote the set of the equivalence classes by $P/_{\sim}$.*

(2) *Then we define the order $\leqq'$ on $P/_{\sim}$ as follows: for $[a], [b] \in P/_{\sim}$*

$$[a] \leqq' [b] \iff a \leqq b.$$

Then this is well-defined, i.e., it does not depend on the representatives a and b.

(3) *The proset $(P_{\sim}, \leqq')$ is a poset, i.e., $[a] \leqq' [b]$ and $[b] \leqq' [a]$ imply that $[a] = [b]$.*

(4) *The projection or quotient map $\pi : (P, \leqq) \to (P_{\sim}, \leqq')$ defined by $\pi(a) := [a]$ is a monotone map.*

Theorem 6.2. *Let $(P, \leqq)$ and $(P_{\sim}, \leqq')$ be as above.*

(1) *For the Alexandroff topologies the quotient map $\pi : (P, \tau_{\leqq}) \to (P_{\sim}, \tau_{\leqq'})$ is an open map. Hence, the Alexandroff topology of the poset $(P_{\sim}, \leqq')$ is the same as the quotient topology of the above quotient map $\pi : (P, \leqq) \to P_{\sim}$.*

(2) *In particular, each equivalence class $\{b \in P \mid a \sim b\}$ of a, i.e., the fiber $\pi^{-1}([a])$, is a locally closed set in the Alexandroff topology of the proset $(P, \leqq)$.*

(3) *In particular, $[a] \leqq' [b]$ if and only if $[a] \subset \overline{[b]}$, where we consider $[a], [b]$ as subsets in P.*

Lemma 6.3. *Let $h\mathcal{T}op$ be the homotopy category of topological spaces and homotopy classes of continuous maps, i.e. the objects of $h\mathcal{T}op$ are all the topological spaces and $hom_{h\mathcal{T}op}(X, Y) = [X, Y]$, the homotopy set of continuous maps, where the homotopy class of a continuous map $f : X \to Y$ is denoted by $[f]$.*

(1) *On the homotopy set $[X, Y]$ we define $[f] \leqq_R [g]$ by $\exists [s] \in [X, X]$ such that $[f] = [g] \circ [s]$. i.e. the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle s}\big\downarrow & \nearrow_{g} & \\ X & & \end{array}$$

commutes up to homotopy. This order is a preorder.

(2) *On the homotopy set $[X, Y]$ we define the relation $[f] \sim_R [g]$ by $[f] \leqq_R [g]$ and $[g] \leqq_R [f]$, which mean that $\exists [s_1], [s_2] \in [X, X]$ such that $[f] = [g] \circ [s_1]$ and $[g] = [f] \circ [s_2]$, i.e., the following diagram commutes up to homotopy:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y. \\ {\scriptstyle s_2}\big\uparrow\big\downarrow{\scriptstyle s_1} & \nearrow_{g} & \\ X & & \end{array}$$

$\sim_R$ is an equivalence relation. The equivalence class of $[f]$ is denoted by $[f]_R$.

(3) *The partial order on the quotient $[X, Y]_R := [X, Y]/_{\sim_R}$ is well-defined as*

$$[f]_R \leqq'_R [g]_R \iff \exists [s] \in [X, X] \text{ such that } [f] = [g] \circ [s].$$

Thus $[X, Y]_R = [X, Y]/_{\sim_R}$ is a poset with the above order.

(4) $\pi_R : ([X,Y], \leqq_R) \to ([X,Y]_R, \leqq'_R)$ *defined by* $\pi_R([f]) := [f]_R$ *is a monotone map.*

Thus from the above Lemma 6.1 and Theorem 6.2 we get the following theorem:

Theorem 6.4. *Let $h\mathcal{T}op$ be the homotopy category and let the set-up be as above.*

(1) *For any objects $X, Y \in \mathcal{O}bj(h\mathcal{T}op)$ the canonical quotient map*

$$\pi_R : ([X,Y], \tau_{\leqq_R}) \to ([X,Y]_R, \leqq'_R)$$

is a poset-stratified space for the Alexandroff topologies.

(2) *In other words, $\mathcal{D} := \{[f]_R\}$ is a decomposition of $[X,Y]$ such that $[f]_R$ (as a subset) is a locally closed set in the Alexandroff space $([X,Y], \tau_{\leqq_R})$.*

(3) *$[f]_R \leqq'_R [g]_R$ if and only if $[f]_R \subset \overline{[g]_R}$ as subsets in $([X,Y], \tau_{\leqq_R})$.*

Corollary 6.5. *Let $h\mathcal{T}op$ be the homotopy category. For any object $S \in \mathcal{O}bj(h\mathcal{T}op)$, we have an associated covariant functor $\mathfrak{st}^S_* : h\mathcal{T}op \to \mathcal{S}trat$ such that*

(1) *for each object $Y \in \mathcal{O}bj(h\mathcal{T}op)$,*

$$\mathfrak{st}^S_*(X) := \Big(([S,X], \tau_{\leqq_R}), ([S,X], \tau_{\leqq_R}) \xrightarrow{\pi_R} ([S,X]_R, \leqq'_R)\Big)$$

(2) *for a morphism $[f] \in [X,Y]$, $\mathfrak{st}^S_*([f])$ is the following commutative diagram:*

$$\begin{array}{ccc} ([S,X], \tau_{\leqq_R}) & \xrightarrow{\pi_R} & ([S,X]_R, \leqq'_R) \\ {\scriptstyle [f]_*}\downarrow & & \downarrow{\scriptstyle [f]_*} \\ ([S,Y], \tau_{\leqq_R}) & \xrightarrow[\pi_R]{} & ([S,Y]_R, \leqq'_R) \end{array}$$

Similarly we can define the following:

Lemma 6.6. *Let $h\mathcal{T}op$ be the homotopy category.*

(1) *On the homotopy set $[X,Y]$ we define the following order $[f] \leqq_L [g]$ by $\exists [t] \in [Y,Y]$ such that $[f] = [t] \circ [g]$. i.e. the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle g}\searrow & \uparrow{\scriptstyle t} \\ & & Y \end{array}$$

commutes up to homotopy. This order is a preorder.

(2) *On the homotopy set $[X,Y]$ we define the relation $[f] \sim_L [g]$ by $[f] \leqq_L [g]$ and $[g] \leqq_L [f]$, which mean that $\exists [t_1], [t_2] \in [Y,Y]$ such that $[f] = [t_1] \circ [g]$ and $[g] = [t_2] \circ [f]$, i.e., the following diagram commutes up to homotopy:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle g}\searrow & {\scriptstyle t_1}\uparrow\downarrow{\scriptstyle t_2} \\ & & Y. \end{array}$$

$\sim_L$ is an equivalence relation. The equivalence class of $[f]$ is denoted by $[f]_L$.

(3) *The partial order on the quotient $[X,Y]_L := [X,Y]/_{\sim_L}$ is well-defined as*

$$[f]_L \leqq'_L [g]_L \iff \exists [t] \in [Y,Y] \ \text{such that} \ [f] = [t] \circ [g].$$

Thus $[X,Y]_L = [X,Y]/_{\sim_L}$ is a poset with the above order.

(4) *$\pi_L : ([X,Y], \leqq_L) \to ([X,Y]_L, \leqq'_L)$ defined by $\pi_L([f]) := [f]_L$ is a monotone map.*

Theorem 6.7. *Let the set-up be as above.*

(1) *For any objects $X, Y \in Obj(h\mathcal{T}op)$ the canonical quotient map*
$$\pi_L : ([X,Y], \tau_{\leqq_L}) \to ([X,Y]_L, \leqq'_L)$$
is a poset-stratified space for the Alexandroff topologies.

(2) *In other words, $\mathcal{D} := \{[f]_L\}$ is a decomposition of $[X,Y]$ such that $[f]_L$ (as a subset) is a locally closed set in the Alexandroff space $([X,Y], \tau_{\leqq_L})$.*

(3) *$[f]_L \leqq'_L [g]_L$ if and only if $[f]_L \subset \overline{[g]_L}$ as subsets in $([X,Y], \tau_{\leqq_L})$.*

Corollary 6.8. *Let $h\mathcal{T}op$ be the homotopy category. For any object $T \in Obj(h\mathcal{T}op)$, we have an associated contravariant functor $\mathfrak{st}^*_T : h\mathcal{T}op \to \mathcal{S}trat$ such that*

(1) *for each object $X \in Obj(h\mathcal{T}op)$,*
$$\mathfrak{st}^*_T(X) := \Big(([X,T], \tau_{\leqq_L}), ([X,T], \tau_{\leqq_L}) \xrightarrow{\pi_L} ([X,T]_L, \leqq'_L) \Big)$$

(2) *for a morphism $[f] \in [X,Y]$, $\mathfrak{st}^*_T([f])$ is the following commutative diagram:*
$$\begin{array}{ccc} ([Y,T], \tau_{\leqq_L}) & \xrightarrow{\pi_L} & ([Y,T]_L, \leqq'_L) \\ {\scriptstyle [f]^*}\downarrow & & \downarrow{\scriptstyle [f]^*} \\ ([X,T], \tau_{\leqq_L}) & \xrightarrow[\pi_L]{} & ([X,T]_L, \leqq'_L) \end{array}$$

If we mix the above two, we get the following:

Lemma 6.9. *Let $h\mathcal{T}op$ be the homotopy category.*

(1) *On the homotopy set $[X,Y]$ we define the order $[f] \leqq_{LR} [g]$ by $\exists [s] \in [X,X]$ and $\exists [t] \in [Y,Y]$ such that $[f] = [t] \circ [g] \circ [s]$. i.e. the diagram*
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle s}\downarrow & & \uparrow{\scriptstyle t} \\ X & \xrightarrow[g]{} & Y \end{array}$$
commutes up to homotopy.

This order is a preorder.

(2) *On the set $[X,Y]$ we define the relation $[f] \sim_{LR} [g]$ by $[f] \leqq_{LR} [g]$ and $[g] \leqq_{LR} [f]$, which mean that $\exists [s_1], [s_2] \in [X,X]$ and $\exists [t_1], [t_2] \in [Y,Y]$ such that $[f] = [t_1] \circ [g] \circ [s_1]$ and $[g] = [t_2] \circ [f] \circ [s_2]$, i.e., the following diagram commutes up to homotopy:*
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle s_2}\uparrow\downarrow{\scriptstyle s_1} & & {\scriptstyle t_1}\uparrow\downarrow{\scriptstyle t_2} \\ X & \xrightarrow[g]{} & Y. \end{array}$$
$\sim_{LR}$ is an equivalence relation. The equivalence class of $[f]$ is denoted by $[f]_{LR}$.

(3) *The partial order on the quotient $[X,Y]_{LR} := [X,Y]/_{\sim_{LR}}$ is well-defined as*
$$[f]_{LR} \leqq'_{LR} [g]_{LR} \Longleftrightarrow \exists [s] \in [X,X],\ \exists [t] \in [Y,Y] \ \text{such that}\ [f] = [t] \circ [g] \circ [s].$$
Thus $[X,Y]_{LR} = [X,Y]/_{\sim_{LR}}$ is a poset with the above order.

(4) *$\pi_{LR} : ([X,Y], \leqq_{LR}) \to ([X,Y]_{LR}, \leqq'_{LR})$ defined by $\pi_{LR}([f]) := [f]_{LR}$ is a monotone map.*

Theorem 6.10. *Let the set-up be as above.*

(1) *For any objects $X, Y \in \mathcal{O}bj(h\mathcal{T}op)$ the canonical quotient map*

$$\pi_{LR} : ([X,Y], \tau_{\leqq_{LR}}) \to ([X,Y]_{LR}, \leqq'_{LR})$$

is a poset-stratified space for the Alexandroff topologies.

(2) *In other words, $\mathcal{D} := \{[f]_{LR}\}$ is a decomposition of $[X,Y]$ such that $[f]_{LR}$ (as a subset) is a locally closed set in the Alexandroff space $([X,Y], \tau_{\leqq_{LR}})$.*

(3) $[f]_{LR} \leqq'_{LR} [g]_{LR}$ *if and only if* $[f]_{LR} \subset \overline{[g]_{LR}}$ *as subsets in* $([X,Y], \tau_{\leqq_{LR}})$.

Remark 6.11. For this mixed situation we cannot get any functor from $\mathcal{C}$ to $\mathcal{S}trat$, unlike the cases of $[X,Y]_R, [X,Y]_L$.

Remark 6.12. For the above relation $[f] \leqq_L [g]$ defined by $\exists t : Y \to Y$ such that $[f] = [t] \circ [g]$, i.e., $f \sim t \circ g$, Jim Stasheff (private communication) informed us that this kind of thing, in a different context, was already considered by Karol Borsuk [9, 10] and Peter Hilton [20] (cf. [21, 22]) in 1950's. According to these papers,

(1) K. Borsuk introduced *dependence of maps*: $f : X \to Y$ is said to *depend on* $g : X \to Y$ if whenever g is extended to $X' \supset X$, so is f. He gave an alternative naming for this notion: *f is a multiple of g* or *g is a divisor of f*. it turned out that this naming was correct, because Borsuk proved that *f depends on g if and only if there exists a map $t : Y \to Y$ such that $f \sim t \circ g$.*

(2) Borsuk defined two maps f and g to be *conjugate* if they depend on each other, i.e., by our notation $[f] \leqq_L [g]$ and $[g] \leqq_L [f]$, i.e., $[f] \sim_L [g]$.

(3) $f : X \to Y$ is said to *co-depend on* $g : X \to Y$ if whenever g lifts to the total space E of a fibration over Y, so does g. Then the dual of the above result is that *f co-depends on g if and only if there exists a map $s : X \to X$ such that $f \sim g \circ s$.* (A remark by the authors: It is natural to define that if f and g co-depends on each other, they are called *co-conjugate*. We are not sure if Borsuk or Hilton defined the notion of co-conjugate.)

(4) *The above results about the co-dependence marks the birth of Eckmann-Hilton duality!*

(5) In fact, *R. Thom* [39] *independently introduced the notion of dependence of cohomology classes*. Thom's dependence is subsumed in Borsuk's dependence.

Thus, using Borsuk's notion, $[X,Y]_R$ and $[X,Y]_L$ are the poset of the homotopy classes of *co-conjugate* maps and *conjugate* maps, resp. Furthermore $[X,Y]_{LR}$ can be considered as the poset of homotopy classes of *conjugate-co-conjugate* maps.

Remark 6.13. The above construction can be done for any locally small category $\mathcal{C}$ instead of the homotopy category $h\mathcal{T}op$ of topological spaces ([45]). In other words, the above is an application of the general construction for any locally small category $\mathcal{C}$ to the case of the homotopy category $h\mathcal{T}op$.

Remark 6.14. For topological spaces X and Y, as seen above, the homotopy set $[X,Y]$ which is the set of homotopy classes of (unbased) continuous maps from X to Y can be considered as a topological space, more precisely an Alexandroff space. In the case of topological space, the homotopy set $[X,Y]$ does have *another canonical topological structure* via the *compact-open topology*. Namely, the set $Map(X,Y)$ is a topological space with the compact-open topology and since the homotopy set $[X,Y] := Map(X,Y)/\sim$ is the quotient of $Map(X,Y)$ via the homotopy equivalence $\sim$ becomes a topological space as a quotient space (with the quotient topology) of the topological space $Map(X,Y)$, using the surjective quotient map $\pi : Map(X,Y) \to [X,Y]$.

If we consider the homotopy set $[X,Y]$ of homotopy classes of based continuous maps from X to Y, a typical example of such a topological space $Map(X,Y)$ is the loop space of X with the base point x_0: $\Omega(X, x_0) = Map((S^1, e), (X, x_0))$. The quotient set $\Omega(X, x_0)/\sim$ becomes a group, i.e. the fundamental group $\pi_1(X, x_0)$. Usually one dose not consider such a topological aspect of the quotient set $\pi_1(X, x_0)$. In fact, in 1950s, in [14] J. Dugundi already considered the fundamental group with such a topology. In [7] D.K. Biss proved that the fundamental group with the induced compact-open topology was a topological group, and he called it *the topological fundamental group* and denoted it by $\pi_1^{top}(X, x_0)$. Later in [16] H. Ghane, Z. Hamed, B. Mashayekhy and H. Mirebrahimi considered topologized higher homotopy groups (called "topological homotopy group" denoted by $\pi_n^{top}(X, x_0)$), generalizing Biss's construction. However, recently in [8] J. Brazas pointed out that there exist some examples of topological spaces whose topologized fundamental group is *not* a topological group, and he showed that Biss's topologized fundamental group is a *quasitopological group*, i.e., a group G with topology such that the inversion $\iota : G \to G, \iota(g) = g^{-1}$ is continuous and that the multiplication $m : G \times G \to G, m(a, b) = ab$ is continuous *in each variable*, i.e., all right and left translations are continuous (in fact homeomorphism). Note that a topological group requires that the multiplication $m : G \times G \to G$ is continuous, where $G \times G$ is the product space. So Biss's topologized fundamental group is now called the *quasitopological fundamental group* and denoted by $\pi_1^{qtop}(X, x_0)$.

7. Applications

In this section we present some applications/examples (for a bit more applications/examples with some details, see [42], [45] and [46]).

Definition 7.1. For a group G let $Sub(G)$ be the set of all subgroups of the group G. For subgroups $A, B \in Sub(G)$ we define the order $A \leqq B$ by the usual inclusion $A \subseteq B$, which is clearly a partial order.

Lemma 7.2. *Let $H_*(-)$ be the homology theory with a coefficient ring R. Then the following maps are well-defined and monotone (order-preserving) maps:*

(1) $\mathrm{Im}_{H_*} : ([X, Y], \leqq_R) \to (Sub(H_*(Y)), \leqq), \mathrm{Im}_{H_*}([f]) := \mathrm{Im}(f_* : H_*(X) \to H_*(Y))$.
(2) $\mathrm{Im}'_{H_*} : ([X, Y]_R, \leqq'_R) \to (Sub(H_*(Y)), \leqq), \mathrm{Im}'_{H_*}([f]_R) := \mathrm{Im}_{H_*}([f])$.

We have the following commutative diagram:

$$\begin{array}{ccc} ([X,Y], \leqq_R) & \xrightarrow{\pi_R} & ([X,Y]_R, \leqq'_R) \\ \Big\downarrow{\scriptstyle \mathrm{id}_{[X,Y]}} & & \Big\downarrow{\scriptstyle \mathrm{Im}'_{H_*}} \\ ([X,Y], \leqq_R) & \xrightarrow[\mathrm{Im}_{H_*}]{} & (Sub(H_*(Y)), \leqq). \end{array}$$

Similarly we get the following:

Lemma 7.3. *Let $H^*(-)$ be the cohomology theory with a coefficient ring R. Then the following maps are well-defined and monotone (order-preserving) maps:*

(1) $\mathrm{Im}_{H^*} : ([X, Y], \leqq_L) \to (Sub(H^*(X)), \leqq), \mathrm{Im}_{H^*}([f]) := \mathrm{Im}(f^* : H^*(Y) \to H_*(X))$.
(2) $\mathrm{Im}'_{H^*} : ([X, Y]_R, \leqq'_L) \to (Sub(H^*(X)), \leqq), \mathrm{Im}'_{H^*}([f]_R) := \mathrm{Im}_{H^*}([f])$.

We have the following commutative diagram:

$$\begin{array}{ccc} ([X,Y], \leqq_L) & \xrightarrow{\pi_L} & ([X,Y]_L, \leqq'_L) \\ \Big\downarrow{\scriptstyle \mathrm{id}_{[X,Y]}} & & \Big\downarrow{\scriptstyle \mathrm{Im}'_{H^*}} \\ ([X,Y], \leqq_L) & \xrightarrow[\mathrm{Im}_{H^*}]{} & (Sub(H^*(X)), \leqq). \end{array}$$

Corollary 7.4. *Let $H_*(-)$ and $H^*(-)$ be as above.*

(1) *For* $\forall S \in Obj(h\mathcal{T}op)$, *we have a* **covariant** *functor* $\mathfrak{st}^S_{H_*} : h\mathcal{T}op \to \mathcal{S}trat$ *such that*

(a) *for each object* $X \in Obj(h\mathcal{T}op)$,

$$\mathfrak{st}^S_{H_*}(X) := \left(([S,X], \tau_{\leqq_R}), ([S,X], \tau_{\leqq_R}) \xrightarrow{\mathrm{Im}_{H_*}} (\mathcal{S}ub(H_*(X)), \leqq) \right).$$

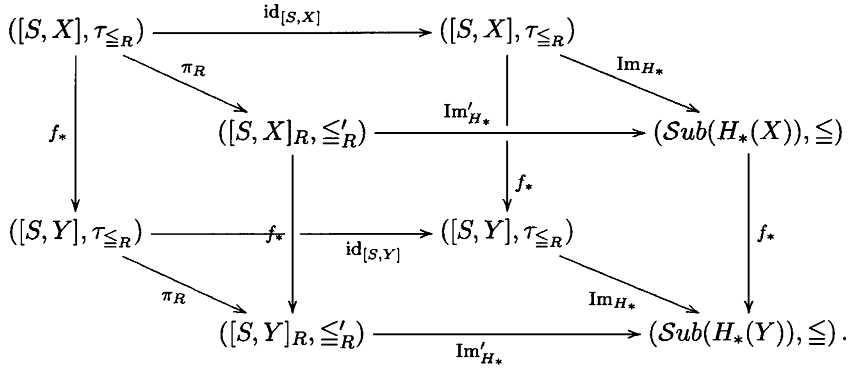
(b) *for a morphism* $[f] \in [X,Y]$, $\mathfrak{st}^S_{H_*}([f])$ *is the following commutative diagram:*

$$\begin{array}{ccc} ([S,X], \tau_{\leqq_R}) & \xrightarrow{\mathrm{Im}_{H_*}} & (\mathcal{S}ub(H_*(X)), \leqq) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ ([S,Y], \tau_{\leqq_R}) & \xrightarrow[\mathrm{Im}_{H_*}]{} & (\mathcal{S}ub(H_*(Y)), \leqq) . \end{array}$$

(2) Im'_{H_*} *gives rise to a natural transformation* $\mathrm{Im}'_{H_*} : \mathfrak{st}^S_*(-) \to \mathfrak{st}^S_{H_*}(-)$, *namely for a morphism* $[f] \in [X,Y]$ *we have the following commutative diagram:*

$$\begin{array}{ccc} \mathfrak{st}^S_*(X) & \xrightarrow{\mathrm{Im}'_{H_*}} & \mathfrak{st}^S_{H_*}(X) . \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathfrak{st}^S_*(Y) & \xrightarrow[\mathrm{Im}'_{H_*}]{} & \mathfrak{st}^S_{H_*}(Y) \end{array}$$

Namely we have the following commutative cube:



(3) *For any object* $T \in Obj(h\mathcal{T}op)$, *we have an associated* **contravariant** *functor* $\mathfrak{st}^{H^*}_T : h\mathcal{T}op \to \mathcal{S}trat$ *such that*

(a) *for each object* $X \in Obj(h\mathcal{T}op)$,

$$\mathfrak{st}^{H^*}_T(X) := \left(([X,T], \tau_{\leqq_L}), ([X,T], \tau_{\leqq_L}) \xrightarrow{\mathrm{Im}_{H^*}} (\mathcal{S}ub(H^*(X)), \leqq) \right)$$

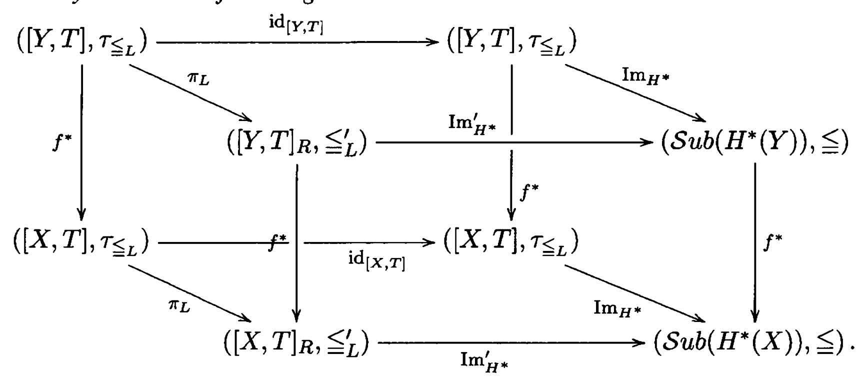
(b) *for a morphism* $[f] \in [X,Y]$, $\mathfrak{st}^{H^*}_T([f])$ *is the following commutative diagram:*

$$\begin{array}{ccc} ([Y,T], \tau_{\leqq_L}) & \xrightarrow{\mathrm{Im}_{H^*}} & (\mathcal{S}ub(H^*(Y)), \leqq) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ ([X,T], \tau_{\leqq_L}) & \xrightarrow[\mathrm{Im}_{H^*}]{} & (\mathcal{S}ub(H^*(X)), \leqq) . \end{array}$$

(4) Im'_{H^*} *gives rise to a natural transformation* $\mathrm{Im}'_{H^*} : \mathfrak{st}^*_T(-) \to \mathfrak{st}^{H^*}_T(-)$, *namely for a morphism* $[f] \in [X, Y]$ *we have the following commutative diagram:*

$$\begin{array}{ccc} \mathfrak{st}^*_T(Y) & \xrightarrow{\mathrm{Im}'_{H^*}} & \mathfrak{st}^{H^*}_T(Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathfrak{st}^*_T(X) & \xrightarrow[\mathrm{Im}'_{H^*}]{} & \mathfrak{st}^{H^*}_T(X). \end{array}$$

Namely we have the following commutative cube:



The case of $\mathrm{Im}_{H^*} : ([X,T], \leqq_L) \to (Sub(H^*(X), \leqq)$ is related to Thom's dependence of cohomology classes [39] mentioned above. To explain that, we recall the definition of dependence of cohomology classes (e.g., see [20]).

Definition 7.5 (R. Thom)**.** The cohomology class $\beta \in H^q(X; B)$ *depends on* the cohomology class $\alpha \in H^p(X; A)$, where A, B are coefficient rings, if, for all (perhaps infinite) polyhedra Y and all maps $f : X \to Y$ such that $\alpha \in f^*(H^p(Y; A))$, we have $\beta \in f^*(H^q(Y; B))$.

We also recall that the cohomology theory is representable by the Eilenberg-Maclane space, i.e., $H^j(X; R) \cong [X, K(R, j)]$ where $K(R, j)$ is the Eilenberg-Maclane space whose homotopy type is completely characterized by the homotopy groups $\pi_j(K(R, j)) = R$ and $\pi_i(K(R, j)) = 0, i \neq j$. Then by the Hurewicz Theorem we have $H_j(K(R, j); \mathbb{Z}) \cong \pi_j(K(R, j)) = R$ and $H_d(K(R, j)) = 0$ for $d < j$. Hence by the universal coefficient theorem (i.e., $H^j(X; R) \cong Hom(H_j(X; \mathbb{Z}) \oplus \mathrm{Ext}^1(H_{j-1}(X; \mathbb{Z}), R))$ we have the isomorphism

$$\Phi : H^j(K(R, j); R) \cong Hom(H_j(K(R, j); \mathbb{Z}), R) \cong Hom(\pi_j(K(R, j)), R) \cong Hom(R, R).$$

Let $u := \Phi^{-1}(\mathrm{id}_R)$ for the identity map $\mathrm{id}_R : R \to R$. Then the above isomorphism $\Theta : [X, K(R, j)] \cong H^j(X, R)$ is obtained by $\Theta([f]) := f^* u$ where $f^* : H^j(K(R, j); R) \to H^j(X, R)$. Thom [39] proves the following proposition (also see [20]).

Proposition 7.6. *Let* $\alpha \in H^p(X; A) \cong [X, K(A, p)]$ *and let* $f_\alpha : X \to K(A, p)$ *be a map such that the homotopy class* $[f_\alpha]$ *corresponds to* α. *Then* $\beta \in H^q(X, B)$ *depends on* α *if and only if* $\beta \in f^*_\alpha(H^q(K(A, p); B))$.

Using this proposition we can get the following result. By the monotone map

$$\mathrm{Im}_{H^*} : ([X, K(A, p)], \leqq_L) \to (Sub(H^*(X; B)), \leqq)$$

the image $\mathrm{Im}_{H^*}([f_\alpha]) = f^*_\alpha((H^q(K(A, p); B))$ is *nothing but the subgroup of all the cohomology classes* $\beta \in H^q(X; B)$ *depending on the cohomology class* α. Now we let $\alpha, \alpha' \in H^p(X, A)$

and let $f_\alpha, f_{\alpha'} : X \to K(A,p)$ be the corresponding maps. Then, if f_α depends on $f_{\alpha'}$, i.e., $[f_\alpha] \leqq_L [f_{\alpha'}]$ by our terminology (in other words, we can define the order of the cohomology classes $\alpha \leqq_L \alpha'$ by this), then we have $(\alpha \in) \operatorname{Im}_{H^*}([f_\alpha]) \subset \operatorname{Im}_{H^*}([f_{\alpha'}])$, i.e., $\operatorname{Im}_{H^*}([f_\alpha]) \leqq \operatorname{Im}_{H^*}([f_{\alpha'}])$. Thus, that α depends on α' is equivalent to that $\operatorname{Im}_{H^*}([f_\alpha]) \leqq \operatorname{Im}_{H^*}([f_{\alpha'}])$.

Here is another application to vector bundles and characteristic classes (e.g., see [31], [19]). Let $\mathrm{Vect}_n(X)$ be the set of isomorphism classes of complex vector bundles of rank n. Then we do know that

$$\mathrm{Vect}_n(X) \cong [X, G_n(\mathbb{C}^\infty)]$$

where $G_n(\mathbb{C}^\infty)$ is the infinite Grassmann manifold of complex planes of dimension n, i.e., the classifying space of complex vector bundles of rank n. This isomorphism is by the correspondence $[E] \longleftrightarrow [f_E]$, where $f_E : X \to G_n(\mathbb{C}^\infty)$ is a classifying map of E, i.e., $E = f_E^*\gamma^n$, where γ^n is the universal complex vector bundle of rank n over $G_n(\mathbb{C}^\infty)$.

By the isomorphism $\mathrm{Vect}_n(X) \cong [X, G_n(\mathbb{C}^\infty)]$ we can consider the preorder of $[E]$ and $[F]$:

$$[E] \leqq_L [F] \Longleftrightarrow [f_E] \leqq_L [f_F],$$

where $f_E, f_F : X \to G_n(\mathbb{C}^\infty)$ are respectively the classifying maps of E and F.

Then we have the following well-defined monotone (order-preserving) map:

$$\operatorname{Im}_{H^*} : (\mathrm{Vect}_n(X), \leqq_L) \to (\mathcal{S}ub(H^*(X;\mathbb{Z})), \leqq)$$

defined by

$$\operatorname{Im}_{H^*}([E]) := \operatorname{Im}\Big(f_E^* : H^*(G_n(\mathbb{C}^\infty);\mathbb{Z}) \to H^*(X;\mathbb{Z})\Big).$$

By the definition of characteristic classes, $\operatorname{Im}\Big(f_E^* : H^*(G_n(\mathbb{C}^\infty);\mathbb{Z}) \to H^*(X;\mathbb{Z})\Big)$ *is nothing but the subring consisting of all the characteristic classes of E*, which is $\mathbb{Z}[c_1(E), c_2(E), \cdots, c_n(E)]$. Let us denote this subring by $\mathrm{Char}(E)$. Therefore we have

$$[E] \leqq_L [F] \Longrightarrow \mathrm{Char}(E) \subseteq \mathrm{Char}(F).$$

We also get that $[E] \sim_L [F] \Longrightarrow \mathrm{Char}(E) = \mathrm{Char}(F)$.

Question 7.7. *Is it true that* $\mathrm{Char}(E) = \mathrm{Char}(F) \Longrightarrow [E] \sim_L [F]$*?*

Even if the above answer is negative, we can introduce the following equivalence relation on $\mathrm{Vect}_n(X)$:

$$[E] \sim_{Char} [F] \Longleftrightarrow \mathrm{Char}(E) = \mathrm{Char}(F).$$

If we denote the image of $\operatorname{Im}_{H^*} : \mathrm{Vect}_n(X) \to \mathcal{S}ub(H^*(X;\mathbb{Z}))$ by $\mathcal{C}har\mathcal{S}ub(H^*(X;\mathbb{Z}))$, then we get the surjective monotone map

$$\operatorname{Im}_{H^*} : \mathrm{Vect}_n(X) \to \mathcal{C}har\mathcal{S}ub(H^*(X;\mathbb{Z})).$$

Then each fiber $\operatorname{Im}_{H^*}^{-1}(\mathrm{Char}(E))$ is the set of the isomorphism classes $[F]$ such that $[E] \sim_{Char} [F]$. If we consider Alexandroff topologies on the proset $(\mathrm{Vect}_n(X), \leqq_L)$ and the poset $(\mathcal{C}har\mathcal{S}ub(H^*(X;\mathbb{Z})), \subseteq)$, we have a continuous map $\operatorname{Im}_{H^*} : \mathrm{Vect}_n(X) \to \mathcal{C}har\mathcal{S}ub(H^*(X;\mathbb{Z}))$, hence each fiber $\operatorname{Im}_{H^*}^{-1}(\mathrm{Char}(E))$ is a locally closed set, because each singleton in a poset with the associated Alexandroff topology is a locally closed set. In other words, the above equivalence relation $\sim_{Char}$ (via the characteristic subring $\mathrm{Char}(E)$) gives rise to a canonical naive "stratification" of $\mathrm{Vect}_n(X)$.

Question 7.8. *Is the above equivalence relation* $\sim_{Char}$ *useful for vector bundles?*

Question 7.9. *If we define* $[E] \sim_{Char\,\mathbb{Q}} [F] \Leftrightarrow \mathrm{Char}(E) \otimes \mathbb{Q} = \mathrm{Char}(F) \otimes \mathbb{Q}(\subset H^*(X;\mathbb{Q}))$, *then we get a larger stratification, i.e., each strata is larger. Is such a larger stratification useful?*

We would like to consider these questions in a different paper.

Remark 7.10. In the case of real vector bundles, the complex infinite Grassmann $G_n(\mathbb{C}^\infty)$, the Chern class c_i and the coefficient ring $\mathbb{Z}$ are respectively replaced by the real infinite Grassmann $G_n(\mathbb{R}^\infty)$, the Stiefell-Whitney class w_i and the coefficient ring $\mathbb{Z}_2$.

Remark 7.11. Instead of homology $H_*(-)$ and cohomology $H^*(-)$, we can consider homotopy version of these, i.e., homotopy groups $\pi_*(-)$ and cohomotopy "groups" $\pi^*(-)$. In this case we consider the based homotopy set $[X,Y]_*$. We note that the cohomotopy set $\pi^p(X) := [X, S^p]$ (e.g., see [23]).

Remark 7.12. In the case of a general locally small category $\mathcal{C}$, if we have reasonable covariant functors $\mathcal{H}_*$ and contra variant functors $\mathcal{H}^*$, we can do similar things to the above.

When it comes to the homotopy groups π_*, we have another application. Let $Map(X,Y:f)$ be the path component of $Map(X,Y)$ containing f. Let $*$ be the base point of X and we consider the evaluation map

$$ev : Map(X,Y:f) \to Y \quad ev(g) := g(*).$$

Definition 7.13 ([40]). For a continuous map $f : X \to Y$, *the n-th evaluation subgroup $G_n(Y,X:f)$* of the n-th homotopy group $\pi_n(Y)$ is defined as follows:

$$G_n(Y,X:f) := \mathrm{Im}\Big(ev_* : \pi_n(Map(X,Y:f)) \to \pi_n(Y)\Big).$$

This is a generalized version of the following Gottlieb group $G_n(X)$ ([17, 18]):

$$G_n(X) := \mathrm{Im}\Big(ev_* : \pi_n(aut_1 X) \to \pi_n(X)\Big),$$

where $aut_1 X = Map(X,X:\mathrm{id}_X)$ and id_X is the identity map.

The n-th evaluation subgroup $G_n(Y,X:f)$ can be described as follows:

Lemma 7.14 ([40]). *The n-th evaluation subgroup of a map $f : X \to Y$ is*

$$G_n(Y,X;f) := \left\{ a \in \pi_n(Y) \;\middle|\; \begin{array}{ccc} X \times S^n & \xleftarrow{\;i_{S^n}\;} & S^n \\ {\scriptstyle i_X}\uparrow & \exists\phi \searrow & \downarrow{\scriptstyle a} \\ X & \xrightarrow[f]{} & Y \end{array} \text{ is homotopy commutative} \right\}$$

from the adjointness.

As to the case of generalized Gottlieb groups, we need to reverse the order.

Proposition 7.15. *The following map (called "the n-th generalized Gottlieb evaluation subgroup map")*

$$\mathfrak{g}_n : [X,Y] \to \mathcal{S}(\pi_n(Y)) \quad G_n([f]) := G_n(Y,X:f)$$

is well-defined, i.e., $f \sim f'$ implies that $G_n(Y,X:f) = G_n(Y,X:f')$.

Proposition 7.16. *The following map (called "the finer n-th generalized Gottlieb evaluation subgroup map")*

$$\mathfrak{g}_n^R : [X,Y]_R \to \mathcal{S}(\pi_n(Y)) \quad \mathfrak{g}_n^R([f]_R) := G_n(Y,X:[f]) = G_n(Y,X:f)$$

is well-defined, i.e., $[f] \sim_R [g]$ implies that $G_n(Y,X;f) = G_n(Y,X;g)$. Namely the following diagram commutes:

$$\begin{array}{ccc} [X,Y] & \xrightarrow{\pi_R} & [X,Y]_R \\ & \searrow^{\mathfrak{g}_n} & \downarrow \mathfrak{g}_n^R \\ & & \mathcal{S}(\pi_n(Y)) \end{array}$$

Corollary 7.17. (1) *If $[f] \leqq_R [g]$, then we have $G_n(Y,X;g) \subset G_n(Y,X;f)$, i.e., $\mathfrak{g}_n([g]) \leqq \mathfrak{g}_n([f])$. Hence*
$\mathfrak{g}_n : ([X,Y], \leqq_R^{op}) \to \mathcal{S}ub(\pi_n(Y), \leqq)$ *is a monotone map.*

(2) *If $[f]_R \leqq [g]_R$, then we have $G_n(Y,X;g) \subset G_n(Y,X;f)$, i.e., $\mathfrak{g}_n^R([g]_R) \leqq \mathfrak{g}_n^R([f]_R)$. Hence*
$\mathfrak{g}_n^R : ([X,Y]_R, \leqq_R^{\prime op}) \to \mathcal{S}ub(\pi_n(Y), \leqq)$ *is a monotone map.*

We also have the following commutative diagram:

$$\begin{array}{ccc} ([X,Y], \leqq_R^{op}) & \xrightarrow{\pi_R} & ([X,Y]_R, \leqq_R^{\prime op}) \\ {\scriptstyle \mathrm{id}_{[X,Y]}} \downarrow & & \downarrow {\scriptstyle \mathfrak{g}_n^R} \\ ([X,Y], \leqq_R^{op}) & \xrightarrow[\mathfrak{g}_n]{} & (\mathcal{S}ub(\pi_n(Y)), \leqq) \end{array} .$$

Corollary 7.18. (1) *For $\forall S \in Obj(h\mathcal{T}op)$, we have a **covariant** functor $\mathfrak{st}_{Gott}^S : h\mathcal{T}op \to \mathcal{S}trat$ such that*

(a) *for each object $X \in Obj(h\mathcal{T}op)$,*
$\mathfrak{st}_{Gott}^S(X) := \Big(([S,X], \tau_{\leqq_R^{op}}), ([S,X], \tau_{\leqq_R^{op}}) \xrightarrow{\mathfrak{g}_n} (\mathcal{S}ub(\pi_n(X)), \leqq) \Big)$.

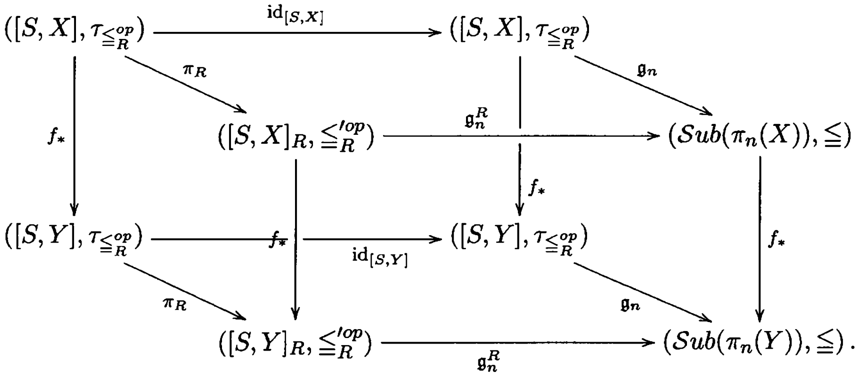
(b) *for a morphism $[f] \in [X,Y]$, $\mathfrak{st}_{Gott}^S([f])$ is the following commutative diagram:*

$$\begin{array}{ccc} ([S,X], \tau_{\leqq_R^{op}}) & \xrightarrow{\mathfrak{g}_n} & (\mathcal{S}ub(\pi_n(X)), \leqq) \\ {\scriptstyle f_*} \downarrow & & \downarrow {\scriptstyle f_*} \\ ([S,Y], \tau_{\leqq_R^{op}}) & \xrightarrow[\mathfrak{g}_n]{} & (\mathcal{S}ub(\pi_n(Y)), \leqq) \end{array} .$$

(2) *$\mathfrak{g}_n^R$ gives rise to a natural transformation $\mathfrak{g}_n^R : \mathfrak{st}_*^S(-) \to \mathfrak{st}_{Gott}^S(-)$, namely for a morphism $[f] \in [X,Y]$ we have the following commutative diagram:*

$$\begin{array}{ccc} \mathfrak{st}_*^S(X) & \xrightarrow{\mathfrak{g}_n^R} & \mathfrak{st}_{Gott}^S(X) \\ {\scriptstyle f_*} \downarrow & & \downarrow {\scriptstyle f_*} \\ \mathfrak{st}_*^S(Y) & \xrightarrow[\mathfrak{g}_n^R]{} & \mathfrak{st}_{Gott}^S(Y) \end{array} .$$

Namely we have the following commutative cube:



Remark 7.19. When it comes to the case $[X, Y]_L$ we do not have similar results as above.

Example 7.20. Let $\mathrm{cat}(f)$ be the Lusternik-Schnirelmann category of a map $f : X \to Y$ ([15, p.352]). Then $\mathrm{cat} : [X, Y] \to (\mathbb{Z}_{\geqq 0}, \leqq)$ is a monotone map. In the case of cat, we have the three finer poset-stratified space structure on the reversed ordered posets $[X, Y]_R$, $[X, Y]_L$ and $[X, Y]_{LR}$ as follows:

(1) If $[g] \leqq_R [f]$, i.e., $g \sim f \circ s$ with $s : X \to X$, then we have ([15, Lemma 27.1(ii)])

$$\mathrm{cat}(g) = \mathrm{cat}(f \circ s) \leqq \min\{\mathrm{cat}(f), \mathrm{cat}(s)\} \leqq \mathrm{cat}(f).$$

Hence we have $\mathrm{cat}(g) \leqq \mathrm{cat}(f)$. Thus there is a poset map $\mathrm{cat}_R : [X, Y]_R \to (\mathbb{Z}_{\geqq 0}, \leqq)$. Here $\mathrm{cat}_R([f]_R) := \mathrm{cat}(f)$.

(2) If $[g] \leqq_L [f]$, i.e., $g \sim t \circ f$ with $t : Y \to Y$, then we have

$$\mathrm{cat}(g) = \mathrm{cat}(t \circ f) \leq \min\{\mathrm{cat}(t), \mathrm{cat}(f)\} \leqq \mathrm{cat}(f).$$

Hence we have $\mathrm{cat}(g) \leqq \mathrm{cat}(f)$. Thus $\mathrm{cat}_L : [X, Y]_L \to (\mathbb{Z}_{\geqq 0}, \leqq)$ is a poset map. Here $\mathrm{cat}_L([f]_R) := \mathrm{cat}(f)$.

(3) If $[g] \leqq_{LR} [f]$, i.e., $g \sim h \circ f \circ s$ with $s : X \to X$ and $t : Y \to Y$, then we have

$$\mathrm{cat}(g) = \mathrm{cat}(t \circ f \circ s) \leq \min\{\mathrm{cat}(t), \mathrm{cat}(f), \mathrm{cat}(s)\} \leqq \mathrm{cat}(f).$$

Hence we have $\mathrm{cat}(g) \leqq \mathrm{cat}(f)$. Thus $\mathrm{cat}_{LR} : [X, Y]_{LR} \to (\mathbb{Z}_{\geqq 0}, \leqq)$ is a poset map. Here $\mathrm{cat}_{LR}([f]_R) := \mathrm{cat}(f)$.

Namely we have the following commutative diagrams:

$$\begin{array}{ccc} [X,Y] & \xrightarrow{\pi_R} & [X,Y]_R \\ {\scriptstyle \mathrm{id}_{[X,Y]}}\downarrow & & \downarrow{\scriptstyle \mathrm{cat}_R} \\ [X,Y] & \xrightarrow[\mathrm{cat}]{} & (\mathbb{Z}_{\geqq 0}, \leqq), \end{array} \quad \begin{array}{ccc} [X,Y] & \xrightarrow{\pi_L} & [X,Y]_L \\ {\scriptstyle \mathrm{id}_{[X,Y]}}\downarrow & & \downarrow{\scriptstyle \mathrm{cat}_L} \\ [X,Y] & \xrightarrow[\mathrm{cat}]{} & (\mathbb{Z}_{\geqq 0}, \leqq), \end{array} \quad \begin{array}{ccc} [X,Y] & \xrightarrow{\pi_{LR}} & [X,Y]_{LR} \\ {\scriptstyle \mathrm{id}_{[X,Y]}}\downarrow & & \downarrow{\scriptstyle \mathrm{cat}_{LR}} \\ [X,Y] & \xrightarrow[\mathrm{cat}]{} & (\mathbb{Z}_{\geqq 0}, \leqq). \end{array}$$

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