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TOPOLOGICAL GROUPS WITH MANY SMALL SUBGROUPS REVISITED

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As usual, $\mathbb{Z}$ and $\mathbb{Q}$ denote the groups of integer numbers and rational numbers respectively, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$.

Definition 0.1. For a subset A of a topological group G, we denote by $\langle A\rangle$ the smallest subgroup of G containing A, and we define

$$\langle A\rangle_k = \left\{\prod_{i=1}^{j} a_i : j \leq k, a_1, \ldots, a_j \in A\right\}$$

for every $k \in \mathbb{N}^+$.

Remark 0.2. If A is a subset of a group G such that $A = A^{-1}$, then $\langle A\rangle = \bigcup_{k\in\mathbb{N}^+} \langle A\rangle_k$.

1. The small subgroup generating property

In the realm of topological groups von Neumann [11] introduced the class of of the *minimally almost periodic groups.* A topological group G satisfies this property if every continuous homomorphism of G to a compact group K is trivial. Examples of minimally almost periodic groups are rather difficult or cumbersome to find; see [5] for the overview of the difficulties involved. Therefore, one approach towards simplifying the search for examples can be done by considering a narrower class of groups such that every group in this class can be easily checked to have the minimal almost periodic property. This was done by Gould in [9] who introduced the *small subgroup generating property* (SSGP) while attempting to study the behavior of well-known examples of minimally almost periodic groups.

Definition 1.1. A topological group G has the *small subgroup generating property*, abbreviated to *SSGP*, provided that for every neighbourhood U of the identity of G, there exists a family $\mathcal{H}$ of subgroups of G such that $\bigcup \mathcal{H} \subseteq U$ and the smallest closed subgroup N of G containing $\bigcup \mathcal{H}$ coincides with G.

One can easily see that SSGP groups are minimally almost periodic:

Proposition 1.2. [1] *If G is an* SSGP *group, then G is minimally almost periodic.*

Proof. Recall that every compact group is is isomorphic to a subgroup of a product of finite-dimensional unitary groups. Thanks to this fact, it suffices to prove that every continuous homomorphism f of G to the n-dimensional unitary group $\mathbb{U}(n)$ is trivial. Recall that $\mathbb{U}(n)$ satisfies the *no small subgroups property*; that is, there exists a neighborhood U of e in $\mathbb{U}(n)$ that contains no non-trivial subgroups. Now $V = f^{-1}(U)$ is an open neighbourhood of e in G. Since G is SSGP, there exists a family $\mathcal{H}$ of subgroups of G such that

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$\bigcup \mathcal{H} \subseteq V$ and G coincides with the smallest closed subgroup of G containing $\bigcup \mathcal{H}$. Since f is an homomorphism, it maps all the subgroups in $\mathcal{H}$ into subgroups of $\mathbb{U}(n)$. However, by definition, $f(\bigcup \mathcal{H}) \subseteq f(V) \subseteq U$, so all these subgroups in $\mathcal{H}$ are mapped to the trivial group $\{e\}$. This implies $f(\bigcup \mathcal{H}) = \{e\}$. Since f is a homomorphism, $f(\langle \bigcup \mathcal{H} \rangle) = \{e\}$. Since G coincides with the closure of $\langle \bigcup \mathcal{H} \rangle$ and f is continuous, G is mapped to the closure of $\{e\}$. Therefore, f is a trivial mapping and the result follows. □

Comfort and Gould established the following fundamental fact:

Theorem 1.3. [1, Corollary 2.23] *A bounded torsion abelian topological group is* SSGP *if and only if it is minimally almost periodic.*

The following explicit examples of abelian groups admitting an SSGP group topology were given [1]:

Theorem 1.4. *The following abelian groups admit an* SSGP *topology:*

(a) *all subgroups $\mathbb{Q}_\pi$ of $\mathbb{Q}$ with infinite π (in particular, $\mathbb{Q}$ itself);*
(b) *$\mathbb{Q}_\pi/\mathbb{Z}$ where π is an infinite set of primes (in particular, $\mathbb{Q}/\mathbb{Z}$);* [1]
(c) *direct sums of the form $\oplus\mathbb{Z}(p_i)$ where the primes p_i all coincide or differ;*
(d) *$\mathbb{Z}^{(\omega)}$ (the direct sum);*
(e) *$\mathbb{Z}^\omega$ (the full product);*
(f) *$G^{(\lambda)}$ for $|G| > 1$ and $\lambda \geq \omega$;*
(g) *F^λ for $1 < |F| < \omega$ and $\lambda \geq \omega$;*
(h) *arbitrary sums and products of groups which admit an* SSGP *group topology.*

Finally, Comfort and Gould [1] showed that two classical groups do not admit an SSGP group topology, while they both admit a minimally almost periodic group topology; see [5].

Example 1.5. [1] The group $\mathbb{Z}$ of integer numbers and the Prüfer group $\mathbb{Z}(p^\infty)$ do not admit an SSGP group topology.

Following [4], for every subset A of a group G, define

$$\mathrm{Cyc}(A) = \{x \in G : \langle x \rangle \subseteq A\}, \tag{1}$$

where $\langle x \rangle$ denotes the smallest subgroup of G containing x, i.e., the cyclic subgroup generated by x.

Using this notation, a simple reformulation of the SSGP property for abelian groups can be given.

Proposition 1.6. [4] *An abelian topological group G has the small subgroup generating property if and only if $\langle \mathrm{Cyc}(U) \rangle$ is dense in G for every neighbourhood U of zero of G.*

A more algebraic reformulation is given in the next proposition which was proved in [12] for abelian groups G.

Proposition 1.7. *A topological group G has the small subgroup generating property if and only if*

$$G = \bigcup_{k \in \mathbb{N}^+} \langle \mathrm{Cyc}(W) \rangle_k W$$

for every neighbourhood W of e_G in G.

Comfort and Gould [1] asked the following question.

Question 1.8. [1, Question 5.2] What are the (abelian) groups which admit an SSGP topology?

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[1] We refer the reader to Definition 4.7 for the notation $\mathbb{Q}_\pi$ appearing in items (a) and (b).

2. The classes SSGP(n)

An infinite sequence of proper subclasses of the class of minimally periodic groups was defined in [1].

Definition 2.1. Let G be a topological group.

(a) G has SSGP(0) if G is the trivial group.
(b) For $n \in \mathbb{N}^+$, G has SSGP(n) provided that, for every neighbourhood U of the identity of G, there exists a family $\mathcal{H}$ of subgroups of G such that $\bigcup \mathcal{H} \subseteq U$ and the smallest closed subgroup N of G containing $\bigcup \mathcal{H}$ is normal in G and G/N has SSGP($n-1$).

One can easily see that the class SSGP(1) coincides with the class of minimally almost periodic groups.

It was proved in [1, Remark 3.4, Theorem 3.5] that

$$\text{(2)}\quad \text{SSGP} = \text{SSGP}(1) \to \text{SSGP}(2) \to \dots \to \text{SSGP}(n) \to \text{SSGP}(n+1) \to \dots \to \text{ minimally almost periodic.}$$

Examples distinguishing all classes in (2) can be found in [1, Corollary 3.14, Theorem 4.6].

The following two examples are of particular interest:

Example 2.2. [1] The group $\mathbb{Z}$ if integer numbers and the Prüfer group $\mathbb{Z}(p^\infty)$ admit no SSGP(n) topology for any $n \in \mathbb{N}^+$.

Comfort and Gould [1] asked the following question.

Question 2.3. [1, Question 5.3] Does every abelian group which for some $n > 1$ admits an SSGP(n) topology also admit an SSGP topology?

3. The classes SSGP(α)

In [4], Dikranjan and the first listed author utilized an operator based approach to further extend the classes SSGP(n) to any ordinal α. We outline this approach here.

Definition 3.1. Let G be a topological group, A be a subset of G and H be a subgroup of G.

(i) N(H) denotes the maximal normal subgroup of G contained in H.[2]
(ii) Cs(A) is the smallest closed subgroup of G containing X.

Note that N(H) is closed in G whenever H is closed in G, as the closure of a normal subgroup of G is a normal subgroup of G.

The operator Cyc defined in (1) behaves as an interior operator on G and Cs behaves as a closure operator on G [4].

Let us consider $\mathbf{S} = \mathrm{N} \circ \mathrm{Cs} \circ \mathrm{Cyc}$ as the composition of the operators Cyc, Cs and N; that is, $\mathbf{S}(X) = \mathrm{N}(\mathrm{Cs}(\mathrm{Cyc}(X)))$ for every $X \subseteq G$.

By transfinite induction, for every ordinal α, define the α's iteration $\mathbf{S}^{(\alpha)}$ of $\mathbf{S}$ as follows. Let $\mathbf{S}^{(0)}(X) = \{e_G\}$ for every $X \subseteq G$. If $\alpha > 0$ is an ordinal and $\mathbf{S}^{(\beta)}(X)$ has already

[2]Its existence follows from the fact that the family $\mathcal{N}$ of normal subgroups of G contained in H is directed, as the product N_1N_2 of two members of $\mathcal{N}$ still belongs to $\mathcal{N}$, so $\mathrm{N}(H) = \bigcup \mathcal{N}$.

been defined for all $\beta < \alpha$, let

$$\mathbf{S}^{(\alpha)}(X) = \mathbf{S}\left(X \cdot \bigcup_{\beta<\alpha} \mathbf{S}^{(\beta)}(X)\right) \quad \text{for every} \quad X \subseteq G.$$

Definition 3.2. [4, Definition 2.3] For an ordinal α, a topological group G is said to be an $\mathtt{SSGP}(\alpha)$ *group* provided that $\mathbf{S}^{(\alpha)}(U) = G$ for every neighbourhood U of the identity of G.

The connection of this definition with Definition 2.1 is seen from the following theorem.

Theorem 3.3. [4, Theorem 6.4] *Let $n \in \mathbb{N}^+$. An abelian topological group is an* SSGP(n) *group in the sense of Definition 2.1 if and only if it is an* $\mathtt{SSGP}(n)$ *group in the sense of Definition 3.2.*

This theorem shows that the sequence of properties $\mathtt{SSGP}(n)$ is a natural extension for the SSGP(n) properties in the realm of Abelian groups. Do note that the use of quotients is replaced with a description that depends only on the operator $\mathbf{S}$.

Now, if α and β are ordinals with $\beta < \alpha$, then $\mathtt{SSGP}(\beta) \to \mathtt{SSGP}(\alpha)$ [4, Proposition 5.1]. In addition, every $\mathtt{SSGP}(\alpha)$ group is minimally almost periodic [4, Proposition 5.3 (iii)]. Thus, we have the following natural extension of inclusions from (2).

(3) $\text{SSGP} = \mathtt{SSGP}(1) \to \mathtt{SSGP}(2) \to \ldots \to \mathtt{SSGP}(\alpha) \to$
$\to \mathtt{SSGP}(\alpha+1) \to \ldots \to$ minimally almost periodic.

It follows from Theorem 1.3 that all the classes in (3) coincide for bounded torsion abelian topological groups.

4. Existence of SSGP topologies in the abelian case

Following [6], for an abelian group G, we denote by $r_0(G)$ the free rank of G, by $r_p(G)$ the p-rank of G, and we let

$$r(G) = \max\left\{r_0(G),\ \sum\{r_p(G) : p \in \mathbb{P}\}\right\},$$

where $\mathbb{P}$ denotes the set of prime numbers.

Definition 4.1. [3, Definition 7.2] For an abelian group G, the cardinal

$$r_d(G) = \min\{r(nG) : n \in \mathbb{N}^+\} \tag{4}$$

is called the *divisible rank* of G.

The notion of the divisible rank was defined, under the name of *final rank*, by Szele [13] for p-groups.

Remark 4.2. An abelian group G satisfies $r_d(G) = 0$ if and only if G is a bounded torsion group; that is, if $nG = \{0\}$ for some $n \in \mathbb{N}^+$.

Recall that a non-trivial bounded torsion abelian group G is a direct sum

$$G = \bigoplus_{p\in\pi(G)} \bigoplus_{i=1}^{m_p} \mathbb{Z}(p^i)^{(\alpha_{p,i})} \tag{5}$$

of cyclic groups, where $\pi(G)$ is a non-empty finite set of prime numbers and the cardinals $\alpha_{p,i}$ are known as *Ulm-Kaplanski invariants* of G. Note that while some of them may be equal to zero, the cardinals α_{p,m_p} must be positive; they are called *leading Ulm-Kaplanski invariants* of G.

Gabriyelyan [7] proved that a non-trivial bounded abelian group admits a minimally almost periodic group topology precisely when all its leading Ulm-Kaplanski invariants are infinite. (An alternative proof of this result can be found also in [5].) Combining this with Theorem 1.3 and Remark 4.2, we get the following corollary.

Corollary 4.3. *A non-trivial abelian group G satisfying $r_d(G) = 0$ admits an* SSGP *group topology if and only if all leading Ulm-Kaplanski invariants of G are infinite.*

Dikranjan and the first author completely resolved Question 1.8 for abelian groups of infinite divisible rank.

Theorem 4.4. [4, Theorem 3.2] *Every abelian group G satisfying $r_d(G) \geq \omega$ admits an* SSGP *group topology.*

In the remaining case $0 < r_d(G) < \omega$, Dikranjan and the first author found a necessary condition on G in order to admit an SSGP group topology.

Theorem 4.5. [4, Theorem 3.8] *Let G be an abelian* SSGP(α) *group for some ordinal α. If $1 \leq r_d(G) < \infty$, then the quotient $H = G/t(G)$ of G with respect to its torsion part*

$$t(G) = \{g \in G : ng = 0 \ \ \textit{for some} \ \ n \in \mathbb{N}^+\}$$

has finite rank $r_0(H)$ and $r(H/A) = \omega$ for some (equivalently, every) free subgroup A of H such that H/A is torsion.

Dikranjan and the first author asked if the necessary condition given in Theorem 4.5 is also sufficient for the existence of an SSGP group topology on G. Moreover, the same authors reduced this problem to the following question:

Question 4.6. [4, Question 13.1] Let $m \in \mathbb{N}^+$ and

$$G = G_0 \times \left(\bigoplus_{i=1}^{k} \mathbb{Z}(p_i^\infty) \right) \times F,$$

where F is a finite group, $k \in \mathbb{N}$, $p_1, p_2, \ldots, p_k$ are (not necessarily distinct) prime numbers, and G_0 is a subgroup of $\mathbb{Q}^m$ containing $\mathbb{Z}^m$ such that $G_0 \not\subseteq \mathbb{Q}_\pi^m$ for every finite set π of prime numbers. Is it true that G admits an SSGP group topology?

The notation $\mathbb{Q}_\pi$ appearing in the above question is given in the next definition.

Definition 4.7. For a set π of prime numbers, we use $\mathbb{Q}_\pi$ to denote the set of all rational numbers q whose irreducible representation $q = z/n$ with $z \in \mathbb{Z}$ and $n \in \mathbb{N}^+$ is such that all prime divisors of n belong to π.

Dikranjan and the first author provisionally provided a theorem completely characterizing abelian groups G admitting an SSGP group topology in the remaining open case $0 < r_d(G) < \omega$ provided that the answer to Question 4.6 is positive [4, Theorem 13.2].

5. Our results in the abelian case

We provide a positive answer to a much more general version of Question 4.6:

Theorem 5.1. [12, Theorem 2.10] *Suppose that $m \in \mathbb{N}^+$ and G_0 is a subgroup of $\mathbb{Q}^m$ containing $\mathbb{Z}^m$ such that $G_0 \not\subseteq \mathbb{Q}_\pi^m$ for every finite set π of prime numbers. Then for each at most countable abelian group H, the product $G = G_0 \times H$ admits a (separable) metric SSGP group topology.*

Having validated a positive answer to Question 4.6, the final statement of [4, Theorem 13.2] becomes as follows.

Theorem 5.2. [12, Theorem 2.9] *For an abelian group G satisfying $1 \leq r_d(G) < \infty$, the following conditions are equivalent:*

(i) *G admits an* SSGP *topology;*
(ii) *G admits an* $\mathtt{SSGP}(\alpha)$ *topology for some ordinal α;*
(iii) *the quotient $H = G/t(G)$ of G with respect to its torsion part $t(G)$ has finite rank $r_0(H)$ and $r(H/A) = \omega$ for some (equivalently, every) free subgroup A of H such that H/A is torsion.*

Corollary 4.3, Theorem 4.4 and Theorem 5.2 provide a complete solution to Question 1.8 in the abelian case.

Theorem 5.3. [12, Corollary 2.11] *For an abelian group G, the following conditions are equivalent:*

(i) *G admits an* SSGP *topology;*
(ii) *G admits an* $\mathtt{SSGP}(\alpha)$ *topology for some ordinal α.*

Proof. By Remark 4.2, an abelian group G satisfies $r_d(G) = 0$ if and only if G is a bounded torsion group. Therefore, in case $r_d(G) = 0$, the conclusion of our theorem follows from Corollary 1.3 and (3). In case $1 \leq r_d(G) < \infty$, the result follows from the equivalence of items (i) and (ii) of Theorem 5.2. Finally, in case $r_d(G) \geq \omega$, the equivalence of items (i) and (ii) of our theorem follows from Theorem 4.4 and (3). □

The next corollary provides a complete solution to Question 2.3 in the abelian case.

Corollary 5.4. [12] *For an abelian group G, the following conditions are equivalent:*

(i) *G admits an* SSGP *topology;*
(ii) *G admits an* SSGP(n) *topology for some $n \in \mathbb{N}$.*

Proof. Since G is an abelian group, it follows from Theorem 3.3 that G is SSGP(n) if and only if it is $\mathtt{SSGP}(n)$. Now the conclusion follows from Theorem 5.3. □

6. Our results in the non-commutative case

As far as the authors know, the non-commutative version of Question 1.8 remains completely open. Our next theorem positively resolves Question 1.8 in the case of free groups with infinitely many generators:

Theorem 6.1. *For an infinite set X, the free group $F(X)$ generated by X admits an* SSGP *topology. Furthermore, if X is countably infinite, then $F(X)$ admits a (separable) metric* SSGP *topology.*

The free group with a single generator is isomorphic to $\mathbb{Z}$, so it does not admit an SSGP topology by Example 1.5. This justifies the following

Question 6.2. Let $n \in \mathbb{N}$ and $n \geq 2$. Does the free group with n generators admit an SSGP topology?

In contrast with Therem 6.1, another highly non-commutative group does not admit SSGP topologies.

Example 6.3. *Let X be a set having at least two elements. Then the symmetric group $S(X)$ of all bijections of X with the composition of maps as group operation does not admit an* SSGP *topology.* Indeed, suppose that $\mathscr{T}$ is a (Hausdorff) group topology on $S(X)$. Assume first that X is finite. Then $S(X)$ is finite as well, so $\mathscr{T}$ is discrete. Since $S(X)$ is non-trivial, $\mathscr{T}$ cannot be SSGP. Suppose now that X is infinite. It is a well-known result of Gaughan that $\mathscr{T}$ is stronger than the topology $\mathscr{T}_p$ of pointwise convergence on $S(X)$, that is, the topology $S(X)$ inherits from the Tychonoff product X^X when X is equipped with the discrete topology [8]. Observe that $(S(X), \mathscr{T}_p)$ has many proper clopen subgroups. For example, $H = \{f \in S(X) : f(x_0) = x_0\}$, where $x_0 \in X$ is a fixed element, is such a subgroup. Since $\mathscr{T}_p \subseteq \mathscr{T}$, H is also $\mathscr{T}$-clopen. Finally, observe that a topological group containing a proper clopen subgroup cannot be SSGP.

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