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TWO OPEN-POINT GAMES RELATED TO SELECTIVE (SEQUENTIAL) PSEUDOCOMPACTNESS, WITH APPLICATION TO 1-CL-STARCOMPACTNESS PROPERTY OF MATVEEV

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ABSTRACT. A topological space X is *selectively sequentially pseudocompact* (*selectively pseudocompact*) if for every sequence $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, one can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ in such a way that the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence (respectively, has an accumulation point in X). It was shown by the authors in [3] that the class of selectively sequentially pseudocompact spaces is closed under taking arbitrary products and continuous images, contains the class of dyadic spaces and forms a proper subclass of the class of selectively pseudocompact spaces. Moreover, the latter class coincides with the class of strongly pseudocompact spaces of García-Ferreira and Ortiz-Castillo [7].

In this paper, we define two topological games closely related to the class of selectively (sequentially) pseudocompact spaces. Let X be a topological space. At round n, Player A chooses a non-empty open subset U_n of X, and Player B responds by selecting a point $x_n \in U_n$. In the selectively sequentially pseudocompact game $Ssp(X)$, Player B wins if the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence; otherwise Player A wins. In the selectively pseudocompact game $Sp(X)$, Player B wins if the sequence $\{x_n : n \in \mathbb{N}\}$ has an accumulation point in X; otherwise Player A wins. The (non-)existence of winning strategies for each player in the game $Ssp(X)$ (in the game $Sp(X)$) defines a compactness-like property of X sandwiched between sequential compactness (countable compactness) and selective sequential pseudocompactness (selective pseudocompactness) of X.

We prove that a topological space X such that Player A does not have a winning strategy in $Sp(X)$, is 1-cl-starcompact in the sense of Matveev. As an application of this result, we give an example of a locally compact, first-countable, zero-dimensional, 1-cl-starcompact space without a dense relatively countably compact subspace. This shows that Theorem 15 in Matveev's survey [10] is not reversible.

All topological spaces are assumed to be Tychonoff.

The symbol $\mathbb{N}$ denotes the set of *positive* natural numbers; that is, $\mathbb{N} = \{1, 2, 3, \dots\}$.

1. SELECTIVE (SEQUENTIAL) PSEUDOCOMPACTNESS

A point x is said to be an *accumulation point* of a sequence $\{x_n : n \in \mathbb{N}\}$ of points of a topological space X provided that the set $\{n \in \mathbb{N} : x_n \in U\}$ is infinite for every open neighbourhood U of x in X.

Let us recall two well-known compactness-type properties.

Definition 1.1. A topological space X is called:

(i) *sequentially compact* if every sequence in X has a convergent subsequence;

(ii) *countably compact* if every sequence in X has an accumulation point in X.

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In the next remark we restate this definition in order to emphasize the selective character of the properties appearing in it.

Remark 1.2. A topological space X is:

(i) sequentially compact if and only if for every sequence $\{A_n : n \in \mathbb{N}\}$ of singletons in X, we can chose a point $x_n \in A_n$ for every $n \in \mathbb{N}$ in such a way that the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence;

(ii) countably compact if and only if for every sequence $\{A_n : n \in \mathbb{N}\}$ of singletons in X, we can chose a point $x_n \in A_n$ for every $n \in \mathbb{N}$ in such a way that the sequence $\{x_n : n \in \mathbb{N}\}$ has an accumulation point in X.

By replacing singletons A_n in Remark 1.2 with non-empty open subsets U_n of X, one naturally obtains two selective properties which are weaker than sequential compactness and countable compactness of X, respectively.

Definition 1.3. [3] A topological space X is:

(i) *selectively sequentially pseudocompact* if for every sequence $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, we can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ in such a way that the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence;

(ii) *selectively pseudocompact* if and only if for every sequence $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, we can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ in such a way that the sequence $\{x_n : n \in \mathbb{N}\}$ has an accumulation point in X.

It was proved in [3, Theorem 2.1] that the property from item (ii) is equivalent to the notion of strong pseudocompactness of García-Ferreira and Ortiz-Castillo introduced in [7].

2. Basic properties of selectively (sequentially) pseudocompact spaces

In this section, we list the most important results from [3] about the basic properties of the class of selectively (sequentially) pseudocompact spaces.

Proposition 2.1. [3] *Let $f : X \to Y$ be a continuous map from a topological space X onto a topological space Y. If X is selectively (sequentially) pseudocompact, then so is Y.*

Lemma 2.2. [3] *Suppose that for every sequence $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of a topological space X, there exists a selectively (sequentially) pseudocompact subspace Y of X such that $U_n \cap Y \neq \emptyset$ for all $n \in \mathbb{N}$. Then X is selectively (sequentially) pseudocompact.*

Corollary 2.3. [3] *If every countable subset of a topological space X is contained in a selectively (sequentially) pseudocompact subspace of X, then X is selectively (sequentially) pseudocompact.*

Corollary 2.4. [3] *If some dense subspace of a topological space X is selectively (sequentially) pseudocompact, then X itself is selectively (sequentially) pseudocompact.*

Definition 2.5. If p is a point in the product $X = \prod_{i \in I} X_i$ of a family $\{X_i : i \in I\}$ of sets, then the subset

(1) $$\Sigma(p, X) = \{f \in X : \text{ the set } \{i \in I : f(i) \neq p(i)\} \text{ is at most countable}\}$$

of X is called the *Σ-product of $\{X_i : i \in I\}$ with the basis point $p \in X$*. If each X_i is a topological space, then we consider $\Sigma(p, X)$ with the subspace topology it inherits from the Cartesian product $X = \prod_{i \in I} X_i$.

Theorem 2.6. [3] *Let $X = \prod_{i\in I} X_i$ be the product of a family $\{X_i : i \in I\}$ of topological spaces and $p \in X$.*

(i) *If all X_i are selectively sequentially pseudocompact, then so is $\Sigma(p, X)$.*
(ii) *If $\prod_{i\in J} X_i$ is selectively pseudocompact for every at most countable set $J \subseteq I$, then so is $\Sigma(p, X)$.*

Since a Σ-product $\Sigma(X, p)$ is dense in the corresponding product X, from Proposition 2.1, Corollary 2.4 and Theorem 2.6, we obtain the following corollary.

Corollary 2.7. [3]

(i) *A product of topological spaces is selectively sequentially pseudocompact if and only if each factor is selectively sequentially pseudocompact.*
(ii) *The product $\prod_{i\in I} X_i$ of a family $\{X_i : i \in I\}$ of topological spaces is selectively pseudocompact if and only if its subproduct $\prod_{i\in J} X_i$ is selectively pseudocompact for every at most countable set $J \subseteq I$.*

Example 2.8. Let X be a countably compact space such that its square X^2 is not pseudocompact; see [6, Example 3.10.9]. Since countably compact spaces are selectively pseudocompact and selectively pseudocompact spaces are pseudocompact, this shows that *selective pseudocompactness is not a productive property.*

Proposition 2.9. [3]

(i) *Every infinite selectively sequentially pseudocompact space has a non-trivial convergent sequence.*
(ii) *Every infinite selectively pseudocompact space contains a countable non-closed subset.*

It should be noted that, in contrast with item (i) of this proposition, even an infinite selectively pseudocompact group need not contain non-trivial convergent sequences [13].

3. A diagram displaying connections between selective (sequential) pseudocompactness and known compactness properties

Definition 3.1. A space X is called *sequentially pseudocompact* if for every family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, there exists an infinite set $J \subseteq \mathbb{N}$ and a point $x \in X$ such that the set $\{n \in J : W \cap U_n = \emptyset\}$ is finite for every open neighborhood W of x.

This notion is mentioned on page 15 of Matveev's survey [10] and attributed to an unpublished manuscript of Reznichenko; see citation no. 152 in [10]. The same notion appeared later in [5, Definition 1.4] under the name sequentially feebly compact. A formally weaker property obtained by requiring the conclusion of Definition 3.1 to hold only for the sequences $\{U_n : n \in \mathbb{N}\}$ consisting of pairwise disjoint non-empty open subsets of X was defined earlier in [1, Definition 1.8]. It was proved in [9, Proposition 1] that these two versions of Definition 3.1 are in fact equivalent. An alternative proof of this fact can be found also in [3, Corollary 1.8].

Diagram 1 below shows the connections between selective (sequential) pseudocompactness, sequential pseudocompactness and known compactness properties.

The next example shows that *none of properties on the right side of Diagram 1 imply any of the properties on the left side.*

Example 3.2. The Stone-Čech compactification $\beta\mathbb{N}$ of the countable discrete space $\mathbb{N}$ is (compact but is) not sequentially pseudocompact by [5, Example 2.9].

$$\begin{array}{ccc}
 & & \text{compact} \\
 & & \downarrow \\
\text{sequentially compact} & \longrightarrow & \text{countably compact} \\
{\scriptstyle 1}\downarrow & & {\scriptstyle 2}\downarrow \\
\text{selectively sequentially pseudocompact} & \longrightarrow & \text{selectively pseudocompact} \\
{\scriptstyle 3}\downarrow & & {\scriptstyle 4}\downarrow \\
\text{sequentially pseudocompact} & \longrightarrow & \text{pseudocompact}
\end{array}$$

Diagram 1.

In the next remark we address the question whether the horizontal arrows in Diagram 1 are reversible for topological groups.

Remark 3.3. (i) The Stone-Čech compactification $\beta\mathbb{N}$ of the countable discrete space $\mathbb{N}$ is homeomorphic to a subspace of $G = \{0,1\}^{\mathfrak{c}}$, where $\mathfrak{c}$ is the cardinality of the continuum, so G is not sequentially compact. Since G is a compact group, the *first horizontal arrow in Diagram 1 is not reversible even for topological groups.*

(ii) Sequentially pseudocompact topological groups need not be selectively sequentially pseudocompact [13]. Therefore, the *second horizontal arrow in Diagram 1 is not reversible even for topological groups.*

(iii) Every pseudocompact group is sequentially pseudocompact [1]. Therefore, the *third horizontal arrow in Diagram 1 is reversible for topological groups.*

Example 3.4. The Mrówka space $X = \mathbb{N} \cup \mathcal{A}$ associated with a maximal almost disjoint family $\mathcal{A}$ on $\mathbb{N}$ is selectively sequentially pseudocompact [3, Example 2.6]. Since X is not countably compact, this shows that *arrows 1 and 2 of Diagram 1 are not reversible.*

Example 3.5. There is a pseudocompact space X such that all its countable subsets are closed and C^*-embedded, see [11]. This X is (pseudocompact but) not selectively pseudocompact by item (ii) of Proposition 2.9. Hence, *arrow 4 of Diagram 1 is not reversible.*

Example 3.6. [3, Example 5.8] In the text preceding Theorem 1.2 of [8], García-Ferreira and Tomita give an example of a selectively pseudocompact subgroup G of $X = \{0,1\}^{\mathfrak{c}}$ which is not countably compact. Since G contains the Σ-product $\Sigma(0, X)$, it is selectively sequentially pseudocompact by Corollary 2.4 and Theorem 2.6(i). Therefore, *a selectively sequentially pseudocompact abelian group need not be countably compact.* This shows that *arrows 1 and 2 of Diagram 1 are not reversible even for topological groups.*

Example 3.7. García-Ferreira and Tomita constructed a pseudocompact group G which is not selectively pseudocompact [8, Example 2.4]. By the result cited in item (iii) of Remark 3.3, G is sequentially pseudocompact. Therefore, *arrows 3 and 4 of Diagram 1 are not reversible even for topological groups.*

We finish this section by showing that arrow 1 of Diagram 1 is reversible for Alexandroff duplicates.

Let X be a space. The *Alexandroff duplicate* of X, see [6, 3.1.26], is denoted by $A(X) = (X \times \{0\}) \cup (X \times \{1\})$, with the topology generated by the base

$$\mathcal{B} = \{\{x\} \times \{0\} : x \in X\} \cup \{(U \times \{1\}) \cup ((U \setminus F) \times \{0\}) : U \in \mathcal{T}(X), F \in [U]^{<\omega}\},$$

where $\mathcal{T}(X)$ is the topology of X and $[U]^{<\omega}$ is the set of all finite subsets of U. It is known and easy to see that X is Tychonoff if and only if $A(X)$ is Tychonoff.

Proposition 3.8. *For every topological space X, the following conditions are equivalent:*

(i) *X is sequentially compact,*
(ii) *$A(X)$ is sequentially compact,*
(iii) *$A(X)$ is selectively sequentially pseudocompact.*

Proof. (i)⇒(ii) Let $S = \{x_n : n \in \mathbb{N}\}$ be a faithfully indexed sequence in $A(X)$. If $S_1 = S \cap (X \times \{1\})$ is infinite, then S_1 has a convergent subsequence in $X \times \{1\}$. If S_1 is finite, then $S \setminus S_1 \subseteq X \times \{0\}$ is infinite, so $Y = \{y \in X : (y, 0) \in S \setminus S_0\}$ is an infinite subset of X. Since X is sequentially compact, Y has a subsequence Y_0 converging to some point $y \in X$. Therefore, the sequence $\{(y, 0) : y \in Y_0\} \subseteq S$ converges to $(y, 1)$ in $A(X)$.

(iii)⇒(i) Let $S = \{x_n : n \in \mathbb{N}\}$ be a faithfully indexed sequence in X. Then $\mathcal{U} = \{\{x_n\} \times \{0\} : n \in \mathbb{N}\}$ is an infinite family of non-empty open subsets in $A(X)$. Since $A(X)$ is selectively sequentially pseudocompact, for every $n \in \mathbb{N}$, there is a point $y_n \in \{x_n\} \times \{0\}$ such that the sequence $T = \{y_n : n \in \mathbb{N}\}$ has a subsequence T_1 converging to some point $y \in A(X)$.

If $y = (x, 0)$ for some point $x \in X$, then the open set $\{(x, 0)\}$ contains all but finitely many elements of T_1. Hence all but finitely many elements of the set $\{x_n \in S : (x_n, 0) \in T_1\}$ are equal to x which is a contradiction since the sequence S is faithfully indexed. Therefore $y = (x, 1)$ for some point $x \in X$ and the sequence $\{x_n : (x_n, 0) \in T_1\} \subseteq S$ converges to x. □

4. An open-point game $OP(X)$ on a topological space X

Consider the following *open-point game $OP(X)$* on a topological space X. In the round n of the play, Player A chooses a non-empty open set $U_n \subseteq X$ and Player B responds by selecting a point x_n in U_n.

Definition 4.1. A *play in $OP(X)$* is an infinite sequence $w = (U_1, x_1, U_2, x_2, \ldots)$ such that U_n is a non-empty open subset of X and $x_n \in U_n$ for every $n \in \mathbb{N}$.

Given a set Y, we use $\mathrm{Seq}(Y)$ to denote the set of all finite sequences $(y_1, \ldots, y_n)$ of elements of Y. We include the empty sequence $\emptyset$ in $\mathrm{Seq}(Y)$.

For a topological space X, the symbol $\mathcal{O}^*(X)$ denotes the family of all non-empty open subsets of X.

Definition 4.2. A function $\sigma : \mathrm{Seq}(X) \to \mathcal{O}^*(X)$ is called a *strategy for Player A in $OP(X)$*. *A strategy for Player B in $OP(X)$* is a function $\tau : \mathrm{Seq}(\mathcal{O}^*(X)) \setminus \{\emptyset\} \to X$ such that

(2) $$\tau(U_1, U_2, \ldots, U_n) \in U_n \text{ for every } (U_1, U_2, \ldots, U_n) \in \mathrm{Seq}(\mathcal{O}^*(X)) \setminus \{\emptyset\}.$$

Definition 4.3. A strategy σ for Player A in $OP(X)$ and a strategy τ for Player B in $OP(X)$ *produce the play*

(3) $$w_{\sigma,\tau} = (U_1, x_1, U_2, x_2, \ldots, U_n, x_n, \ldots)$$

in $OP(X)$ as follows. Player A starts with

$$U_1 = \sigma(\emptyset), \tag{4}$$

and Player B responds with

$$x_1 = \tau(U_1). \tag{5}$$

At the nth move, for $n \geq 2$, Player A selects

$$U_n = \sigma(x_1, \ldots, x_{n-1}) \tag{6}$$

and Player B responds with

$$x_n = \tau(U_1, \ldots, U_n). \tag{7}$$

5. Topological games $Ssp(X)$ and $Sp(X)$ associated with selective (sequential) pseudocompactness

The selective properties from Definition 1.3 naturally lead to two versions of the game $OP(X)$, the *selectively sequentially pseudocompact game* $Ssp(X)$ and the *selectively pseudocompact game* $Sp(X)$. These games differ only in the way the winner is declared.

Definition 5.1. Given a play $w = (U_1, x_1, U_2, x_2, \ldots)$ in $OP(X)$, we say that:

(i) *Player B wins w in $Ssp(X)$* if the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence in X; otherwise, *Player A wins w in $Ssp(X)$*.

(ii) *Player B wins w in $Sp(X)$* if the sequence $\{x_n : n \in \mathbb{N}\}$ has an accumulation point in X; otherwise, *Player A wins w in $Sp(X)$*.

Definition 5.2. We say that a strategy σ for Player A in $OP(X)$ is:

(i) a *winning strategy in $Ssp(X)$* if Player A wins $w_{\sigma,\tau}$ in $Ssp(X)$, for every strategy τ for Player B in $OP(X)$.

(ii) a *winning strategy in $Sp(X)$* if Player A wins $w_{\sigma,\tau}$ in $Sp(X)$, for every strategy τ for Player B in $OP(X)$.

Definition 5.3. We say that a strategy τ for Player B in $OP(X)$ is:

(i) a *winning strategy in $Ssp(X)$* if Player B wins $w_{\sigma,\tau}$ in $Ssp(X)$, for every strategy σ for Player A in $OP(X)$.

(ii) a *winning strategy in $Sp(X)$* if Player B wins $w_{\sigma,\tau}$ in $Sp(X)$, for every strategy σ for Player A in $OP(X)$.

If either Player A or Player B has a winning strategy in $Ssp(X)$ (respectively, in $Sp(X)$) we say that the game $Ssp(X)$ (respectively, $Sp(X)$) is *determined*.

A strategy σ for Player A in $OP(X)$ is *stationary* if

$$\sigma(x_1, x_2, \ldots, x_n) = \sigma(x_n) \text{ for every } (x_1, x_2, \ldots, x_n) \in \mathrm{Seq}(X) \setminus \{\emptyset\}. \tag{8}$$

The following fundamental theorem connects games $Ssp(X)$ and $Sp(X)$ on a topological space X with selective (sequential) pseudocompactness of X.

Theorem 5.4. *Let X be a topological space.*

(i) *If X is not selectively sequentially pseudocompact, then Player A has a stationary winning strategy in the selectively sequentially pseudocompact game $Ssp(X)$ on X.*

(ii) *If X is not selectively pseudocompact, then Player A has a stationary winning strategy in the selectively pseudocompact game $Sp(X)$ on X.*

Proof. We consider two cases.

Case 1: *X is not selectively sequentially pseudocompact.* In this case, we use Proposition [3, Proposition 2.4] to fix a family $\{V_n : n \in \mathbb{N}\}$ of pairwise disjoint non-empty open subsets of X such that if $x_n \in V_n$ for every $n \in \mathbb{N}$, then the sequence $\{x_n : n \in \mathbb{N}\}$ does not have a convergent subsequence in X.

Case 2: *X is not selectively pseudocompact.* In this case, we use [3, Proposition 2.1] to fix a family $\{V_n : n \in \mathbb{N}\}$ of pairwise disjoint non-empty open subsets of X such that if $x_n \in V_n$ for every $n \in \mathbb{N}$, then the sequence $\{x_n : n \in \mathbb{N}\}$ does not have an accumulation point in X.

Now we follow the same proof in both cases. Since the family $\{V_n : n \in \mathbb{N}\}$ consists of pairwise disjoint non-empty subsets of X, for every $x \in \bigcup_{n\in\mathbb{N}} V_n$ there exists exactly one $n \in \mathbb{N}$ such that $x \in V_n$. We denote this n by $m(x)$. For $x \in X \setminus \bigcup_{n\in\mathbb{N}} V_n$, we let $m(x) = 0$.

Define the strategy $\sigma : \mathrm{Seq}(X) \to \mathcal{O}^*(X)$ for Player A by $\sigma(\emptyset) = V_1$ and

$$\sigma(x_1, x_2, \dots, x_n) = V_{m(x_n)+1} \text{ for } (x_1, x_2, \dots, x_n) \in \mathrm{Seq}(X) \setminus \{\emptyset\}. \tag{9}$$

Assume that $\tau : \mathrm{Seq}(\mathcal{O}^*(X)) \setminus \{\emptyset\}$ is an arbitrary a strategy for Player B. Let $w_{\sigma,\tau} = (U_1, x_1, U_2, x_2, \dots)$ be the play produced by following strategies σ and τ; see Definition 4.3.

Claim 1. $x_n \in U_n = V_n$ for every $n \in \mathbb{N}$.

Proof. We shall prove our claim by induction on $n \in \mathbb{N}$.

We have $U_1 = \sigma(\emptyset)$ by (4) and $V_1 = \sigma(\emptyset)$ by the definition of σ, so $U_1 = V_1$. Since $x_1 \in U_1$ by (2) and (5), our claim holds for $n = 1$.

Suppose that $n \in \mathbb{N}$, $n \geq 2$ and our claim holds for $n - 1$; that is, $x_{n-1} \in U_{n-1} = V_{n-1}$. Then $m(x_{n-1}) = n - 1$ by the definition of $m(x_{n-1})$, so

$$\sigma(x_1, x_2, \dots, x_{n-1}) = V_{m(x_{n-1})+1} = V_n$$

by (9). On the other hand, $\sigma(x_1, x_2, \dots, x_{n-1}) = U_n$ by (6). This establishes the equality $U_n = V_n$. Finally, $x_n \in U_n$ by (2) and (7). □

It follows from Claim 1 that $x_n \in V_n$ for every $n \in \mathbb{N}$. From this and the choice of the sequence $\{V_n : n \in \mathbb{N}\}$, we conclude that the sequence $\{x_n : n \in \mathbb{N}\}$ does not have a subsequence converging to some point of X (in Case 1) or does not have an accumulation point in X (in Case 2). According to Definition 5.1, this means that Player A wins the play $w_{\sigma,\tau}$ in $Ssp(X)$ (in Case 1) or $Sp(X)$ (in Case 2). Since τ was an arbitrary strategy on $OP(X)$, from Definition 5.2 we conclude that σ is a winning strategy in $Ssp(X)$ (in Case 1) or in $Sp(X)$ (in Case 2). □

A strategy τ for Player B in $OP(X)$ is *stationary* if

$$\tau(U_1, U_2, \dots, U_n) = \tau(U_n) \text{ for every } (U_1, U_2, \dots, U_n) \in \mathrm{Seq}(\mathcal{O}^*(X)) \setminus \{\emptyset\}. \tag{10}$$

The next theorem gives an internal characterization of spaces X such that Player B has a stationary winning strategy in the games $Ssp(X)$ and $Sp(X)$, respectively.

Theorem 5.5. [4] *Let X be a topological space.*

(i) *Player B has a stationary winning strategy in $Ssp(X)$ if and only if X has a dense subspace D which is relatively sequentially compact in X; that is, every sequence of points of D has a subsequence which converges to some point of X.*

(ii) *Player B has a stationary winning strategy in $Sp(X)$ if and only if X has a dense subspace D which is relatively countably compact in X; that is, every sequence of points of D has an accumulation point in X.*

Since every dyadic space has a dense sequentially compact subspace, from item (i) of Theorem 5.5 we obtain the following

Corollary 5.6. *For every dyadic space X, Player B has a stationary winning strategy in the selectively sequentially pseudocompact game $Ssp(X)$ on X.*

Since compact groups are dyadic, the following particular case of the above corollary deserves explicit mentioning:

Corollary 5.7. *For every compact group G, Player B has a stationary winning strategy in the selectively sequentially pseudocompact game $Ssp(G)$ on G.*

Corollaries 5.6 and 5.7 strengthen [3, Corollary 4.6]; see Remark 6.5.

6. Compactness properties defined by games $Ssp(X)$ and $Sp(X)$

The next diagram clarifies the fine structure of the interval between sequential compactness and selective sequential pseudocompactness, as well as the interval between countable compactness and selective pseudocompactness.

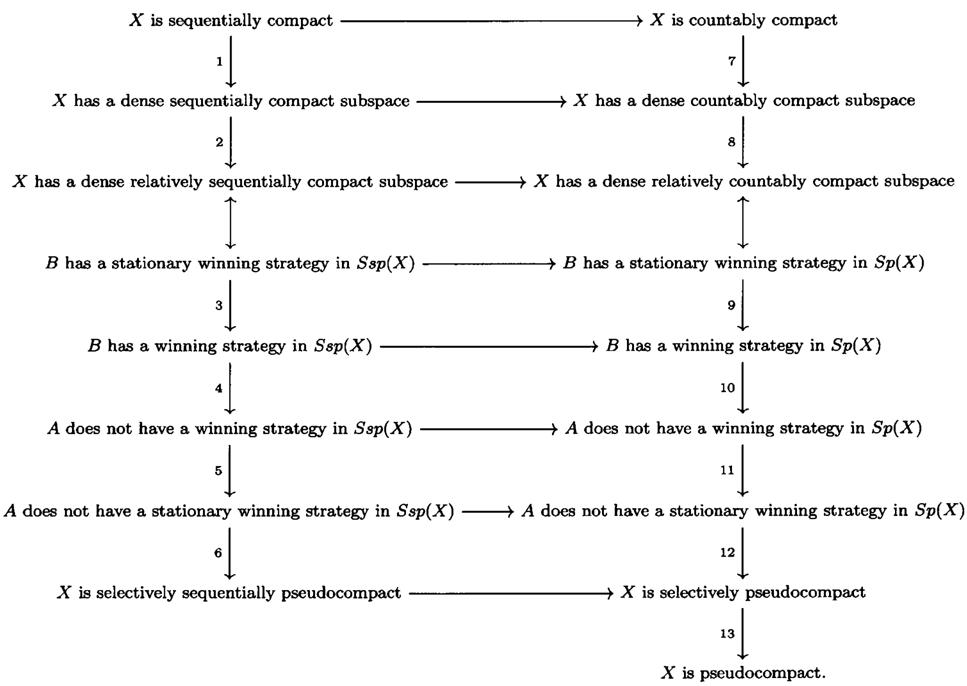

Diagram 2.

The Stone-Čech compactification of the countably infinite discrete space is a compact space which is not selectively sequentially pseudocompact [3, Example 2.5]. Hence, *none of the properties on the right side of Diagram 2 imply any of the properties on its left side.*

The next two examples are well known.

Example 6.1. Let α be an ordinal and $[0,\alpha)$ be the space of all ordinals less than α with the order topology. The space $T = [0,\omega+1) \times [0,\omega_1+1) \setminus \{(\omega,\omega_1)\}$ *has a dense sequentially compact subspace but is not countably compact.*

Example 6.2. Let $\mathcal{A}$ be an arbitrary maximal almost disjoint family of subsets of $\mathbb{N}$. Consider the Mrówka space $X = \mathbb{N} \cup \mathcal{A}$ associated with $\mathcal{A}$ [6, 3.6.I]. Then X *has a dense relatively sequentially compact subspace, yet does not contain any dense countably compact subspace.*

Example 6.1 shows that *arrows 1 and 7 of Diagram 2 are not reversible*, while Example 6.2 shows that *arrows 2 and 8 of Diagram 2 are not reversible.*

The next theorem shows that *arrows 3 and 9 of Diagram 2 are not reversible.*

Theorem 6.3. [4] *There exists a locally compact, first-countable, zero-dimensional space X such that Player B has a winning strategy in $Ssp(X)$ but does not have a stationary winning strategy even in $Sp(X)$.*

The next theorem shows that either arrow 5 or arrow 6 is not reversible, and either arrow 11 or arrow 12 is not reversible. Exactly which of these four arrows are not reversible remains unclear; see Problem 8.2.

Theorem 6.4. [4] *There exists a selectively sequentially pseudocompact space X such that Player A has a winning strategy in $Sp(X)$.*

We do not know if arrows 4 and 10 are reversible; see Problem 8.1. Arrow 13 coincides with arrow 4 of Diagram 1, so it is not reversible; see Examples 3.5 and 3.7.

Remark 6.5. As we have seen above, the existence of a stationary winning strategy for Player B in the selectively sequentially pseudocompact game $Ssp(X)$ on X is much stronger than selective sequential pseudocompactness of X. Therefore, Corollaries 5.6 and 5.7 significantly strengthen [3, Corollary 4.6].

7. The game $Sp(X)$ and a starcompact property of Matveev

Let us recall a definition due to M. Matveev [10]:

Definition 7.1. A topological space X is said to be *1-cl-starcompact* provided that for every open cover $\mathcal{U}$ of X there exists a finite subset A of X such that

$$St(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$$

is dense in X.

The next theorem highlights a connection of our game $Sp(X)$ with this property of M. Matveev.

Theorem 7.2. *If X is a topological space such that Player A does not have a winning strategy in $Sp(X)$, then X is 1-cl-starcompact.*

Proof. We shall prove a contraposition of the implication stated in our proposition; that is, we assume that X is not 1-cl-starcompact, and then we shall define a winning strategy for Player A in the game $Sp(X)$.

Since X is assumed to be not 1-cl-starcompact, we can fix an open cover $\mathcal{U}$ of X such that $St(A,\mathcal{U})$ is dense in X for no finite subset A of X. This implies that for every finite set $A \subseteq X$, the set $X \setminus St(A,\mathcal{U})$ has non-empty interior $\mathrm{Int}_X(X \setminus St(A,\mathcal{U}))$; in particular, $\mathrm{Int}_X(X \setminus St(A,\mathcal{U})) \in \mathcal{O}^*(X)$ for every finite set $A \subseteq X$. This allows us to define a strategy $\sigma : \mathrm{Seq}(X) \to \mathcal{O}^*(X)$ for Player A in $Sp(X)$ by $\sigma(\emptyset) = X$ and

(11) $\quad \sigma(x_1, \ldots, x_n) = \mathrm{Int}_X(X \setminus St(\{x_1, \ldots, x_n\},\mathcal{U}))$ for $(x_1, \ldots, x_n) \in \mathrm{Seq}(X) \setminus \{\emptyset\}$.

Let us prove that σ is a winning strategy for Player A in $Sp(X)$. Let $\tau : \mathrm{Seq}(\mathcal{O}^*(X)) \setminus \{\emptyset\} \to X$ be an arbitrary strategy for Player B in $OP(X)$, and let

$$w_{\sigma,\tau} = (U_1, x_1, U_2, x_2, \ldots)$$

be the play produced by following the strategies σ and τ; see Definition 4.3. By Definition 5.2(ii), we have to check that Player A wins the play $w_{\sigma,\tau}$ in $Sp(X)$. In turn, to do this we must show that the sequence $\{x_n : n \in \mathbb{N}\}$ does not have an accumulation point in X; see Definition 5.1(ii).

Let $x \in X$. Since $\mathcal{U}$ is a cover of X, there exists $U \in \mathcal{U}$ such that $x \in U$. We are going to show that the set $\{n \in \mathbb{N} : x_n \in U\}$ is finite, thereby showing that x is not an accumulation point of the sequence $\{x_n : n \in \mathbb{N}\}$. If $x_n \in U$ for no $n \in \mathbb{N}$, we are done.

Suppose now that $x_m \in U$ for some $m \in \mathbb{N}$. Assume also that $n \in \mathbb{N}$ and $n > m$. Then $n \geq 2$ and $x_m \in U \cap \{x_1, \ldots, x_{n-1}\}$, so $U \subseteq St(\{x_1, \ldots, x_{n-1}\},\mathcal{U})$, as $U \in \mathcal{U}$. This implies $U \cap (X \setminus St(\{x_1, \ldots, x_{n-1}\},\mathcal{U})) = \emptyset$ and $U \cap \mathrm{Int}_X(X \setminus St(\{x_1, \ldots, x_{n-1}\},\mathcal{U})) = \emptyset$. Therefore, $U \cap U_n = U \cap \sigma(x_1, \ldots, x_{n-1}) = \emptyset$ by (6) and (11). Since $x_n \in U_n$ by (2) and (7), we obtain $x_n \notin U$. Therefore, $\{n \in \mathbb{N} : x_n \in U\} \subseteq \{1, \ldots, m\}$, so the former set is finite. □

Corollary 7.3. *There exists a locally compact, first-countable, zero-dimensional, 1-cl-starcompact space without a dense relatively countably compact subspace.*

Proof. Let X be a space from Theorem 6.3. Since Player B has a winning strategy in $Ssp(X)$, it has a winning strategy also in the game $Sp(X)$. Therefore, Player A does not have a winning strategy in $Sp(X)$. Applying Theorem 7.2, we conclude that X is 1-cl-starcompact.

Since Player B does not have a stationary winning strategy in the game $Sp(X)$, we can apply Theorem 5.5(ii) to conclude that X does not have a dense relatively countably compact subspace. □

Remark 7.4. Corollary 7.3 shows that Theorem 15 in [10] is not reversible.

Remark 7.5. It is shown in [10, Proposition 13] and [10, Proposition 14] that every 1-cl-starcompact space is pseudocompact.

The following diagram highlights connections of 1-cl-starcompactness with the properties considered in Diagram 2.

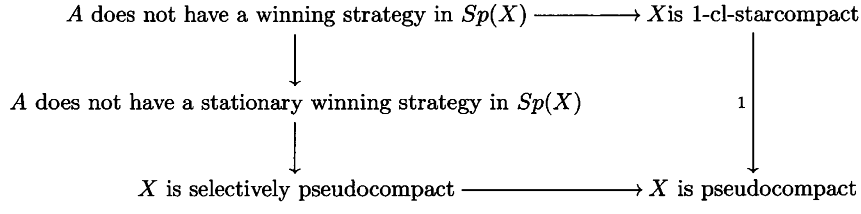

Diagram 3.

Example 7.6. I. J. Tree constructed in [14] a pseudocompact space X which is not 2-starcompact. It is shown in [10, Proposition 13] that 1-cl-starcompactness implies 2-starcompactness. Hence, *X is a pseudocompact space which is not 1-cl-starcompact.* Therefore, *arrow 1 in Diagram 3 is not reversible.*

8. Open problems

Problem 8.1. (i) Is the selectively sequentially pseudocompact game $Ssp(X)$ on each topological space X determined? In other words, is arrow 4 of Diagram 2 reversible?
(ii) Is the selectively pseudocompact game $Sp(X)$ on each topological space X determined? In other words, is arrow 10 of Diagram 2 reversible?

Problem 8.2. (i) Is arrow 5 of Diagram 2 reversible?
(ii) Is arrow 6 of Diagram 2 reversible?
(iii) Is arrow 11 of Diagram 2 reversible?
(iv) Is arrow 12 of Diagram 2 reversible?

Problem 8.3. Which arrows of Diagram 2 are reversible for (locally) compact spaces?

Problem 8.4. Which arrows of Diagram 2 are reversible for topological groups?

We refer the reader to [12] for the definition of function spaces $C_p(X, G)$, for a topological group G.

Problem 8.5. Which arrows of Diagram 2 are reversible for function spaces $C_p(X, G)$, for a topological group G and a topological space X such that $C_p(X, G)$ is dense in the Tychonoff product G^X?

Since all spaces with any of the properties from Diagram 2 are pseudocompact, similarly to the argument in [3, Remark 8.4], one shows that the topological group G in Problem 8.5 should be assumed to be pseudocompact.

Corollary 5.7 justifies the following question:

Question 8.6. Let G be a group such that the closure of every countable subgroup of G is compact. Does Player B have a stationary winning strategy in the selectively sequentially pseudocompact game $Ssp(G)$ on G?

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