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Resolvents of convex functions and the shrinking projection method on geodesic spaces
凸関数のリゾルベントと測地距離空間における収縮射影法

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Abstract

We apply the shrinking projection method to the convex minimization problems on a complete geodesic space by using the notion of resolvents for convex functions. We consider the calculation errors for metric projections onto the closed convex sets, and obtain the upper bound of the error for the limit of approximation sequence. We also discuss the choice of the coefficient of convex combination used in the iterative process.

1 Introduction

In the fixed point theory, the approximation methods for fixed points is one of the most important branches in the study of nonexpansive and other nonlinear mappings as well as the existence of fixed points. We will focus on the shrinking projection method, which was originally proved in 2008 by Takahashi, Takeuchi, and Kubota [3].

Theorem 1 (Takahashi, Takeuchi, and Kubota [3]). *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping such that the set* $\mathrm{Fix}T$ *of fixed points of T is nonempty. Let $\{\alpha_n\}$ be a sequence in $[0,1[$ with $\alpha_0 = \sup_{n\in\mathbb{N}} \alpha_n < 1$. For a given point $u \in H$, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$, and*

$$
\begin{aligned}
y_n &= \alpha_n x_n + (1-\alpha_n)Tx_n, \\
C_{n+1} &= \{z \in C : \|y_n - z\| \le \|x_n - z\|\} \cap C_n, \\
x_{n+1} &= P_{C_{n+1}} u
\end{aligned}
$$

for $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to* $P_{\mathrm{Fix}\,T}u \in C$, *where* P_K *is the metric projection of* H *onto a nonempty closed convex subset* K *of* H.

We note that the original result deals with the family of nonexpansive mappings.

In this paper, we will apply this method to the convex minimization problems on complete CAT(1) spaces by using the notion of resolvents for convex functions. The concept of resolvent on complete CAT(1) spaces was recently proposed by Kimura and Kohsaka [2]. We consider the calculation errors for metric projections onto the closed convex sets, and obtain the upper bound of the error for the limit of approximation sequence. We also discuss the choice of the coefficient of convex combination used in the iterative process.

2 Preliminaries

Let X be a metric space. For $x, y \in X$ and $l \geq 0$, we call a mapping $c : [0, l] \to X$ a geodesic with endpoints x, y if it satisfies that $c(0) = x$, $c(l) = y$, and $d(c(t), c(s)) = |t - s|$ for every $t, s \in [0, l]$. If a geodesic exists for every $x, y \in X$, X is called a geodesic space. In what follows, we assume that a geodesic is always unique for every choice endpoints. The image of a geodesic c with endpoints $x, y \in X$ is called a geodesic segment joining x and y, and we denote it by $[x, y]$. For $x, y \in X$ and $\alpha \in [0, 1]$, we define $z = \alpha x \oplus (1 - \alpha)y$ as a unique point $z \in [x, y]$ such that

$$d(z, y) = \alpha d(x, y), \ d(x, z) = (1 - \alpha)d(x, y).$$

Let X be a geodesic space such that $d(u, v) < \pi/2$ for all $u, v \in X$. A geodesic triangle for $x, y, z \in X$ is defined as $\triangle(x, y, z) = [y, z] \cup [z, x] \cup [x, y]$. A CAT(1) space is usually defined by using the notion of comparison triangle defined in the model space having the curvature 1. For the details, see [1]. We remark that we may characterize a geodesic space X to be a CAT(1) space by that the inequality

$$\begin{aligned} &\cos d(\alpha x \oplus (1 - \alpha)y, z) \sin d(x, y) \\ &\qquad \geq \cos d(x, z) \sin(\alpha d(x, y)) + \cos d(y, z) \sin((1 - \alpha)d(x, y)) \end{aligned}$$

holds for every $x, y, z \in X$ and $\alpha \in [0, 1]$. In particular, since $\sin td(x, y) \leq t \sin d(x, y)$ for $t \in [0, 1]$, it holds that

$$\cos d(\alpha x \oplus (1 - \alpha)y, z) \geq \alpha \cos d(x, z) + (1 - \alpha) \cos d(y, z)$$

for any $\alpha \in [0, 1]$ and x, y, z in a CAT(1) space.

Let C be a nonempty closed convex subset of a complete CAT(1) space X such that $d(u, v) < \pi/2$ for all $u, v \in X$. For any $x \in X$, there exists a unique $y_x \in C$ such that $d(x, y_x) = d(x, C) = \inf_{y \in C} d(x, y)$. We define the metric projection $P_C : X \to C$ by $P_C x = y_x$ for $x \in X$.

Let X be a complete CAT(1) space and let $f : X \to \,]-\infty, +\infty]$ be a proper lower semicontinuous convex function. Then, for $x \in X$, the function $g_x : X \to \,]-\infty, +\infty]$

defined by $g_x(y) = f(y) + \tan d(y, x) \sin d(y_x, x)$ has the unique minimizer. We define a resolvent $J_f : X \to X$ by

$$J_f x = \operatorname*{argmin}_{y \in X} g_x(y) = \operatorname*{argmin}_{y \in X}(f(y) + \tan d(y, x) \sin d(y, x))$$

for $x \in X$. It is known that the set $\operatorname{Fix} J_f = \{z \in X : J_f z = z\}$ of fixed points of J_f coincides with the set $\operatorname{argmin}_{y \in X} f(y)$ of minimizers of f on X. We also know that the resolvent operator is firmly spherically nonspreading in the sense that

$$(\cos d(J_f x, x) + \cos d(J_f y, y)) \cos^2 d(J_f x, J_f y) \geq 2 \cos d(J_f x, y) \cos d(x, J_f y)$$

for all $x, y \in X$. In particular, it is quasinonexpansive;

$$d(J_f x, z) \leq d(x, z)$$

for all $x \in X$ and $z \in \operatorname{Fix} J_f$. For a nonempty closed convex subset $C \subset X$, if f is the indicator function i_C of C defined by

$$i_C(x) = \begin{cases} \infty & (x \in C), \\ 0 & (x \notin C), \end{cases}$$

then, J_f is the metric projection onto C. For more details of the resolvent operators, see [2].

3 Approximation of the minimizer of the function

In this section, we prove an approximation theorem to a solution to a convex minimization problem for a convex function defined on a complete CAT(1) space.

Theorem 2. *Let X be a complete* CAT(1) *space such that $d(u, v) < \pi/2$ for any $u, v \in X$ and that a subset $\{z \in X : d(v, z) \leq d(u, z)\}$ is convex for any $u, v \in X$. Let $f : X \to]-\infty, +\infty]$ be a proper lower semicontinuous convex function such that the set $S = \operatorname{argmin}_{x \in X} f(x)$ of its minimizers is nonempty. Let $\{\alpha_n\}$ be a sequence in $[0, 1[$ with $\alpha_0 = \sup_{n \in \mathbb{N}} \alpha_n < 1$, $\{\epsilon_n\}$ be a sequence in $[0, \pi/4[$, and $\epsilon_0 = \limsup_{n \to \infty} \epsilon_n$. Let J_f be the resolvent for f. For given $u \in X$, generate a sequence $\{x_n\} \subset X$ and $\{C_n\}$ as follows: $x_1 = u$, $C_1 = X$, and*

$$\begin{aligned} y_n &= \alpha_n x_n \oplus (1 - \alpha_n) J_f x_n, \\ C_{n+1} &= \{z \in X : d(y_n, z) \leq d(x_n, z)\} \cap C_n, \\ x_{n+1} &\in C_{n+1} \text{ such that } \cos d(u, x_{n+1}) \geq \cos d(u, C_{n+1}) \cos \epsilon_{n+1}, \end{aligned}$$

for each $n \in \mathbb{N}$. Then

$$\limsup_{n \to \infty} d(x_n, J_f x_n) \leq \frac{2\epsilon_0}{1 - \alpha_0}$$

and

$$f(p) \le \liminf_{n\to\infty} f(J_f x_n) \le \limsup_{n\to\infty} f(J_f x_n) \\ \le f(p) + \pi\left(\sec^2 \frac{2\epsilon_0}{1-\alpha_0} + 1\right) \sin\frac{\epsilon_0}{1-\alpha_0}.$$

Moreover, if $\epsilon_0 = 0$*, then* $\{x_n\}$ *converges to* $P_S u$*, where* P_S *is the metric projection of* X *onto* S*.*

Proof. We first show that each C_n is a closed convex subset containing S by induction. It is obvious that $C_1 = X$ satisfies the conditions. Suppose that C_k is a nonempty closed convex subset of X and $S \subset C_k$ for fixed $k \in \mathbb{N}$. To show $S \subset C_{k+1}$, let $z \in S$. Then, since J_f is quasinonexpansive with $\operatorname{Fix} J_f = S$, we have that

$$\begin{aligned} \cos d(y_k, z) &= \cos d(\alpha_k x_k \oplus (1-\alpha_k) J_f x_k, z) \\ &\ge \alpha_k \cos d(x_k, z) + (1-\alpha_k) \cos d(J_f x_k, z) \\ &= \cos d(x_k, z), \end{aligned}$$

which implies $d(y_k, z) \le d(x_k, z)$, and hence $z \in S \subset C_{k+1}$. We also get that C_{k+1} is closed from the continuity of the metric d. The convexity of C_{k+1} is obtained from the assumption of the space. We also know that, there exists at least one point $y \in C_k$ such that $\cos d(u, y) \ge \cos d(u, C_k) \cos \epsilon_k$. In fact, the point $P_{C_k} u \in X$ satisfies that

$$\cos d(u, P_{C_k} u) = \cos d(u, C_k) \ge \cos d(u, C_k) \cos \epsilon_k.$$

Consequently, we have that C_n is a closed convex subset containing S for all $n \in \mathbb{N}$ and thus $\{x_n\}$ is well defined. Since $x_n \in C_n$, we have that

$$\cos d(u, x_n) \ge \cos d(u, C_n) \cos \epsilon_n$$

for every $n \in \mathbb{N}$.

Let $p_n = P_{C_n} u$, where P_{C_n} is the metric projection of X onto C_n for $n \in \mathbb{N}$, and let $C_0 = \bigcap_{n=1}^{\infty} C_n$. Then, since each C_n is convex, we have that $\alpha p_n \oplus (1-\alpha) x_n \in C_n$ for $\alpha \in \,]0, 1[$. Therefore,

$$\begin{aligned} &\sin d(p_n, x_n) \cos d(p_n, u) \\ &\ge \sin d(p_n, x_n) \cos d(\alpha p_n \oplus (1-\alpha) x_n, u) \\ &\ge \sin(\alpha d(p_n, x_n)) \cos d(p_n, u) + \sin((1-\alpha) d(p_n, x_n)) \cos d(x_n, u) \end{aligned}$$

and thus we have that

$$\begin{aligned} &\sin d(p_n, x_n) - \sin(\alpha d(p_n, x_n)) \\ &\ge \sin((1-\alpha) d(p_n, x_n)) \frac{\cos d(x_n, u)}{\cos d(p_n, u)} \end{aligned}$$

$$= 2\cos\left(\frac{1-\alpha}{2}d(p_n,x_n)\right)\sin\left(\frac{1-\alpha}{2}d(p_n,x_n)\right)\frac{\cos d(x_n,u)}{\cos d(p_n,u)}.$$

We also have that

$$\sin d(p_n,x_n) - \sin(\alpha d(p_n,x_n)) = 2\cos\left(\frac{1+\alpha}{2}d(p_n,x_n)\right)\sin\left(\frac{1-\alpha}{2}d(p_n,x_n)\right).$$

Suppose that $p_n \neq x_n$. Then these inequalities imply that

$$\cos\left(\frac{1+\alpha}{2}d(p_n,x_n)\right) \geq \cos\left(\frac{1-\alpha}{2}d(p_n,x_n)\right)\frac{\cos d(x_n,u)}{\cos d(p_n,u)}.$$

Tending $\alpha \to 1$, we have that

$$\cos d(p_n,x_n) \geq \frac{\cos d(x_n,u)}{\cos d(p_n,u)} = \frac{\cos d(x_n,u)}{\cos d(u,C_n)} \geq \cos\epsilon_n,$$

and it follows that $d(p_n,x_n) \leq \epsilon_n$ for every $n \in \mathbb{N}$. If $p_n = x_n$, this inequality is trivially true. Thus it holds for all $n \in \mathbb{N}$. Since $p_{n+1} \in C_{n+1}$, $d(y_n,x_n) = (1-\alpha_n)d(J_f x_n, x_n)$, and $\alpha_0 = \sup_{n\in\mathbb{N}} \alpha_n < 1$, we have that

$$\begin{aligned}
d(J_f x_n, x_n) &= \frac{1}{1-\alpha_n}d(y_n,x_n) \\
&\leq \frac{1}{1-\alpha_0}(d(y_n,p_{n+1}) + d(p_{n+1},x_n)) \\
&\leq \frac{2}{1-\alpha_0}d(x_n,p_{n+1}) \\
&\leq \frac{2}{1-\alpha_0}(d(x_n,p_n) + d(p_n,p_{n+1})) \\
&\leq \frac{2}{1-\alpha_0}(\epsilon_n + d(p_n,p_{n+1}))
\end{aligned}$$

for every $n \in \mathbb{N}$. Tending $n \to \infty$, we obtain that

$$\limsup_{n\to\infty} d(J_f x_n, x_n) \leq \frac{2\epsilon_0}{1-\alpha_0}.$$

Let $p = P_S u \in S$ and $z_\alpha = \alpha J_f x_n \oplus (1-\alpha)p$ for $\alpha \in]0,1[$. Then, we have that

$$\begin{aligned}
&f(J_f x_n) + \tan d(J_f x_n, x_n)\sin d(J_f x_n, x_n) \\
&\leq f(z_\alpha) + \tan d(z_\alpha, x_n)\sin d(z_\alpha, x_n) \\
&\leq \alpha f(J_f x_n) + (1-\alpha)f(p) + \tan d(z_\alpha, x_n)\sin d(z_\alpha, x_n).
\end{aligned}$$

Since $\tan t \sin t = \sec t - \cos t = 1/\cos t - \cos t$, we have that

$$(1-\alpha)(f(J_f x_n) - f(p))$$

$$
\begin{aligned}
&\le \tan d(z_\alpha, x_n)\sin d(z_\alpha, x_n) - \tan d(J_f x_n, x_n)\sin d(J_f x_n, x_n)\\
&= \left(\frac{1}{\cos d(z_\alpha, x_n)} - \frac{1}{\cos d(J_f x_n, x_n)}\right) - (\cos d(z_\alpha, x_n) - \cos d(J_f x_n, x_n))\\
&= \left(\frac{1}{\cos d(z_\alpha, x_n)\cos d(J_f x_n, x_n)} + 1\right)(\cos d(J_f x_n, x_n) - \cos d(z_\alpha, x_n))\\
&= E_{\alpha,n}(\cos d(J_f x_n, x_n) - \cos d(z_\alpha, x_n)),
\end{aligned}
$$

where

$$
E_{\alpha,n} = \frac{1}{\cos d(z_\alpha, x_n)\cos d(J_f x_n, x_n)} + 1.
$$

Let $D_n = d(J_f x_n, p)$ for $n \in \mathbb{N}$. If $D_{n_0} = 0$ for some $n_0 \in \mathbb{N}$, then

$$
J_f x_{n_0} = p \in S = \operatorname{Fix} J_f.
$$

This implies that $x_n = x_{n_0} = p$ for all $n \ge n_0$ and hence $\{x_n\}$ converges to $p = P_S u$, the conclusions of the theorem obviously hold. Therefore, we suppose that $D_n > 0$ for all $n \in \mathbb{N}$. We have that

$$
\begin{aligned}
&(\cos d(J_f x_n, x_n) - \cos d(z_\alpha, x_n))\sin D_n\\
&= \cos d(J_f x_n, x_n)\sin D_n - \cos d(\alpha J_f x_n \oplus (1-\alpha)p, x_n)\sin D_n\\
&\le \cos d(J_f x_n, x_n)\sin D_n\\
&\quad - \cos d(J_f x_n, x_n)\sin(\alpha D_n) - \cos d(p, x_n)\sin((1-\alpha)D_n)\\
&= \cos d(J_f x_n, x_n)(\sin D_n - \sin(\alpha D_n)) - \cos d(p, x_n)\sin((1-\alpha)D_n)\\
&\le (\sin D_n - \sin(\alpha D_n)) - \cos d(p, x_n)\sin((1-\alpha)D_n)\\
&\le 2\cos\frac{(1+\alpha)D_n}{2}\sin\frac{(1-\alpha)D_n}{2} - \cos d(p, x_n)\sin((1-\alpha)D_n),
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&(f(J_f x_n) - f(p))\frac{\sin D_n}{D_n}\\
&= \frac{E_{\alpha,n}}{(1-\alpha)D_n}(\cos d(J_f x_n, x_n) - \cos d(z_\alpha, x_n))\sin D_n\\
&\le \frac{E_{\alpha,n}}{(1-\alpha)D_n}2\cos\frac{(1+\alpha)D_n}{2}\sin\frac{(1-\alpha)D_n}{2} - \cos d(p, x_n)\sin((1-\alpha)D_n)\\
&= E_{\alpha,n}\left(\frac{\sin((1-\alpha)D_n/2)}{(1-\alpha)D_n/2}\cos\frac{(1+\alpha)D_n}{2} - \frac{\sin((1-\alpha)D_n)}{(1-\alpha)D_n}\cos d(p, x_n)\right).
\end{aligned}
$$

Since $E_{\alpha,n} \to 1/\cos^2 d(J_f x_n, x_n) + 1 = \sec^2 d(J_f x_n, x_n) + 1$ as $\alpha \uparrow 1$, we have that

$$
(f(J_f x_n) - f(p))\frac{\sin D_n}{D_n}
$$

$$\begin{aligned}&\le (\sec^2 d(J_f x_n, x_n) + 1)(\cos D_n - \cos d(p, x_n))\\&= (\sec^2 d(J_f x_n, x_n) + 1)(\cos d(J_f x_n, p) - \cos d(x_n, p)).\end{aligned}$$

Further we have that

$$\begin{aligned}&\cos d(J_f x_n, p) - \cos d(x_n, p)\\&= 2\sin\frac{d(x_n,p) + d(J_f x_n, p)}{2}\sin\frac{d(x_n,p) - d(J_f x_n,p)}{2}\\&\le 2\sin\frac{d(x_n,p) - d(J_f x_n,p)}{2}\\&\le 2\sin\frac{d(J_f x_n, x_n)}{2}.\end{aligned}$$

Since $\sin D_n / D_n \ge 2/\pi$ for all $n \in \mathbb{N}$, we get that

$$\begin{aligned}\frac{2}{\pi}(f(J_f x_n) - f(p)) &\le (f(J_f x_n) - f(p))\frac{\sin D_n}{D_n}\\&\le 2(\sec^2 d(J_f x_n, x_n) + 1)\sin\frac{d(J_f x_n, x_n)}{2}.\end{aligned}$$

Tending $n \to \infty$, we have that

$$\begin{aligned}&\frac{2}{\pi}\left(\limsup_{n\to\infty} f(J_f x_n) - f(p)\right)\\&\le 2\limsup_{n\to\infty}(\sec^2 d(J_f x_n, x_n) + 1)\sin\frac{d(J_f x_n, x_n)}{2}\\&\le 2\left(\sec^2\frac{2\epsilon_0}{1-\alpha_0} + 1\right)\sin\frac{\epsilon_0}{1-\alpha_0}.\end{aligned}$$

Hence we obtain that

$$f(x_0) - f(p) \le \limsup_{n\to\infty} f(J_f x_n) - f(p) \le \pi\left(\sec^2\frac{2\epsilon_0}{1-\alpha_0} + 1\right)\sin\frac{\epsilon_0}{1-\alpha_0},$$

and therefore

$$\begin{aligned}f(p) &\le \liminf_{n\to\infty} f(J_f x_n) \le \limsup_{n\to\infty} f(J_f x_n)\\&\le f(p) + \pi\left(\sec^2\frac{2\epsilon_0}{1-\alpha_0} + 1\right)\sin\frac{\epsilon_0}{1-\alpha_0},\end{aligned}$$

which is the desired result.

For the latter part of the theorem, suppose $\epsilon_0 = 0$. Then, since $d(x_n, p_n) \le \epsilon_n$ and $\lim_{n\to\infty}\epsilon_n = \epsilon_0 = 0$, we get that both $\{x_n\}$ and $\{p_n\}$ converges to $p_0 = P_{C_0}u$. Moreover, since

$$\limsup_{n\to\infty} d(J_f x_n, x_n) \le \frac{2\epsilon_0}{1-\alpha_0} = 0,$$

$\{J_f x_n\}$ also converges to p_0. Using the lower semicontinuity of f, we have that

$$\begin{aligned} f(p) &\leq f(p_0) \\ &\leq \liminf_{n\to\infty} f(J_f x_n) \\ &\leq f(p) + \pi \left(\sec^2 0 + 1\right) \sin 0 \\ &= f(p). \end{aligned}$$

Therefore $p_0 \in \operatorname{argmin}_{x\in X} f(x) = S$. Since $S \subset C_0$ and $p_0 = P_{C_0}u$, we have that $p_0 = P_S u$, which completes the proof. □

Let us consider the values

$$\frac{2\epsilon_0}{1-\alpha_0} \text{ and } \pi\left(\sec^2 \frac{2\epsilon_0}{1-\alpha_0} + 1\right) \sin \frac{\epsilon_0}{1-\alpha_0}$$

of the upper bounds for $\limsup_{n\to\infty} d(x_n, J_f x_n)$ and $\limsup_{n\to\infty} |f(p) - f(J_f x_n)| = \limsup_{n\to\infty} |\min_{x\in X} f(x) - f(J_f x_n)|$. It is easy to see that these values are increasing with respect to $\alpha_0 \in [0, 1[$. Therefore, if we wish to make them as small as possible, the best choice is α_0 to be 0. Letting simply $\alpha_n = 0$ for all $n \in \mathbb{N}$, we obtain the following iterative scheme: $x_1 = u$, $C_1 = X$, and

$$\begin{aligned} &C_{n+1} = \{z \in X : d(J_f x_n, z) \leq d(x_n, z)\} \cap C_n, \\ &x_{n+1} \in C_{n+1} \text{ such that } \cos d(u, x_{n+1}) \geq \cos d(u, C_{n+1}) \cos \epsilon_{n+1}, \end{aligned}$$

for each $n \in \mathbb{N}$. Then, by the main result, we obtain that $\limsup_{n\to\infty} d(x_n, J_f x_n) \leq 2\epsilon_0$ and

$$f(p) \leq \liminf_{n\to\infty} f(J_f x_n) \leq \limsup_{n\to\infty} f(J_f x_n) \leq f(p) + \pi\left(\sec^2(2\epsilon_0) + 1\right) \sin \epsilon_0,$$

which is much simpler consequence and the best choice of the coefficients $\{\alpha_n\}$ from the view of error estimate.

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