数理解析研究所講究録
第2065巻 2018年 9-18



# Optimality and duality for a class of nonsmooth fractional multiobjective optimization problems

Do Sang Kim[1] and Zhe Hong[2]

[1]Department of Applied Mathematics
Pukyong National University, Republic of Korea
E-mail: dskim@pknu.ac.kr

[2]Department of Applied Mathematics
Pukyong National University, Republic of Korea
email: hong_chul@163.com

**Abstract.** In this paper, we establish necessary optimality conditions for (weakly) efficient solutions of a nonsmooth fractional multiobjective optimization problem with inequality and equality constraints by employing some advanced tools of variational analysis and generalized differentiation. Sufficient optimality conditions for such solutions to the considered problem are also provided by means of introducing (strictly) convex-affine functions. Along with optimality conditions, we formulate a dual problem to the primal one and explore weak, strong and converse duality relations between them under assumptions of (strictly) convex-affine functions.

## 1 Introduction and Preliminaries

Optimality conditions and duality for (weakly) Pareto/efficient solutions in *fractional multiobjective optimization problems* have been investigated intensively by many researchers; see e.g., [2, 3, 5–11, 14, 15, 17] and the references therein. One of the main tools used to examine a fractional multiobjective optimization problem is that one employs the separation theorem of *convex sets* (see e.g., [16]) to provide necessary conditions for (weakly) efficient solutions of the considered problem and exploits various kinds of (generalized) convex/or invex functions to formulate sufficient conditions for the existence of such solutions.

It should be noted further that since the kinds of (generalized) invex functions mentioned above have been constructed via the Clarke subdifferential of locally Lipschitz functions, we therefore have to remain using tacitly the separation theorem of convex sets in the schemes of proof. In fact, a characteristic of a fractional multiobjective optimization problem is that its objective function is generally *not* a convex function. Even under more restrictive concavity/convexity assumptions fractional multiobjective optimization problems are generally *nonconvex* ones.

Besides, the (approximate) *extremal principle* [13], which plays a key role in variational analysis and generalized differentiation, has been well-recognized as a variational counterpart of the separation theorem for *nonconvex* sets. Hence using the extremal principle and other advanced techniques of variational analysis and generalized differentiation to establish optimality conditions seems to be suitable for *nonconvex/nonsmooth* fractional multiobjective optimization problems.

In this work, we employ some advanced tools of variational analysis and generalized differentiation (e.g., the nonsmooth version of Fermat's rule, the sum rule and the quotient rule for the limiting/Mordukhovich subdifferential, and the intersection rule for the normal/Mordukhovich cone) to establish necessary conditions for (weakly) efficient solutions of a nonsmooth fractional multiobjective optimization problem with inequality and equality constraints.

Since the limiting/Mordukhovich subdifferential of a real-valued function at a given point is *contained in* the Clarke subdifferential of such a function at the corresponding point (cf. [13]), the necessary conditions formulated in terms of the limiting/Mordukhovich subdifferential are *sharper* than the corresponding ones expressed in terms of the Clarke subdifferential. Sufficient conditions for the existence of such solutions to the considered problem are also provided by means of introducing (strictly) convex-affine functions defined in terms of the limiting subdifferential for locally Lipschitz functions.

Along with optimality conditions, we state a dual problem to the primal one and explore weak, strong and converse duality relations under assumptions of (strictly) convexity-affineness. Furthermore, examples are given for analyzing and illustrating the obtained results.

Throughout the paper we use the standard notation of variational analysis; see e.g., [13]. Unless otherwise specified, all spaces under consideration are assumed to be *Asplund* (i.e., Banach spaces whose separable subspaces have separable duals). The canonical pairing between space $X$ and its topological dual $X^*$ is denoted by $\langle\cdot,\cdot\rangle$, while the symbol $\|\cdot\|$ stands for the norm in the considered space. As usual, the *polar cone* of a set $\Omega\subset X$ is defined by

$$\Omega^\circ := \{x^*\in X^* \mid \langle x^*,x\rangle \le 0 \quad \forall x\in\Omega\}. \tag{1.1}$$

Also, for each $m\in\mathbb{N}:=\{1,2,\dots\}$, we denote by $\mathbb{R}^m_+$ the nonnegative orthant of $\mathbb{R}^m$.

Given a multifunction $F\colon X \rightrightarrows X^*$, we denote by

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{x^*\in X^* \Big|\ \exists \text{ sequences } x_n\to\bar{x} \text{ and } x_n^* \xrightarrow{w^*} x^* \text{ with } x_n^*\in F(x_n) \text{ for all } n\in\mathbb{N}\Big\}$$

the *sequential Painlevé-Kuratowski upper/outer limit* of $F$ as $x\to\bar{x}$, where the notation $\xrightarrow{w^*}$ indicates the convergence in the weak* topology of $X^*$.

Given $\Omega\subset X$ and $\varepsilon\ge 0$, define the collection of *$\varepsilon$-normals* to $\Omega$ at $\bar{x}\in\Omega$ by

$$\widehat{N}_\varepsilon(\bar{x};\Omega) := \Big\{x^*\in X^* \Big| \limsup_{x\xrightarrow{\Omega}\bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \le \varepsilon\Big\}, \tag{1.2}$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. When $\varepsilon = 0$, the set $\widehat{N}(\bar{x};\Omega) := \widehat{N}_0(\bar{x};\Omega)$ in (1.2) is a cone called the *Fréchet normal cone* to $\Omega$ at $\bar{x}$. If $\bar{x} \notin \Omega$, we put $\widehat{N}_\varepsilon(\bar{x};\Omega) := \emptyset$ for all $\varepsilon \geq 0$.

The *limiting/Mordukhovich normal cone* $N(\bar{x};\Omega)$ at $\bar{x} \in \Omega$ is obtained from $\widehat{N}_\varepsilon(x;\Omega)$ by taking the sequential Painlevé-Kuratowski upper limits as

$$N(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{\Omega} \bar{x} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x;\Omega), \tag{1.3}$$

where $\varepsilon \downarrow 0$ signifies $\varepsilon \to 0$ and $\varepsilon \geq 0$. If $\bar{x} \notin \Omega$, we put $N(\bar{x};\Omega) := \emptyset$. Note that one can put $\varepsilon := 0$ in (1.3) when $\Omega$ is (locally) *closed around* $\bar{x}$, i.e., there is a neighborhood $U$ of $\bar{x}$ such that $\Omega \cap \operatorname{cl} U$ is closed.

For an extended real-valued function $\varphi : X \to \overline{\mathbb{R}} := [-\infty,\infty]$, we set

$$\operatorname{gph}\varphi := \{(x,\mu) \in X \times \mathbb{R} \mid \mu = \varphi(x)\}, \quad \operatorname{epi}\varphi := \{(x,\mu) \in X \times \mathbb{R} \mid \mu \geq \varphi(x)\}.$$

The *limiting/Mordukhovich subdifferential* of $\varphi$ at $\bar{x} \in X$ with $|\varphi(\bar{x})| < \infty$ is defined by

$$\partial\varphi(\bar{x}) := \{x^* \in X^* \mid (x^*,-1) \in N((\bar{x},\varphi(\bar{x}));\operatorname{epi}\varphi)\}. \tag{1.4}$$

If $|\varphi(\bar{x})| = \infty$, then one puts $\partial\varphi(\bar{x}) := \emptyset$. It is known (cf. [13]) that when $\varphi$ is a convex function, the above-defined subdifferential coincides with the subdifferential in the sense of convex analysis [16].

Considering the indicator function $\delta(\cdot;\Omega)$ defined by $\delta(x;\Omega) := 0$ for $x \in \Omega$ and by $\delta(x;\Omega) := \infty$ otherwise, we have a relation between the Mordukhovich normal cone and the limiting subdifferential of the indicator function as follows (see [13, Proposition 1.79]):

$$N(\bar{x};\Omega) = \partial\delta(\bar{x};\Omega) \quad \forall \bar{x} \in \Omega. \tag{1.5}$$

The nonsmooth version of Fermat's rule (see e.g., [13, Proposition 1.114]), which is an important fact for many applications, can be formulated as follows: If $\bar{x} \in X$ is a *local minimizer* for $\varphi : X \to \overline{\mathbb{R}}$, then

$$0 \in \partial\varphi(\bar{x}). \tag{1.6}$$

The following limiting subdifferential sum rule is needed for our study.

**Lemma 1.1** (See [13, Theorem 3.36]) *Let $\varphi_i : X \to \overline{\mathbb{R}}, i = 1,2,...,n, n \geq 2$, be lower semicontinuous around $\bar{x} \in X$, and let all these functions except, possibly, one be Lipschitz continuous around $\bar{x}$. Then one has*

$$\partial(\varphi_1 + \varphi_2 + \cdots + \varphi_n)(\bar{x}) \subset \partial\varphi_1(\bar{x}) + \partial\varphi_2(\bar{x}) + \cdots + \partial\varphi_n(\bar{x}). \tag{1.7}$$

Combining this limiting subdifferential sum rule with the quotient rule (cf. [13, Corollary 1.111(ii)]), we get an estimate for the limiting subdifferential of quotients.

**Lemma 1.2** *Let $\varphi_i : X \to \overline{\mathbb{R}}, i = 1,2,$ be Lipschitz continuous around $\bar{x}$. Assume that $\varphi_2(\bar{x}) \neq 0$. Then one has*

$$\partial\left(\frac{\varphi_1}{\varphi_2}\right)(\bar{x}) \subset \frac{\partial\big(\varphi_2(\bar{x})\varphi_1\big)(\bar{x}) + \partial\big(-\varphi_1(\bar{x})\varphi_2\big)(\bar{x})}{[\varphi_2(\bar{x})]^2}. \tag{1.8}$$

Recall [13] that a set $\Omega \subset X$ is *sequentially normally compact* (SNC) at $\bar{x} \in \Omega$ if for any sequences

$$\varepsilon_k \downarrow 0,\ x_k \xrightarrow{\Omega} \bar{x},\ \text{and } x_k^* \xrightarrow{w^*} 0 \text{ with } x_k^* \in \widehat{N}_{\varepsilon_k}(x_k;\Omega),$$

one has $\|x_k^*\| \to 0$ as $k \to \infty$. Here, $\varepsilon_k$ can be omitted when $\Omega$ is closed around $\bar{x}$. Obviously, this SNC property is automatically satisfied in finite dimensional spaces. A function $\varphi : X \to \mathbb{R}$ is called *sequentially normally compact* (SNC) at $\bar{x} \in X$ if $\operatorname{gph}\varphi$ is SNC at $(\bar{x}, \varphi(\bar{x}))$. According to [13, Corollary 1.69(i)], $\varphi$ is SNC at $\bar{x} \in X$ if it is Lipschitz continuous around $\bar{x}$.

In what follows, we also need the intersection rule for the normal cones under the fulfillment of the SNC condition.

**Lemma 1.3** (See [13, Corollary 3.5]) *Assume that $\Omega_1, \Omega_2 \subset X$ are closed around $\bar{x} \in \Omega_1 \cap \Omega_2$ and that at least one of $\{\Omega_1, \Omega_2\}$ is* SNC *at this point. If*

$$N(\bar{x};\Omega_1) \cap \big(-N(\bar{x};\Omega_2)\big) = \{0\},$$

*then*

$$N(\bar{x};\Omega_1 \cap \Omega_2) \subset N(\bar{x};\Omega_1) + N(\bar{x};\Omega_2).$$

# 2 Optimality Conditions in Fractional Multiobjective Optimization

This section is devoted to studying optimality conditions for fractional multiobjective optimization problems. More precisely, by using the nonsmooth version of Fermat's rule, the sum rule and the quotient rule for the limiting subdifferentials, and the intersection rule for the Mordukhovich cones, we first establish necessary conditions for (weakly) efficient solutions of a fractional multiobjective optimization problem. Then by imposing assumptions of (strictly) convexity-affineness, we give sufficient conditions for the existence of such solutions.

Let $\Omega$ be a nonempty locally closed subset of $X$, and let $K = \{1,...,m\}$, $I = \{1,...,n\} \cup \emptyset$ and $J = \{1,...,l\} \cup \emptyset$ be index sets. In what follows, $\Omega$ is always assumed to be SNC at the point under consideration. This assumption is automatically fulfilled when $X$ is a finite dimensional space.

We consider the following *fractional multiobjective optimization problem* (P):

$$\min\nolimits_{\mathbb{R}^m_+} \left\{ f(x) := \left( \frac{p_1(x)}{q_1(x)}, \cdots, \frac{p_m(x)}{q_m(x)} \right) \Big| \, x \in C \right\}, \tag{2.9}$$

where the constraint set $C$ is defined by

$$C := \{x \in \Omega \mid g_i(x) \le 0,\, i \in I,$$
$$h_j(x) = 0,\, j \in J\}, \tag{2.10}$$

and the functions $p_k, q_k, k \in K, g_i, i \in I$, and $h_j, j \in J$ are locally Lipschitz on $X$. For the sake of convenience, we further assume that $q_k(x) > 0, k \in K$ for all $x \in \Omega$, and that $p_k(\bar{x}) \le 0, k \in K$ for the reference point $\bar{x} \in \Omega$. Also, we use hereafter the notation $g := (g_1, ..., g_n)$, $h := (h_1, ..., h_l)$ and $f := (f_1, ..., f_m)$, where $f_k := \frac{p_k}{q_k}$, $k \in K$.

**Definition 2.1** (i) We say that $\bar{x} \in C$ is an *efficient solution* of problem (2.9), and write $\bar{x} \in \mathcal{S}(P)$, iff

$$\forall x \in C, \quad f(x) - f(\bar{x}) \notin -\mathbb{R}^m_+ \setminus \{0\}.$$

(ii) A point $\bar{x} \in C$ is called a *weakly efficient solution* of problem (2.9), and write $\bar{x} \in \mathcal{S}^w(P)$, iff

$$\forall x \in C, \quad f(x) - f(\bar{x}) \notin -\operatorname{int} \mathbb{R}^m_+.$$

For $\bar{x} \in \Omega$, let us put

$$I(\bar{x}) := \{i \in I \mid g_i(\bar{x}) = 0\}, \quad J(\bar{x}) := \{j \in J \mid h_j(\bar{x}) = 0\}.$$

**Definition 2.2** We say that condition (CQ) is satisfied at $\bar{x} \in \Omega$ if there do not exist $\beta_i \ge 0, i \in I(\bar{x})$ and $\gamma_j \ge 0, j \in J(\bar{x})$, such that $\sum_{i \in I(\bar{x})} \beta_i + \sum_{j \in J(\bar{x})} \gamma_j \neq 0$ and

$$0 \in \sum_{i \in I(\bar{x})} \beta_i \partial g_i(\bar{x}) + \sum_{j \in J(\bar{x})} \gamma_j \big(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\big) + N(\bar{x}; \Omega).$$

It is worth to mention here that when considering $\bar{x} \in C$ defined in (2.10) with $\Omega = X$ in the smooth setting, the above-defined (CQ) is guaranteed by the Mangasarian-Fromovitz constraint qualification; see e.g., [13] for more details.

The following theorem gives a Karush-Kuhn-Tucker type necessary condition for (weakly) efficient solutions of problem (2.9).

**Theorem 2.1** *Let the* (CQ) *be satisfied at* $\bar{x} \in \Omega$. *If* $\bar{x} \in \mathcal{S}^w(P)$, *then there exist* $\lambda := (\lambda_1, ..., \lambda_m) \in \mathbb{R}^m_+ \backslash \{0\}, \beta := (\beta_1, ..., \beta_n) \in \mathbb{R}^n_+$, *and* $\gamma = (\gamma_1, ..., \gamma_l) \in \mathbb{R}^l_+$ *such that*

$$0 \in \sum_{k \in K} \lambda_k \left( \partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{i \in I} \beta_i \partial g_i(\bar{x})$$
$$+ \sum_{j \in J} \gamma_j \big( \partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x}) \big) + N(\bar{x}; \Omega), \quad \beta_i g_i(\bar{x}) = 0, \quad i \in I. \tag{2.11}$$

A simple example below shows that the conclusion of Theorem 2.1 may fail if the (CQ) is not satisfied at the point in question.

**Example 2.1** Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x) := \left( \frac{p_1(x)}{q_1(x)}, \frac{p_2(x)}{q_2(x)} \right),$$

where $p_1(x) = p_2(x) := x, q_1(x) = q_2(x) := x^2 + 1, x \in \mathbb{R}$, and let $g, h : \mathbb{R} \to \mathbb{R}$ be given by $g(x) := x^2$, $h(x) := 0, x \in \mathbb{R}$. We consider problem (2.9) with $m := 2$ and $\Omega := (-\infty, 0] \subset \mathbb{R}$. Then $C = \{0\}$ and thus, $\bar{x} := 0 \in \mathcal{S}^w(P)(= \mathcal{S}(P))$. In this setting, we have $N(\bar{x}; \Omega) = [0, +\infty)$. Now, we can check that condition (CQ) is not satisfied at $\bar{x}$. Meantime, $\bar{x}$ does not satisfy (2.11) either.

We refer the reader to a result [1, Theorem 4.2] about necessary conditions for a more general multiobjective fractional program with *equilibrium constraints* by way of a different approach.

The next example illustrates that Theorem 2.1 works better in comparison with some of the existing results about optimality conditions for fractional multiobjective optimization problems, for instance, in [5].

**Example 2.2** Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x) := \left( \frac{p_1(x)}{q_1(x)}, \frac{p_2(x)}{q_2(x)} \right),$$

where $p_1(x) = p_2(x) := |x|, q_1(x) = q_2(x) := -|x| + 1, x \in \mathbb{R}$, and let $g, h : \mathbb{R} \to \mathbb{R}$ be given by $g(x) := -x - 1$, $h(x) = 0$, $x \in \mathbb{R}$. Let us consider problem (2.9) with $K := \{1, 2\}, I := \{1\}, J := \emptyset$, and $\Omega := (-1, 1) \subset \mathbb{R}$. It is easy to check that $\bar{x} := 0 \in \mathcal{S}^w(P)$ and the (CQ) is satisfied at this point. So, in this setting we can apply Theorem 2.1 to conclude that $\bar{x}$ satisfies condition (2.11). Meanwhile, since the functions $q_1, q_2$ are not differentiable at $\bar{x}$, [5, Theorem 2.2] is not applicable to this problem.

It should be noted further that, in general, a feasible point of problem (2.9) satisfying condition (2.11) is not necessarily to be a weakly efficient solution even in the smooth case. This will be illustrated by the following example.

**Example 2.3** Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x) := \left(\frac{p_1(x)}{q_1(x)}, \frac{p_2(x)}{q_2(x)}\right),$$

where $p_1(x) = p_2(x) := x^3 - 1, q_1(x) = q_2(x) := x^2 + 1, x \in \mathbb{R}$, and let $g, h : \mathbb{R} \to \mathbb{R}$ be given by $g(x) := -x^2$, $h(x) := 0$, $x \in \mathbb{R}$. Let us consider problem (2.9) with $m := 2$ and $\Omega := (-\infty, 1] \subset \mathbb{R}$. Then $C = \Omega$ and thus, $\bar{x} := 0 \in C$. In this setting, we have $N(\bar{x}; \Omega) = \{0\}$. Observe that $\bar{x}$ satisfies condition (2.11). However, $\bar{x} \notin \mathcal{S}^w(P)$.

By virtue of Example 2.3, obtaining sufficient conditions for (weakly) efficient solutions of problem (2.9) requires concepts of convexity-affineness-type for locally Lipschitz functions on $\Omega$, here $\Omega$ is a convex set. Note that if $\Omega$ is nonconvex set, then some results can be referred to [4].

**Definition 2.3** (i) We say that $(f, g; h)$ is *convex-affine* on $\Omega$ at $\bar{x} \in \Omega$ if for any $x \in \Omega$, $u_k^* \in \partial p_k(\bar{x})$, $v_k^* \in \partial q_k(\bar{x})$, $k \in K$, $x_i^* \in \partial g_i(\bar{x}), i \in I$, and $y_j^* \in \partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x}), j \in J$,

$$\begin{aligned} p_k(x) - p_k(\bar{x}) &\geq \langle u_k^*, x - \bar{x}\rangle, k \in K, \\ q_k(x) - q_k(\bar{x}) &\geq \langle v_k^*, x - \bar{x}\rangle, k \in K, \\ g_i(x) - g_i(\bar{x}) &\geq \langle x_i^*, x - \bar{x}\rangle, i \in I, \\ h_j(x) - h_j(\bar{x}) &= \omega_j \langle y_j^*, x - \bar{x}\rangle, j \in J, \end{aligned}$$

where $\omega_j = 1$ (respectively, $\omega_j = -1$) whenever $y_j^* \in \partial h_j(\bar{x})$ (respectively, $y_j^* \in \partial(-h_j)(\bar{x})$).

(ii) We say that $(f, g; h)$ is *strictly convex-affine* on $\Omega$ at $\bar{x} \in \Omega$ if for any $x \in \Omega\backslash\{\bar{x}\}$, $u_k^* \in \partial p_k(\bar{x})$, $v_k^* \in \partial q_k(\bar{x})$, $k \in K$, $x_i^* \in \partial g_i(\bar{x}), i \in I$, and $y_j^* \in \partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x}), j \in J$,

$$\begin{aligned} p_k(x) - p_k(\bar{x}) &> \langle u_k^*, x - \bar{x}\rangle, k \in K, \\ q_k(x) - q_k(\bar{x}) &\geq \langle v_k^*, x - \bar{x}\rangle, k \in K, \\ g_i(x) - g_i(\bar{x}) &\geq \langle x_i^*, x - \bar{x}\rangle, i \in I, \\ h_j(x) - h_j(\bar{x}) &= \omega_j \langle y_j^*, x - \bar{x}\rangle, j \in J, \end{aligned}$$

where $\omega_j = 1$ (respectively, $\omega_j = -1$) whenever $y_j^* \in \partial h_j(\bar{x})$ (respectively, $y_j^* \in \partial(-h_j)(\bar{x})$).

We are now in a position to provide sufficient conditions for a feasible point of problem (2.9) to be a *weakly efficient* (or *efficient*) solution.

**Theorem 2.2** *Assume that* $\bar{x} \in C$ *satisfies condition* (2.11).

(i) *If* $(f, g; h)$ *is convex-affine on* $\Omega$ *at* $\bar{x}$, *then* $\bar{x} \in \mathcal{S}^w(P)$.

(ii) *If* $(f, g; h)$ *is strictly convex-affine on* $\Omega$ *at* $\bar{x}$, *then* $\bar{x} \in \mathcal{S}(P)$.

# 3 Duality in Fractional Multiobjective Optimization

In this section we propose a dual problem to the primal one in the sense of Mond-Weir [12] and examine weak, strong, and converse duality relations between them. Note further that another dual problem formulated in the sense of Wolfe [18] can be similarly treated.

Let $z \in X, \lambda := (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m_+ \setminus \{0\}$, $\mu := (\mu_1, \dots, \mu_n) \in \mathbb{R}^n_+$, and $\gamma := (\gamma_1, \dots, \gamma_l) \in \mathbb{R}^l_+$. In connection with the fractional multiobjective optimization problem (P) given in (2.9), we consider a *fractional multiobjective dual problem* of the form (D):

$$\max_{\mathbb{R}^m_+} \left\{ \bar{f}(z, \lambda, \mu, \gamma) := \left( \frac{p_1(z)}{q_1(z)}, \cdots, \frac{p_m(z)}{q_m(z)} \right) \;\Big|\; (z, \lambda, \mu, \gamma) \in C_D \right\}. \quad (3.12)$$

Here the constraint set $C_D$ is defined by

$$C_D := \Big\{ (z, \lambda, \mu, \gamma) \in \Omega \times (\mathbb{R}^m_+ \setminus \{0\}) \times \mathbb{R}^n_+ \times \mathbb{R}^l_+ \mid 0 \in \sum_{k \in K} \lambda_k \left( \partial p_k(z) - \frac{p_k(z)}{q_k(z)} \partial q_k(z) \right)$$
$$+ \sum_{i \in I} \mu_i \partial g_i(z) + \sum_{j \in J} \gamma_j \left( \partial h_j(z) \cup \partial(-h_j)(z) \right) + N(z; \Omega),$$
$$\langle \mu, g(z) \rangle + \langle \sigma, h(z) \rangle \geq 0 \quad \forall \sigma \in \mathbb{S}(0, ||\gamma||) \Big\},$$

where $\mathbb{S}(0, ||\gamma||) := \{ \sigma \in \mathbb{R}^l \mid ||\sigma|| = ||\gamma|| \}$.

We need to address here that an *efficient solution* (resp., a *weakly efficient solution*) of the dual problem in (3.12) is similarly defined as in Definition 2.1 by replacing $-\mathbb{R}^m_+$ (resp., $\operatorname{int} \mathbb{R}^m_+$) by $\mathbb{R}^m_+$ (resp., $-\operatorname{int} \mathbb{R}^m_+$). Also, we denote the set of efficient solutions (resp., weakly efficient solutions) of problem (3.12) by $\mathcal{S}(D)$ (resp., $\mathcal{S}^w(D)$).

In what follows, we use the following notation for convenience.

$$u \prec v \Leftrightarrow u - v \in -\operatorname{int} \mathbb{R}^m_+, \; u \nprec v \text{ is the negation of } u \prec v,$$

$$u \preceq v \Leftrightarrow u - v \in -\mathbb{R}^m_+ \setminus \{0\}, \; u \npreceq v \text{ is the negation of } u \preceq v.$$

The first theorem in this section describes weak duality relations between the primal problem (P) in (2.9) and the dual problem (D) in (3.12).

**Theorem 3.1** (Weak Duality) *Let $x \in C$ and let $(z, \lambda, \mu, \gamma) \in C_D$.*

(i) *If $(f, g; h)$ is convex-affine on $\Omega$ at $z$, then*

$$f(x) \nprec \bar{f}(z, \lambda, \mu, \gamma).$$

(ii) *If $(f, g; h)$ is strictly convex-affine on $\Omega$ at $z$, then*

$$f(x) \npreceq \bar{f}(z, \lambda, \mu, \gamma).$$

Strong duality relations between the primal problem (P) in (2.9) and the dual problem (D) in (3.12) read as follows.

**Theorem 3.2** (Strong Duality) *Let $\bar{x} \in \mathcal{S}^w(P)$ be such that the* (CQ) *is satisfied at this point. Then there exists $(\bar{\lambda}, \bar{\mu}, \bar{\gamma}) \in (\mathbb{R}^m_+ \backslash \{0\}) \times \mathbb{R}^n_+ \times \mathbb{R}^l_+$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma}) \in C_D$ and $f(\bar{x}) = \bar{f}(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma})$.*

(i) *If in addition $(f, g; h)$ is convex-affine on $\Omega$ at any $z \in \Omega$, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma}) \in \mathcal{S}^w(D)$.*

(ii) *If in addition $(f, g; h)$ is strictly convex-affine on $\Omega$ at any $z \in \Omega$, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma}) \in \mathcal{S}(D)$.*

We close this section by presenting converse-like duality relations between the primal problem (P) in (2.9) and the dual problem (D) in (3.12).

**Theorem 3.3** (Converse Duality) *Let $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma}) \in C_D$.*
(i) *If $\bar{x} \in C$ and $(f, g; h)$ is convex-affine on $\Omega$ at $\bar{x}$, then $\bar{x} \in \mathcal{S}^w(P)$.*
(ii) *If $\bar{x} \in C$ and $(f, g; h)$ is strictly convex-affine on $\Omega$ at $\bar{x}$, then $\bar{x} \in \mathcal{S}(P)$.*

# References


[1] T. Q. Bao, P. Gupta, B. S. Mordukhovich, Necessary conditions in multiobjective optimization with equilibrium constraints, J. Optim. Theory Appl. 135 (2007), no. 2, 179–203.

[2] C. R. Bector, S. Chandra, I. Husain, Optimality conditions and duality in subdifferentiable multiobjective fractional programming, J. Optim. Theory Appl. 79 (1993) 105–125.

[3] X. Chen, Optimality and duality for the multiobjective fractional programming with the generalized $(F, \rho)$ convexity, J. Math. Anal. Appl. 273 (2002), no. 1, 190–205.

[4] T. D. Chuong, D. S. Kim, Optimality conditions and duality in nonsmooth multiobjective optimization problems, Ann. Oper. Res. 217 (2014), 117–136.

[5] D. S. Kim, S. J. Kim, M. H. Kim, Optimality and duality for a class of nondifferentiable multiobjective fractional programming problems, J. Optim. Theory Appl. 129 (2006) 131–146.

[6] H. Kuk, G. M. Lee, T. Tanino, Optimality and duality for nonsmooth multiobjective fractional programming with generalized invexity, J. Math. Anal. Appl., 262 (2001), 365–375.

[7] H.-C. Lai, S.-C. Ho, Optimality and duality for nonsmooth multiobjective fractional programming problems involving exponential $V$-$r$-invexity, Nonlinear Anal. 75 (2012), no. 6, 3157–3166.

[8] J.-C. Lee, H.-C. Lai, Parameter-free dual models for fractional programming with generalized invexity, Ann. Oper. Res. 133 (2005), 47–61.

[9] J. C. Liu, Optimality and duality for multiobjectional fractional programming involving nonsmooth $(F, \rho)$-convex functions, Optimization 36 (1996) 333–346.

[10] S. Liu, E. Feng, Optimality conditions and duality for a class of nondifferentiable multiobjective fractional programming problems, J. Global Optim. 38 (2007) 653–666.

[11] X. J. Long, Optimality conditions and duality for nondifferentiable multiobjective fractional programming problems with $(C, \alpha, \rho, d)$-convexity, J. Optim. Theory Appl. 148 (2011), no. 1, 197–208.

[12] B. Mond and T. Weir, Generalized concavity and duality, in: S. Schaible, W.T. Ziemba (Eds.), Generalized Concavity in Optimization and Economics, Academic Press, New York, 1981, pp. 263–279.

[13] B. S. Mordukhovich, Variational Analysis and Generalized Differentiation. I: Basic Theory, Springer, Berlin, 2006.

[14] C. Niculescu, Optimality and duality in multiobjective fractional programming involving $\rho$-semilocally type I-preinvex and related functions, J. Math. Anal. Appl. 335 (2007), no. 1, 7–19.

[15] S. Nobakhtian, Optimality and duality for nonsmooth multiobjective fractional programming with mixed constraints, J. Global Optim. 41 (2008) 103–115.

[16] R. T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, NJ, 1970.

[17] M. Soleimani-damaneh, Optimality for nonsmooth fractional multiple objective programming, Nonlinear Anal. 68 (2008), no. 10, 2873–2878.

[18] P. Wolfe, A duality theorem for nonlinear programming, Quarterly of Applied Mathematics 19 (1961), 239–244.