Some use of weak topologies in the KKM theory

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Abstract

Since the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply KKM theorem) appeared in 1929, a large number of its generalizations and modifications followed. Many of them are stated using the so-called weak topology. In the present article, we show that such KKM type theorems are consequences of one of our previous KKM theorems for abstract convex spaces. We also add some examples of papers adopting weak topologies.

1. Introduction

Since the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply KKM theorem) appeared in 1929 [10], a large number of its generalizations and modifications followed. Many of them are stated using the so-called weak topology. In the present article, we show that such KKM type theorems are consequences of one of our previous KKM theorems for abstract convex spaces. We also add some examples of papers adopting weak topologies.

Our aim in this article is two folds: First, we show examples of the usage of weak topologies in various KKM type theorems and others. Second, many of such KKM type theorems are consequences of one of our KKM theorems given in [20].

Section 2 deals with one of our generalized KKM theorem on our abstract convex spaces [20]. In Section 3, for a collection $\mathfrak{F}$ of nonempty subsets of a topological space X, we say X is $\mathfrak{F}$-generated if it has a weak topology coherent with the collection. We add several examples of particular types of $\mathfrak{F}$-generated spaces. Section 4 deals with some examples of articles on various KKM type theorems and others adopting weak topologies in the chronological order. We state key results in most of articles with some comments.

For some related history, see [17].

2. A KKM theorem on abstract convex spaces

For the concept of our abstract convex spaces $(E, D; \Gamma)$, see [18-23]. Let $\langle D\rangle$ be the class of all nonempty finite subsets of a set D.

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Consider the following related four conditions for a map $G : D \multimap Z$ to a topological space Z:

(a) $\bigcap_{y\in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y\in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y\in D} \overline{G(y)} = \overline{\bigcap_{y\in D} G(y)}$ (G is *intersectionally closed-valued* [13]).

(c) $\bigcap_{y\in D} \overline{G(y)} = \bigcap_{y\in D} G(y)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

In [13], Luc et al. noted that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d), and gave examples of multimaps satisfying (b) but not (c). According to Luc, the concept of (b) is due to Rockafellar in 1970.

The following KKM theorem is due to ourselves; see [18-23]:

Theorem A. *Let $(E, D; \Gamma)$ be a partial KKM space [resp. KKM space] and $G : D \multimap E$ a KKM map such that*

(1) *G is closed-valued [resp. open-valued].*

Then the family $\{G(z) \mid z \in D\}$ has the finite intersection property.

Moreover, suppose that

(2) *there exists a nonempty compact subset K of E such that one of the following holds:*

(i) *$K = E$;*

(ii) *$K = \bigcap\{\overline{G(z)} \mid z \in M\}$ for some $M \in \langle D\rangle$; or*

(iii) *for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and $\overline{L_N} \cap \bigcap_{z\in D'} \overline{G(z)} \subset K$.*

Then $K \cap \bigcap\{\overline{G(z)} \mid z \in D\} \neq \emptyset$.

Furthermore,

(α) *if G is transfer closed-valued, then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$;*

(β) *if G is intersectionally closed-valued, then $\bigcap\{G(z) \mid z \in D\} \neq \emptyset$.*

Cases (i) and (ii) are immediate routine consequences of the definition of partial KKM spaces. Theorem A implies the Fan matching property, some geometric property, the Fan-Browder fixed point property, some minimax inequality, several variational inequalities, von Neumann type minimax theorems, Nash equilibrium theorems, and many others. See [18] with some corrections in [22] and the references therein.

3. Weak extensions of a topology

We begin with the following new definition:

Definition. Let X be a topological space and $\mathfrak{F}$ be a collection of nonempty subsets of X (having a certain property $\mathcal{P}$, sometimes). Then X is said to be $\mathfrak{F}$-generated if it has a weak topology coherent with the collection $\mathfrak{F}$; i.e., if a subset A of X intersects each set $C \in \mathfrak{F}$ in a closed set, then A is closed.

Note that any topological space is $\mathfrak{F}$-generated when $\mathfrak{F}$ is the collection of all closed subsets, i.e., its topology.

Each of the following articles and many others are concerned with particular types of $\mathfrak{F}$-generated spaces.

Park 1968 [14]: $\mathfrak{F}$ is all closed subsets of a topological space X which possesses the *admissible* property $\mathcal{P}$, i.e., it is inherited by closed sets.

Park 1970 [15]: $\mathfrak{F}$ is a collection of closed subsets of a topological space X such that every relatively closed subset of an element of $\mathfrak{F}$ is also in $\mathfrak{F}$.

Brézis-Nirenberg-Stampacchia 1972 [2]: In a topological vector space E, $\mathfrak{F}$ is all finite dimensional subspaces.

Dugundji-Granas 1978 [3]: In a topological vector space E, $\mathfrak{F}$ is the class of all finite dimensional flats $L \subset E$ having the Euclidean topology.

Lassonde 1983 [12]: In a convex space X, $\mathfrak{F}$ is the class of convex hulls of its finite subsets with the Euclidean topology.

Lassonde 1983 [12]: In a k-space X, $\mathfrak{F}$ is the class of compact subsets of X.

Park-Kim 1987 [25]: In a t.v.s. X, $\mathfrak{F}$ is the class of finite dimensional subsets of X.

Khamsi 1996 [9]: In a metric space H, $\mathfrak{F}$ is the family of intersections of closed balls containing finite subsets.

Isac-Yuan 1999 [8], Yuan 1999 [28]: In a metric space H, $\mathfrak{F}$ is the family of intersections of closed balls.

Kirk-Sims-Yuan 2000 [11]: In a metric space H, $\mathfrak{F}$ is the family of intersections of closed balls.

4. Examples of various KKM type theorems and others

In this section, we list some articles on various KKM type theorems and others adopting weak topologies in the chronological order. In most cases, we state key results in each article with some comments.

The numbers attached to Definitions or Theorems are the ones given in the original source.

Knaster-Kuratowski-Mazurkiewicz 1929 — FM 14 [10]

Knaster, Kuratowski, and Mazurkiewicz [10] obtained the following so-called KKM theorem from the Sperner lemma:

Theorem. *Let A_i $(0 \leq i \leq n)$ be $n+1$ closed subsets of an n-simplex $p_0p_1\cdots p_n$. If the inclusion relation*

$$p_{i_0}p_{i_1}\cdots p_{i_k} \subset A_{i_0} \cup A_{i_1} \cup \cdots \cup A_{i_k}$$

holds for all faces $p_{i_0}p_{i_1}\cdots p_{i_k}$ $(0 \leq k \leq n,\ 0 \leq i_0 < i_1 < \cdots < i_k \leq n)$, then $\bigcap_{i=0}^{n} A_i \neq \emptyset$.

Comments: It is known later that all A_i can be also open sets; see [17]. Let Δ_n be a standard n-simplex, V the set of its vertices, and $\Gamma = \mathrm{co}$ is the convex hull operator.

Then $(\Delta_n, V; \Gamma)$ is a KKM space and Δ_n is $\mathfrak{F}$-generated for $\mathfrak{F} = \{\Delta_n\}$. Therefore, the KKM theorem follows from Theorem A(i).

Fan 1961 — Math. Ann. 142 [4]

Ky Fan [4] extended the KKM theorem to infinite dimensional spaces as follows, and applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a compact convex set into a uniform space.

Lemma. *Let X be an arbitrary set in a Hausdorff topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *The convex hull of any finite subset $\{x_1, x_2, \cdots, x_n\}$ of X is contained in $\bigcup_{i=1}^n F(x_i)$.*

(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

Comments: This is usually known as the KKMF theorem. Later it was known that the Hausdorffness is redundant by Lassonde and that Lemma has numerous applications; see [18, 21].

Note that $(Y, X; \mathrm{co})$ is a KKM space and Y is $\mathfrak{F}$-generated for $\mathfrak{F} = \{Y\}$. Therefore, the KKMF theorem follows from Theorem A(ii).

Park 1968 — JKMS 5 [14]

In [14] we generalized the concepts of compactly generated spaces (or Hausdorff k-spaces) and reflexive compact mappings. Using these concepts we obtain sufficient conditions for mappings to generate upper-semicontinuous (u.s.c.) decompositions of certain types of spaces with coherent topologies. We also show that some of the results generalize similar situations which are given previously.

Throughout [14], a property $\mathcal{P}$ is said to be *admissible* if it is inherited by closed sets.

Definition 3.1. Let X be a topological space and $\mathcal{P}$ an admissible property. A $\mathcal{P}$-set in X is a closed subset of X which possesses the property $\mathcal{P}$. X is said to be $\mathcal{P}$-generated if it has a topology coherent with the collection of its $\mathcal{P}$-sets; i.e., if a subset A of X intersects each $\mathcal{P}$-set in a closed set, then A is closed.

Comments: $\mathcal{P}$-generated spaces are $\mathfrak{F}$-generated when $\mathfrak{F}$ is the collection of all $\mathcal{P}$-sets.

Park 1970 — JKMS 7 [15]

In [15], we first consider some properties of spaces with weak topologies with respect to appropriate families of subspaces and weak topologies finer than given topologies, and show that the collection of such weak extensions of a topology forms a complete lattice.

An *admissible family* $\mathcal{A}$ of a topological space X is a collection of closed subsets of X such that every relatively closed subset of an element of $\mathcal{A}$ is also in $\mathcal{A}$. The elements of $\mathcal{A}$ is called $\mathcal{A}$-sets. The space X is called *$\mathcal{A}$-generated* if and only if the original topology of X is the weak topology with respect to $\mathcal{A}$. In the literature this topology is often called the coherent topology with $\mathcal{A}$ [26].

Let $(X, \mathcal{T})$ be a topological space and $\mathcal{A}$ an admissible family of $(X, \mathcal{T})$. The *weak extension* of $\mathcal{T}$ with respect to $\mathcal{A}$ or simply *$\mathcal{A}$-extension* of $\mathcal{T}$ is defined to be the family $\mathcal{T}(\mathcal{A})$ of all subsets U of X such that, for every S in $\mathcal{A}$, $U \cap S$ is open in S. Equivalently, C is $\mathcal{T}(\mathcal{A})$-closed if and only if $C \cap S$ is closed in S.

Comments: The paper [15] gives particular examples of $\mathfrak{F}$-generated spaces.

Brézis-Nirenberg-Stampacchia 1972 — Boll.U.M.I. (4)6 [2]

From the text: This paper [2] is based on a lemma which generalizes a finite dimensional result of Knaster-Kuratowski-Mazurkiewicz [10]. The following is slightly more general than the 1961 KKM Lemma due to Ky Fan [4].

Lemma 1. *Let X be an arbitrary set in a Hausdorff topological vector space E, and $F : X \multimap E$ a map satisfying*

(a) $\overline{F(x_0)} = L$ *is compact for some* $x_0 \in X$.

(b) $\mathrm{co}A \subset F(A)$ *for each* $A \in \langle X \rangle$.

(c) *For every $x \in X$, the intersection of $F(x)$ with any finite dimensional subspace is closed.*

(d) *For every convex subset D of E we have*

$$\overline{\bigcap_{x \in X \cap D} F(x)} \cap D = \bigcap_{x \in X \cap D} F(x) \cap D.$$

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

Comments: Other results in [2] are consequences of Lemma 1. In Lemma 1, the closure operation on E is given with respect to its original topology, and (c) is for $\mathfrak{F}$-generated one where $\mathfrak{F}$ is the family of finite dimensional subspaces.

However, we can prove Lemma 1 without assuming Hausdorffness of E and (c). Moreover we can replace (d) by the following particular case for $D = E$:

(d') (transfer closednes) $\overline{\bigcap_{x \in X} F(x)} = \bigcap_{x \in X} F(x)$.

Proof of Lemma 1: Recall that $(E, X; \mathrm{co})$ is a KKM space and note that $\overline{F} : X \multimap E$ is a closed-valued KKM map by (b). Then $\{\overline{F}(x) \mid x \in X\}$ has the finite intersection property by the first part of Theorem A. Moreover, $K := L = \overline{F(x_0)}$ is compact. Hence, by Theorem A(ii), we have $\bigcap\{\overline{F}(x) \mid x \in X\} \neq \emptyset$. Moreover, by (d'), F is intersectionally closed-valued. Therefore, by Theorem A for the case (β), we have $\bigcap_{x \in X} F(x) \neq \emptyset$. This completes our proof.

Lassonde 1983 — JMAA 97 [12]

From the text:

Definition 2. A *convex space X* is a convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets.

Theorem 000. *Let D be any subset of a convex space X and $G : D \to 2^X$ a KKM multifunction with closed values. If $G(x)$ is compact for at least one $x \in D$, then* $\bigcap\{G(x) \mid x \in D\} \neq \emptyset$.

Definition 3. Let X be a convex space. A nonempty set $K \subset X$ is called a *c-compact set* if for each finite subset $\mathcal{F} \subset X$ there is a compact convex set $K_{\mathcal{F}} \subset X$ such that $K \cup \mathcal{F} \subset K_{\mathcal{F}}$.

Definition 4. Let Y be a topological space. A set $B \subset Y$ is said to be *compactly closed* (*open*, respectively) in Y if for every compact set $L \subset Y$ the set $B \cap L$ is closed (open, respectively) in L.

Theorem I. *Let D be an arbitrary set in a convex space X, Y any topological space, and $F : D \multimap 2^Y$ a multifunction having the following properties*

(i) *For each $x \in D$, $F(x)$ is compactly closed in Y.*

(ii) *For some continuous map $s : X \to Y$, the multifunction $G : D \multimap 2^X$ given by $G(x) = s^{-1}(F(x))$ is KKM.*

(iii) *For some c-compact set $K \subset X$, $\bigcap\{F(x) \mid x \in K \cap D\}$ is compact.*

Then $\bigcap\{F(x) \mid x \in D\} \neq \emptyset$.

Comments: Every convex space is a KKM space. In Theorem I, we may assume Y is a k-space. Note that condition (iii) of Theorem I is a particular case of (iii) in Theorem A. Consequently, Theorem I follows from Theorem A.

Fan 1984 — Math. Ann. 266 [6]

Fan [5,6] introduced a KKM theorem with a more general coercivity (or compactness) condition for noncompact convex sets as follows.

The 1984 KKM Theorem. [6] *In a Hausdorff topological vector space, let Y be a convex set and $\emptyset \neq X \subset Y$. For each $x \in X$, let $F(x)$ be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^n F(x_i)$. If there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x \in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y, then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

Comments: This was first introduced in 1979 [5] without proof and was proved in 1984 [6] via an equivalent matching theorem for open covers of convex sets. The 1984 theorem follows from Theorem A. See also Park [20].

Park-Kim 1987 — JKMS 24 [25]

From the text: We introduce more general closedness conditions (in KKM type theorems), which are relative versions of Dugundji-Granas [3] and Lassonde [12].

Definition. Let Y be a nonempty subset of a topological vector space E. A set $X \subset Y$ is called a *finitely relatively closed* subset of Y if the intersection of X with any finite dimensional subspace F of E is a relatively closed subset of $Y \cap F$. A set $X \subset Y$ is called a *compactly relatively closed* subset of Y if the intersection of X with any compact subset K of E is a relatively closed subset of $Y \cap K$.

Note that every finitely closed subset of E is necessarily finitely relatively closed, and every compactly closed subset of E is also compactly relatively closed. Moreover, every relatively closed subset is also finitely relatively closed and compactly relatively closed.

Note that if Y is closed, then the relative versions of Definition are equivalent to the corresponding ones in [12].

Lemma. *Let Y be a convex subset of a Hausdorff topological vector space E, and $\emptyset \neq X \subset Y$. Let $T : X \to 2^E$ be a KKM-map such that each $T(x)$ is a relatively closed subset of Y. Furthermore, assume that there exists a nonempty subset $X_0 \subset X$, contained in some precompact convex subset Y_0 of Y, such that $\overline{\bigcap_{x \in X_0} T(x)}$ is a compact subset of Y. Then $\bigcap_{x \in X} T(x) \neq \emptyset$.*

Theorem 1. *Let Y be a convex subset of a Hausdorff topological vector space E, and $\emptyset \neq X \subset Y$. Let $T : X \multimap E$ be a KKM-map such that each Tx is finite relatively closed subset of Y. Furthermore, assume the following:*

(1) *There exists a nonempty finite dimensional set $X_0 \subset X$, contained in some compact convex subset of Y, such that $\overline{\bigcap_{x \in X_0} Tx}$ is a compact subset of Y.*

(2) *For every line segment L of E we have*

$$\overline{\bigcap_{x \in X \cap L} Tx} \cap L = \bigcap_{x \in X \cap L} Tx \cap L.$$

Then $\bigcap_{x \in X} Tx \neq \emptyset$.

Comments: The above definitions are theoretically possible, but seems to be not practical. All results in this paper is based on Lemma, whose proof is based on the 1961 KKM Lemma of Ky Fan. Consequently, all results in this article follow from Theorem A.

Khamsi 1996 — JMAA 204 [9]

From the text: In hyperconvex metric spaces, we introduce KKM mappings. Then we prove an analogue to Ky Fan's fixed point theorem in hyperconvex metric spaces.

The following is due to Aronszajn and Panitchpakdi [1]:

Definition 1. A metric space (M, d) is said to be *hyperconvex* if $\bigcap_\alpha B(x_\alpha, r_\alpha) \neq \emptyset$ for any collection $\{B(x_\alpha, r_\alpha)\}$ of closed balls in M for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$.

Here we use $B(x, r)$ for the closed ball about $x \in M$ and of radius $r > 0$.

Definition 2. Let (M, d) be a metric space and $A \subset M$ a nonempty bounded subset. Set:

$\mathrm{BI}(A) = \bigcap\{B \mid B$ is a closed ball such that $A \subset B\}$;

$\mathcal{A}(M) = \{A \subset M \mid A = \mathrm{BI}(A)\}$, i.e., $A \in \mathcal{A}(M)$ iff A is an intersection of balls. In this case we will say A is an *admissible* subset of M.

A subset A of a metric space M is called *finitely closed* if for every $x_1, x_2, \cdots, x_n \in M$ the set $\mathrm{BI}\{x_i\} \cap A$ is closed.

Definition 3. Let H be a metric space and $X \subset H$. A multivalued mapping $G : X \to 2^H$ is called a KKM-map if $\mathrm{BI}\{x_1, \ldots, x_n\} \subset G\{x_1, \ldots, x_n\}$ for any $x_1, \ldots, x_n \in X$.

Theorem 3. (KKM-Map Principle) *Let H be a hyperconvex metric space, X an arbitrary subset of H, and $G : X \to 2^H$ a KKM-map such that each $G(x)$ is finitely closed. Then the family $\{G(x) \mid x \in X\}$ has the finite intersection property.*

Theorem 4. *Let H be a hyperconvex metric space and $X \subset H$ an arbitrary subset. Let $G : X \multimap H$ be a KKM map such that $G(x)$ is closed for any $x \in X$ and $G(x_0)$ is compact for some $x_0 \in X$. Then we have*

$$\bigcap_{x \in X} G(x) \neq \emptyset.$$

Comments: It is known that a normed vector space E is not hyperconvex in general, and the spaces $(\mathbb{R}^n, ||\cdot||_\infty)$, l^∞, L^∞ and $\mathbb{R}$-trees are hyperconvex.

Since each admissible subset of a hyperconvex metric space is hyperconvex and hence contractible, the following is due to Horvath [7]:

Lemma 1. *Any hyperconvex metric space H is a c-space $(H;\Gamma)$, where Γ_A=BI(A) for each $A \in \langle H \rangle$.*

From Lemma 1 and our KKM theory, we have the following:

Lemma 2. *Every hyperconvex metric space is a KKM metric space, that is, a metric space satisfying the KKM principle.*

Based on the partial KKM principle (Theorem 3) on hyperconvex metric spaces, Khamsi obtained a KKM theorem (Theorem 4), a Fan type best approximation lemma, and a Fan type fixed point theorem for such spaces. Here the basic KKM theorem is a particular form of Theorem A for hyperconvex metric spaces.

Yuan 1999 — JMAA 235 [28]

Abstract: In this note, the characterization for a set-valued mapping with finitely metrically open values being a generalized metric KKM mapping in hyperconvex metric spaces is established. This result could be regarded as a dual form of corresponding results for the Fan-KKM principle in hyperconvex metric spaces obtained recently by Khamsi [9] and Kirk-Sims-Yuan [11]. Then we show that the finite intersection property of generalized metric KKM mappings with finitely metrically open values indeed is equivalent to the finite intersection property of generalized metric KKM mappings with finitely metrically closed values in hyperconvex metric spaces. As applications, we first obtain Ky Fan type matching theorems for both closed and open covers in hyperconvex metric spaces, which, in turn, are used to establish fixed point theorems for set-valued mappings in hyperconvex metric spaces.

From the text: The following are due to Khamsi [9] and Yuan [28]:

Definition. Let (M, d) be a metric space. A subset $S \subset M$ is said to be *finitely metrically closed* [resp. *finitely metrically open*] if for each $F \in \mathcal{A}(M)$, the set $F \cap S = \mathrm{BI}(F) \cap S$ is closed [resp. open]. Note that BI(F) is always defined and belongs to $\mathcal{A}(M)$. Thus if S is closed [resp. open] in M, it is obviously finitely metrically closed [resp. open].

Theorem 5. (KKM-Map Principle) [9] *Let H be a hyperconvex metric space, X an arbitrary subset of H, and $G : X \multimap H$ a KKM map such that each $G(x)$ is finitely metrically closed [resp. finitely metrically open]. Then the family $\{G(x) : x \in X\}$ has the finite intersection property.*

Comments: This paper [28] begins with a characterization for a generalized metric KKM map in hyperconvex metric spaces established by Kirk-Sims-Yuan [11] and its open-valued version. This characterization is extended to KKM spaces by the present author. Recall that partial KKM principle implies the KKM principle for hyperconvex metric spaces, that is, hyperconvex metric spaces with finitely metric topology are simply KKM spaces. Therefore, most results in this paper are simple consequences of the corresponding known ones for KKM spaces, for example, in Park [18]. Moreover, the author sometimes assumes superfluous restrictions and some of his proofs are unnecessarily lengthy and complicated.

Theorem 5 shows that any hyperconvex metric space having *finitely metric topology* is a KKM space and follows from Theorem A. Hence such space satisfies all results in [18].

Note that H in [9, Theorem 4] can have the finitely metric topology. Therefore it is natural, but not practical, to assume that *every hyperconvex metric space has the finitely metric topology.* This assumption simplifies the texts of [9, 11, 27].

Isac-Yuan 1999 — Discuss. Math. 19 [8]

Abstract: We first establish the dual form of KKM principle which is a hyperconvex version of corresponding result due to Shih. Then Ky Fan type matching theorem for finitely closed and open covers are given. As applications we establish some intersection theorems which are hyperconvex version of of corresponding results due to Alexandroff and Pasynkoff, Fan, Klee, Horvath and Lassonde. Then Ky Fan type best approximation theorem and Schauder-Tychonoff fixed point theorem for set-valued maps (i.e., Fan-Glicksberg fixed point theorem)in hyperconvex spaces are also developed, and finally one unified form of Browder-Fan fixed point theorem for set-valued maps in hyperconvex spaces is given. These results include corresponding results in the literature due to Khamsi, Kirk and Shin, Kirk et al. as special cases.

Comments: The authors establish the dual (open-valued) form of the KKM principle, which is a hyperconvex version of a corresponding earlier result due to M. H. Shih. Some related results are also obtained.

Kirk-Sims-Yuan 2000 — NA 39 [11]

From the text: In [11], the authors first establish a characterization of the KKM principle in hyperconvex metric spaces which in turn leads to a characterization theorem for a family of subsets with the finite intersection property in such a setting. As applications we give hyperconvex versions of Fan's celebrated minimax principle and Fan's best approximation theorem for set-valued mappings. These in turn are applied to obtain formulations of the Browder-Fan fixed point theorem and the Schauder - Tychonoff fixed point theorem in hyperconvex metric spaces for set-valued mappings. Finally, existence theorems for saddle points, intersection theorems and Nash equilibria are also obtained. Our results unify and extend several of the results cited above.

Definition 2.1. Let X be any nonempty set and let M be a metric space. A set-valued mapping $G : X \to 2^M \setminus \{\emptyset\}$ is said to be a *generalized metric KKM mapping* (GMKKM) if for each nonempty finite set $\{x_1, \ldots, x_n\} \subset X$, there exists a set $\{y_1, \ldots, y_n\}$ of points of M, not necessarily all different, such that for each subset $\{y_{i_1}, \ldots, y_{i_k}\}$ of $\{y_1, \ldots, y_n\}$, we have

$$\mathrm{BI}\{y_{i_j} \mid j = 1, \ldots, k\} \subset \bigcup_{j=1}^{k} G(x_{i_j}).$$

As a special case of a generalized metric KKM mapping, we have the following definition of KKM mappings given essentially by Khamsi in [9].

Definition 2.2. Let X be a nonempty subset of a metric space M. Suppose $G : X \to 2^M$ is a set-valued mapping with nonempty values. Then G is said to be a *metric KKM* (MKKM) mapping if for each finite subset $A \in \langle X \rangle$, $\mathrm{BI}(A) \subset G(A)$.

Theorem 2.1. *Let X be a nonempty set and let M be a hyperconvex metric space. Suppose $G : X \to 2^M \setminus \{\emptyset\}$ has finitely metrically closed values. Then the family $\{G(x) \mid x \in X\}$ has the finite intersection property if and only if the mapping G is a generalized metric KKM mapping.*

Theorem 2.2. *Let X be a non-empty set and M be a hyperconvex metric space. Suppose $G : X \to 2^M \setminus \{\emptyset\}$ is a set-valued mapping with nonempty closed values and suppose there exists $x_0 \in X$ such that $G(x_0)$ is compact. Then $\bigcap_{x \in X} G(x) \neq \emptyset$ if and only if the mapping G is a generalized metric KKM mapping.*

Comments: Note that $(M, X; \Gamma)$ with $\Gamma : \langle X \rangle \multimap M$ and $\Gamma_A := \mathrm{BI}(A)$ for each $A \in \langle X \rangle$ is an H-space (since each Γ_A is contractible).

In Definition 2.1, $(M, A; \Gamma)$ with $A := \{x_1, \ldots, x_n\}$ and $\Gamma : \langle A \rangle \multimap M$ such that $\Gamma\{x_{i_j} \mid j = 1, \ldots, k\} := \mathrm{BI}\{y_{i_j} \mid j = 1, \ldots, k\} \subset M$ is an H-space. Therefore, a GMKKM map $G : A \multimap M$ simply reduces to a MKKM map on $(M, A; \Gamma)$.

The authors stated a characterization, Theorem 2.1, for a generalized metric KKM map with closed values on hyperconvex metric spaces. In Theorem 2.1, the hyperconvex metric space with finitely metric topology is a (partial) KKM space. Note that the necessity of Theorem 2.1 is trivial. Hence Theorem 2.1 follows from our KKM theorem A.

Theorem 2.2 is a Fan type KKM theorem for a generalized metric KKM map on hyperconvex metric KKM maps on hyperconvex metric spaces. Some variants of Theorem 2.2 are added. Since any hyperconvex metric space is an H-space, the sufficiency of Theorem 2.2 follows from Theorem A. The necessity is trivial and well-known for G-convex spaces.

Note that all of the other results in [11] follow from Theorems 2.1 and 2.2. From these results, the authors follow the routine way in the KKM theory to establish a minimax inequality, a best approximation theorem, a Fan-Browder fixed point theorem, a maximal element theorem, a Fan type geometric property, and a Shauder-Tychonoff type fixed point property. The authors add non-compact versions of some of the fore-mentioned theorems, and applications to saddle points and Nash equilibria. The essence of such development is the abstract approach we have shown in [18].

We already noted that every hyperconvex metric space can be assumed to have the finitely metric topology. Therefore, the expressions like 'metric KKM', 'finitely metrically,' etc. can be eliminated in Yuan [28], Kirk et al. [11], and Tarafdar-Yuan [27].

Park 2011 — NA 7 [19]

Abstract: In the KKM theory, some authors adopt the concepts of the compact closure (ccl), compact interior (cint), transfer compactly closed-valued multimap, transfer compactly l.s.c. multimap, and transfer compactly local intersection property, respectively, instead of the closure, interior, closed-valued multimap, l.s.c. multimap, and possession of a finite open cover property. In [19], we show that such adoption is inappropriate and artificial. In fact, any theorem with a term with "transfer" attached is equivalent to the corresponding one without "transfer". Moreover, we can invalidate terms with "compactly" attached by giving a finer topology on the underlying space. In such ways, we obtain simpler formulations of KKM type theorems, Fan-Browder type fixed point theorems, and other results in the KKM theory on abstract convex spaces.

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