STRONG CONVERGENCE THEOREMS FOR ACCRETIVE OPERATORS AND NONEXPANSIVE MAPPINGS IN BANACH SPACES

JONG SOO JUNG

DEPARTMENT OF MATHEMATICS, DONG-A UNIVERSITY

ABSTRACT. In this paper, we introduce two new iterative algorithms (one implicit and one explicit) for finding a common point of the set of zeros of an accretive operator and the set of fixed points of a nonexpansive mapping in a real uniformly convex Banach space having a uniformly Gâteaux differentiable norm. Then under suitable control conditions, we establish strong convergence of sequence generated by proposed algorithm to a common point of above two sets, which is a solution of a ceratin variational inequality. The main theorems develop and complement some well-known results in the literature.

1. INTRODUCTION

Let E be a real Banach space with norm $\|\cdot\|$ and the dual space E^*. The value of $x^*\in E^*$ at $y\in E$ is denoted by $\langle y,x^*\rangle$ and the normalized duality mapping $\mathcal{J}$ from E into 2^{E^*} is defined by

$$\mathcal{J}(x)=\{x^*\in E^*:\langle x,x^*\rangle=\|x\|\|x^*\|,\ \|x\|=\|x^*\|\},\quad \forall x\in E.$$

Recall that a (possibly multivalued) operator $A\subset E\times E$ with the domain $D(A)$ and the range $R(A)$ in E is *accretive* if , for each $x_i\in D(A)$ and $y_i\in Ax_i$ $(i=1,\ 2)$, there exists a $j\in\mathcal{J}(x_1-x_2)$ such that $\langle y_1-y_2,j\rangle\geq 0$. (Here $\mathcal{J}$ is the normalized duality mapping.) In a Hilbert space, an accretive operator is also called monotone operator.

Interest in accretive operators stems mainly from their firm connection with evolution equations. It is well-known that many physically significant problems can be modeled by initial-value problems of the form

$$\frac{dx(t)}{dt}+Ax(t)\ni 0,\quad x(0)=x_0, \tag{1.1}$$

where A is an an accretive operator in a ceratin Banach space. Typical examples where such evolution equations occurs can be found in the heat, wave, or Schrodinger equations. If in (1.1), $x(t)$ is independent of t, then (1.1) reduces $Az\ni 0$ whose solutions correspond to the equilibrium points of system (1.1). Consequently, the iterative algorithms of Halpern type, Mann type, and Rockafellar type have extensively been studied over the last forty years for constructions of zeros of accretive operators (see, e,g., [2, 3, 4, 8, 9, 10, 12, 13, 14, 15, 16, 18, 20, 21, 22, 27, 28] and the references therein). As an original one, the following iterative algorithm in Hilbert spaces or Banach spaces was considered by many authors: for resolvent J_{r_n} of m-accretive operator A,

$$x_{n+1}=J_{r_n}x_n,\quad \forall n\geq 0,$$

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The results presented in this lecture are collected mainly from the work [11] by the author of this report.

where the initial guess $x_0 \in E$ is chosen arbitrarily (see, e.g., [12, 13, 18] and the references therein). In particular, in order to find a zero of a monotone operator A, Rockafellar [20] introduced a powerful and successful algorithm which is recognized as Rockafellar proximal point algorithm in Hilbert space H : for any initial point $x_0 \in H$, a sequence $\{x_n\}$ is generated by

$$x_{n+1} = J_{r_n}(x_n + e_n), \quad \forall n \geq 0,$$

where $J_r = (I + rA)^{-1}$, for $r > 0$, is the resolvent of A and $\{e_n\}$ is an error sequence in H.

Xu [24] in 2006 and Song and Yang [23] in 2009 obtained the strong convergence of the regularization method for Rockafellar's proximal point algorithm in a Hilbert space H: for any initial point $x_0 \in H$

$$x_{n+1} = J_{r_n}(\alpha_n u + (1 - \alpha_n)x_n + e_n), \quad \forall n \geq 0,$$

where $\{\alpha_n\} \subset (0,1)$, $\{e_n\} \subset H$ and $\{r_n\} \subset (0, \infty)$.

On the other hand, in 2011, He *et al.* [6] studied the following iterative algorithm for finding a common point of the set of zeros of accretive operator A such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$ and the set of fixed points of a nonexpansive mapping S in a real reflexive Banach space E having a weakly sequentially continuous duality mapping:

$$\begin{cases} x_0 = x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S J_{r_n} x_n, \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset [0,1]$, $\lim_{n\to\infty} r_n = r$ and $f : C \to C$ is a contractive mapping. Under the suitable conditions $\{\alpha_n\}$ and $\{\beta_n\}$, they also showed that the sequence $\{x_n\}$ generated by (1.2) converges strongly to a common point in $F(S) \cap A^{-1}0$, which is a solution of a certain variational inequality.

Inspired and motivated by the above-mentioned results, in this paper, we introduce new implicit and explicit algorithms for finding a common point of the set of zeros of accretive operator A and the set of fixed points of a nonexpansive mapping S in a real uniformly convex Banach space E having a uniformly Gâteaux differentiable norm. Under suitable control conditions, we prove that the sequence generated by proposed iterative algorithm converge strongly to a common point in $A^{-1}0 \cap F(S)$, which is a solution of a certain variational inequality. The main results develop and supplement the corresponding results of He *et al.* [6] as well as Xu [24] and Song and Yang [23] and the reference therein.

2. Preliminaries and Lemmas

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. Let C be a nonempty subset of E. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f \rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ ($x_n \rightharpoonup x$) will denote strong (weak) convergence of the sequence $\{x_n\}$ to x. For the mapping $S : C \to C$, $F(S)$ will denote the set of fixed point of S; that is, $F(S) = \{x \in C : Sx = x\}$.

A Banach space E is said to be *uniformly convex* if for all $\varepsilon \in [0,2]$, there exists $\delta_\varepsilon > 0$ such that

$$\|x\| = \|y\| = 1 \text{ implies } \frac{\|x + y\|}{2} < 1 - \delta_\varepsilon \text{ whenever } \|x - y\| \geq \varepsilon.$$

Let $l > 1$ and $M > 0$ be two fixed real numbers. Then a Banach space is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^l \leq \lambda\|x\|^l + (1 - \lambda)\|y\|^l - \omega(\lambda)g(\|x - y\|), \tag{2.1}$$

for all $x,\ y \in B_M(0) = \{x \in E : \|x\| \leq M\}$, where $\omega(\lambda) = \lambda^l(1 - \lambda) + \lambda(1 - \lambda)^l$. For more detail, see Xu [25].

The norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t} \tag{2.2}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. Such an E is said to be *smooth* Banach space. The norm is said to be *uniformly Gâteaux differentiable* if for $y \in U$, the limit is attained uniformly for $x \in U$. The space E is said to have a *uniformly Fréchet differentiable norm* (and E is said to be *uniformly smooth*) if the limit in (2.2) is attained uniformly for $(x, y) \in U \times U$. It is known that E is smooth if and only if the normalized duality mapping $\mathcal{J}$ is single-valued. Also, it is well-known that if E has a uniformly Gâteaux differentiable norm, $\mathcal{J}$ is norm to weak* uniformly continuous on each bounded subsets of E. The following property of the normalized duality mapping $\mathcal{J}$ is well-known: $\mathcal{J}(-x) = -\mathcal{J}(x)$ for all $x \in E$ ([1]).

An accretive operator A is said to satisfy *the range condition* if $\overline{D(A)} \subset R(I+rA)$ for all $r > 0$, where I is an identity operator of E and $\overline{D(A)}$ denotes the closure of the domain $D(A)$ of A. An accretive operator A is called *m-accretive* if $R(I+rA) = E$ for each $r > 0$. If A is an accretive operator which satisfies the range condition, then we can define, for each $r > 0$ a mapping $J_r : R(I+rA) \to D(A)$ defined by $J_r = (I+rA)^{-1}$, which is called the *resolvent* of A. We know that J_r is nonexpansive (i.e., $\|J_rx - J_ry\| \leq \|x - y\|$, $\forall x, y \in R(I+rA)$) and $A^{-1}0 = F(J_r) = \{x \in D(J_r) : J_rx = x\}$ for all $r > 0$. For these facts, see [1].

We need the following lemmas for the proof of our main results. We refer to [1] for Lemma 2.1, Lemma 2.2, and Lemma 2.3.

Lemma 2.1. *If E be a real smooth Banach space, then one has*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, \mathcal{J}(x+y)\rangle, \quad \forall x,\ y \in E,$$

where $\mathcal{J}$ is the normalized duality mapping of E.

Lemma 2.2 (The Resolvent Identity). *For $\lambda > 0$, $\mu > 0$ and $x \in E$,*

$$J_\lambda x = J_\mu\Big(\frac{\mu}{\lambda} + \Big(1 - \frac{\mu}{\lambda}\Big)J_\lambda x\Big).$$

Lemma 2.3. *Let E be a real Banach space having a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E, and let $\{y_n\}$ be a bounded sequence in E. Let* LIM *be a Banach limit and $q \in C$. Then*

$$\mathrm{LIM}\|y_n - q\|^2 = \min_{x\in C} \mathrm{LIM}\|x_n - x\|^2$$

if and only if

$$\mathrm{LIM}\langle x - q, \mathcal{J}(y_n - q)\rangle \leq 0, \quad \forall x \in C,$$

where $\mathcal{J}$ is the normalized duality mapping of E.

The following lemma is given in [26].

Lemma 2.4 ([26]). *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n\delta_n + \gamma_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \lambda_n|\delta_n| < \infty$;
(iii) $\gamma_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Finally, we will use the next lemma which is of fundamental importance for our proof.

Lemma 2.5 ([17]). *Let $\{s_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that $s_{n_i} < s_{n_i+1}$ for all $i \ge 0$. For every $n \ge n_0$, define the sequence of integers $\{\tau(n)\}$ by*

$$\tau(n) := \max\{k \le n : s_k < s_{k+1}\}.$$

Then $\{\tau(n)\}_{n\ge n_0}$ is a nondecreasing sequence verifying

$$\lim_{n\to\infty} \tau(n) = \infty,$$

and, for all $n \ge n_0$, the following two estimates hold:

$$s_{\tau(n)} \le s_{\tau(n)+1}, \quad s_n \le s_{\tau(n)+1}.$$

3. Main results

Throughout the rest of this paper, we always assume the following:

- E is a real Banach space;
- $\mathcal{J}$ is the normalized duality mapping of E;
- C is a nonempty closed convex subset of E;
- $A \subset E \times E$ is an accretive operator in E such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$;
- J_r is the resolvent of A for each $r > 0$;
- $S : C \to C$ is a nonexpansive mapping with $F(S) \cap A^{-1}0 \neq \emptyset$;
- $f : C \to C$ is a contractive mapping with a constant $k \in (0, 1)$.

In this section, we introduce the following algorithm that generates a net $\{x_t\}_{t\in(0,1)}$ in an implicit way:

$$x_t = J_r(tfx_t + (1-t)Sx_t). \tag{3.1}$$

We prove strong convergence of $\{x_t\}$ as $t \to 0$ to a point q in $A^{-1}0 \cap F(S)$ which is a solution of the following variational inequality:

$$\langle (I - f)q, \mathcal{J}(q - p)\rangle \ge 0, \quad \forall p \in A^{-1}0 \cap F(S). \tag{3.2}$$

We also propose the following algorithm which generates a sequence in an explicit way:

$$x_{n+1} = J_{r_n}(\alpha_n f x_n + (1-\alpha_n)Sx_n), \quad \forall n \ge 0,$$

where $\{\alpha_n\} \subset (0, 1)$, $\{r_n\} \subset (0, \infty)$ and $x_0 \in C$ is an arbitrary initial guess, and establish the strong convergence of this sequence to a point q in $A^{-1}0 \cap F(S)$, which is also a solution of the variational inequality (3.2).

3.1. Strong convergence of the implicit algorithm.

Now, for $t \in (0, 1)$, consider a mapping $Q_t : C \to C$ defined by

$$Q_t x = J_r(tfx + (1-t)Sx), \quad \forall x \in C.$$

It is easy to see that Q_t is a contractive mapping with constant $1 - (1-k)t$. Indeed, we have

$$\begin{aligned} \|Q_t x - Q_t y\| &\le t\|fx - fy\| + \|(1-t)Sx - (1-t)Sy\| \\ &\le tk\|x - y\| + (1-t)\|x - y\| \\ &= (1 - (1-k)t)\|x - y\|. \end{aligned}$$

Hence Q_t has a unique fixed point, denoted x_t, which uniquely solves the fixed point equation (3.1).

We summarize the basic properties of $\{x_t\}$ and $\{y_t\}$, where $y_t = tfx_t + (1-t)Sx_t$ for $t \in (0,1)$.

Proposition 3.1. *Let E be a uniformly convex Banach space. Let the net $\{x_t\}$ be defined by (3.1), and let $\{y_t\}$ be a net defined by $y_t = tfx_t + (1-t)Sx_t$ for $t \in (0,1)$. Then*

(1): *$\{x_t\}$ and $\{y_t\}$ are bounded for $t \in (0,1)$;*
(2): *x_t defines a continuous path from $(0,1)$ in C and so does y_t;*
(3): $\lim_{t\to 0} \|y_t - Sx_t\| = 0$;
(4): $\lim_{t\to 0} \|y_t - J_r y_t\| = 0$;
(5): $\lim_{t\to 0} \|x_t - y_t\| = 0$;
(6): $\lim_{t\to 0} \|y_t - Sy_t\| = 0$;

We establish the strong convergence of the net $\{x_t\}$ as $t \to 0$, which guarantees the existence of solutions of the variational inequality (3.2).

Theorem 3.2. Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm. Let $\{x_t\}$ be a net defined by (3.1), and let $\{y_t\}$ be a net defined by $y_t = tfx_t + (1-t)Sx_t$ for $t \in (0,1)$. Then the nets $\{x_t\}$ and $\{y_t\}$ converge strongly to a point $q \in A^{-1}0 \cap F(S)$ as $t \to 0$, which is the unique solution of the variational inequality (3.2).

Corollary 3.3. Let E be a uniformly convex and uniformly smooth Banach space. Let $\{x_t\}$ be a net defined by (3.1), and let $\{y_t\}$ be a net defined by $y_t = tfx_t + (1-t)Sx_t$ for $t \in (0,1)$. Then the nets $\{x_t\}$ and $\{y_t\}$ converge strongly to a point $q \in A^{-1}0 \cap F(S)$ as $t \to 0$, which is the unique solution of the variational inequality (3.2).

3.2. Strong convergence of the explicit algorithm. Now, using Theorem 3.2, we show the strong convergence of the sequence generated by the explicit algorithm (3.3) to a point $q \in A^{-1}0 \cap F(S)$, which is the unique solution of the variational inequality (3.2).

Theorem 3.4. *Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm. Let $\{\alpha_n\} \in (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy the conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C3) $|\alpha_{n+1} - \alpha_n| \le o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$ *(the perturbed control condition);*
(C4) $r_n \ge \varepsilon > 0$ *for* $n \ge 0$ *and* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Let $x_0 = x \in C$ be chosen arbitrarily, and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = J_{r_n}(\alpha_n fx_n + (1-\alpha_n)Sx_n), \quad \forall n \ge 0. \tag{3.3}$$

Let $\{y_n\}$ be a sequence defined by $y_n = \alpha_n fx_n + (1-\alpha_n)Sx_n$. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in A^{-1}0 \cap F(S)$, where q is the unique solution of the variational inequality (3.2).

Proof. First, we note that by Theorem 3.2, there exists the unique solution q of the variational inequality

$$\langle (I-f)q, \mathcal{J}(q-p)\rangle \le 0, \quad \forall p \in A^{-1}0 \cap F(S),$$

where $q = \lim_{t\to 0} x_t = \lim_{t\to 0} y_t$ being defined by $x_t = J_r(tfx_t + (1-t)Sx_t)$ and $y_t = tfx_t + (1-t)Sx_t$ for $0 < t < 1$, respectively.

We divide the proof into several steps.

Step 1. We show that $\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{1}{1-k}\|fp - p\|\}$ for all $n \ge 0$ and all $p \in A^{-1}0 \cap F(S)$, and so $\{x_n\}$, $\{y_n\}$, $\{J_{r_n}x_n\}$, $\{Sx_n\}$, $\{J_{r_n}y_n\}$, $\{Sy_n\}$ and $\{fx_n\}$ are bounded. Indeed, let $p \in A^{-1}0 \cap F(S)$. From $A^{-1}0 = F(J_r)$ for each $r > 0$, we know $p = Sp = J_{r_n}p$. Then we have

$$\begin{aligned}
\|x_{n+1} - p\| \le \|y_n - p\| &= \|\alpha_n(fx_n - p) + (1-\alpha_n)(Sx_n - Sp)\| \\
&\le \alpha_n\|fx_n - p\| + (1-\alpha_n)\|x_n - p\| \\
&\le \alpha_n(\|fx_n - fp\| + \|fp - p\|) + (1-\alpha_n)\|x_n - p\| \\
&\le \alpha_n k\|x_n - p\| + \alpha_n\|fp - p\| + (1-\alpha_n)\|x_n - p\| \\
&= (1-(1-k)\alpha_n)\|x_n - p\| + (1-k)\alpha_n\frac{\|fp - p\|}{1-k} \\
&\le \max\left\{\|x_n - p\|, \frac{1}{1-k}\|f(p) - p\|\right\}.
\end{aligned}$$

Using an induction, we obtain

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{1}{1-k}\|fp - p\|\right\}.$$

Hence $\{x_n\}$ is bounded. Also for $p \in A^{-1}0 \cap F(S)$, we get

$$\begin{aligned}
\|y_n - p\| &\le \alpha_n\|fx_n - fp\| + (1-\alpha_n)\|Sx_n - Sp\| + \alpha_n\|fp - p\| \\
&\le \alpha_n k\|x_n - p\| + (1-\alpha_n)\|x_n - p\| + \alpha_n\|fp - p\| \\
&= (1-(1-k)\alpha_n)\|x_n - p\| + (1-k)\alpha_n\frac{\|fp - p\|}{1-k} \\
&\le \max\left\{\|x_n - p\|, \frac{\|fp - p\|}{1-k}\right\},
\end{aligned}$$

and so $\{y_n\}$ is bounded, and so are $\{y_n\}$, $\{J_{r_n}y_n\}$, $\{Sx_n\}$, $\{Sy_n\}$ and $\{fx_n\}$. Moreover, it follows from condition (C1) that

$$\|y_n - Sx_n\| = \alpha_n\|fx_n - Sx_n\| \le \alpha_n(\|fx_n\| + \|Sx_n\|) \to 0 \quad (n \to \infty). \tag{3.4}$$

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. First, from Lemma 2.2 (Resolvent identity), we observe that

$$\begin{aligned}
\|J_{r_n}y_n - J_{r_{n-1}}y_{n-1}\| &= \left\|J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}y_n + \left(1 - \frac{r_{n-1}}{r_n}\right)J_{r_n}y_n\right) - J_{r_{n-1}}y_{n-1}\right\| \\
&\le \left\|\frac{r_{n-1}}{r_n}y_n + \left(1 - \frac{r_{n-1}}{r_n}\right)J_{r_n}y_n\right) - y_{n-1}\right\| \\
&\le \|y_n - y_{n-1}\| + \left|1 - \frac{r_{n-1}}{r_n}\right|(\|y_n - y_{n-1}\| + \|J_{r_n}y_n - y_{n-1}\|) \\
&\le \|y_n - y_{n-1}\| + \left|\frac{r_n - r_{n-1}}{\varepsilon}\right|M_1,
\end{aligned} \tag{3.5}$$

where $M_1 = \sup_{n\ge 0}\{\|J_{r_n}y_n - y_{n-1}\| + \|y_n - y_{n-1}\|\}$. Since

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)Sx_n, \\ y_{n-1} = \alpha_{n-1}f(x_{n-1}) + (1-\alpha_{n-1})Sx_{n-1}, \quad \forall n \ge 1, \end{cases}$$

by (3.5), we have for $n \geq 1$,

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|J_{r_n} y_n - J_{r_{n-1}} y_{n-1}\| \leq \|y_n - y_{n-1}\| + \left|\frac{r_n - r_{n-1}}{\varepsilon}\right| M_1 \\
&= \|(1-\alpha_n)(Sx_n - Sx_{n-1}) + \alpha_n(fx_n - fx_{n-1}) \\
&\quad + (\alpha_n - \alpha_{n-1})(fx_{n-1} - Sx_{n-1})\| + \left|\frac{r_n - r_{n-1}}{\varepsilon}\right| M_1 \\
&\leq (1-\alpha_n)\|x_n - x_{n-1}\| + k\alpha_n\|x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}| M_2 + \left|1 - \frac{r_{n-1}}{r_n}\right| M_1 \\
&\leq (1-(1-k)\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| M_2 + \left|\frac{r_n - r_{n-1}}{\varepsilon}\right| M_1,
\end{aligned} \tag{3.6}$$

where $M_2 = \sup\{\|f(x_n) - Sx_n\| : n \geq 0\}$. Thus, by (C3) we have

$$\|x_{n+1} - x_n\| \leq (1-(1-k)\alpha_n)\|x_n - x_{n-1}\| + M_2(o(\alpha_n) + \sigma_{n-1}) + M_1\left|\frac{r_n - r_{n-1}}{\varepsilon}\right|.$$

In (3.6), by taking $s_{n+1} = \|x_{n+1} - x_n\|$, $\lambda_n = (1-k)\alpha_n$, $\lambda_n\delta_n = M_2 o(\alpha_n)$ and

$$\gamma_n = M_1\left|\frac{r_n - r_{n-1}}{\varepsilon}\right| + M_2\sigma_{n-1},$$

we have

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n\delta_n + \gamma_n.$$

Hence, by the conditions (C1), (C2), (C3), (C4) and Lemma 2.4, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Now, in order to prove that $\lim_{n\to\infty} \|x_n - q\| = 0$, we consider two possible cases as in [10] and [27].

Case 1. Assume that $\{\|x_n - q\|\}$ is a monotone sequence. In other words, for n_0 large enough, $\{\|x_n - q\|\}$ is either nondecreasing or nonincreasing. Hence $\{\|x_n - q\|\}$ converges (since $\{\|x_n - q\|\}$ is bounded).

Step 3. We show that $\lim_{n\to\infty} \|y_n - J_{r_n} y_n\| = 0$. First, from Lemma 2.2 (Resolvent Identity), we know that

$$J_{r_n} y_t = J_{\frac{r_n}{2}}\left(\frac{1}{2}y_n + \frac{1}{2}J_{r_n} y_n\right).$$

Then we have

$$\|J_{r_n} y_n - q\| = \left\|J_{\frac{r}{2}}\left(\frac{1}{2}y_n + \frac{1}{2}J_{r_n} y_n\right) - q\right\| \leq \left\|\frac{1}{2}\left(y_n - q\right) + \frac{1}{2}\left(J_{r_n} y_n - q\right)\right\|.$$

By the inequality (2.1) ($l = 2, \lambda = \frac{1}{2}$), we obtain that

$$\begin{aligned}
\|J_{r_n} y_n - q\|^2 &\leq \left\|J_{\frac{r_n}{2}}\left(\frac{1}{2}y_n + \frac{1}{2}J_{r_n} y_n\right) - q\right\|^2 \\
&\leq \left\|\frac{1}{2}\left(y_n - q\right) + \frac{1}{2}\left(J_{r_n} y_n - q\right)\right\|^2 \\
&\leq \frac{1}{2}\|y_n - q\|^2 + \frac{1}{2}\|J_{r_n} y_n - q\|^2 - \frac{1}{4}g(\|y_n - J_{r_n} y_n\|) \\
&\leq \frac{1}{2}\|y_n - q\|^2 + \frac{1}{2}\|y_n - q\|^2 - \frac{1}{4}g(\|y_n - J_{r_n} y_n\|) \\
&= \|y_n - q\|^2 - \frac{1}{4}g(\|y_n - J_{r_n} y_n\|)
\end{aligned} \tag{3.7}$$

Thus, from (3.3), the convexity of the real function $\psi(t) = t^2$ ($t \in (-\infty, \infty)$) and the inequality (3.7), we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \|J_{r_n} y_n - q\|^2 \\ &\le \|y_n - q\|^2 - \frac{1}{4} g(\|y_n - J_{r_n} y_n\|) \\ &= \|\alpha_n (f x_n - q) + (1 - \alpha_n)(S x_n - q)\|^2 - \frac{1}{4} g(\|y_n - J_{r_n} y_n\|) \\ &\le \alpha_n \|f x_n - q\|^2 + (1 - \alpha_n)\|x_n - q\|^2 - \frac{1}{4} g(\|y_n - J_{r_n} y_n\|)\end{aligned}$$

and hence

$$\frac{1}{4} g(\|y_n - J_{r_n} y_n\|)) - \alpha_n \|f x_n - q\|^2 \le \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

Since $\{\|x_n - q\|\}$ converges, by condition (C1), we obtain

$$\lim_{n \to \infty} g(\|y_n - J_{r_n} y_n\|) = 0.$$

Thus, from the property of the function g in (2.1), it follows that

$$\lim_{n \to \infty} \|y_n - J_{r_n} y_n\| = 0.$$

Step 4. We show that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Indeed, from Step 2 and Step 3, it follows that

$$\begin{aligned}\|x_n - y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ &\le \|x_n - x_{n+1}\| + \|J_{r_n} y_n - y_n\| \to 0, \quad (n \to \infty).\end{aligned}$$

Step 5. We show that $\lim_{n\to\infty} \|y_n - S y_n\| = 0$. In fact, by (3.4) and Step 4, we have

$$\begin{aligned}\|y_n - S y_n\| &\le \|y_n - S x_n\| + \|S x_n - S y_n\| \\ &\le \|y_n - S x_n\| + \|x_n - y_n\| \to 0 \quad (n \to \infty).\end{aligned}$$

Step 6. We show that $\lim_{n\to\infty} \|y_n - J_r y_n\| = 0$ for $r > 0$. Indeed, from Lemma 2.2 (Resolvent identity), we obtain

$$\begin{aligned}\|J_{r_n} y_n - J_r y_n\| &= \left\| J_r \left(\frac{r}{r_n} y_n + \left(1 - \frac{r}{r_n}\right) J_{r_n} y_n \right) - J_r y_n \right\| \\ &\le \left\| \left(\frac{r}{r_n} y_n + \left(1 - \frac{r}{r_n}\right) J_{r_n} y_n \right) - y_n \right\| \\ &\le \left| 1 - \frac{r}{r_n} \right| \|y_n - J_{r_n} y_n\| \to 0 \quad (n \to \infty).\end{aligned} \tag{3.8}$$

Hence, by Step 3 and (3.8) we have

$$\|y_n - J_r y_n\| \le \|y_n - J_{r_n} y_n\| + \|J_{r_n} y_n - J_r y_n\| \to 0 \;\; (n \to \infty).$$

Step 7. We show that $\limsup_{n\to\infty} \langle (I-f)q, \mathcal{J}(q-y_n)\rangle \le 0$. To prove this, let a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ be such that

$$\limsup_{n\to\infty} \langle (I - f)q, \mathcal{J}(q - y_n)\rangle = \lim_{j\to\infty} \langle (I - f)q, \mathcal{J}(q - y_{n_j})\rangle$$

and $y_{n_j} \rightharpoonup z$ for some $z \in E$. From Step 5 and Step 6, it follows that $\lim_{j\to\infty} \|y_{n_j} - S y_{n_j}\| = 0$ and $\lim_{j\to\infty} \|y_{n_j} - J_r y_{n_j}\| = 0$ for $r > 0$.

Now let $q = \lim_{t\to 0} x_t = \lim_{t\to 0} y_t$ where $y_t = t f x_t + (1 - t) S x_t$ and $x_t = J_r y_t$ for $r > 0$. Then we can write

$$y_t - y_{n_j} = t(f x_t - y_{n_j}) + (1 - t)(S x_t - y_{n_j})$$

and

$$\|x_t - y_{n_j}\| = \|J_r y_t - y_{n_j}\| \le \|y_t - y_{n_j}\| + \|J_r y_{n_j} - y_{n_j}\|.$$

Putting

$$a_j(t) = (1-t)^2\|Sy_{n_j} - y_{n_j}\|(2\|x_t - y_{n_j}\| + \|Sy_{n_j} - y_{n_j}\|) \to 0 \quad (j \to \infty)$$

and

$$b_j(t) = \|J_r y_{n_j} - y_{n_j}\|(2\|y_t - y_{n_j}\| + \|J_r y_{n_j} - y_{n_j}\|) \to 0 \quad (j \to \infty)$$

by Step 5 and Step 6, and using Lemma 2.1, we obtain

$$\begin{aligned}
\|x_t - y_{n_j}\|^2 &\le \|y_t - y_{n_j}\|^2 + b_j(t) \\
&\le (1-t)^2\|Sx_t - y_{n_j}\|^2 + 2t\langle fx_t - y_{n_j}, \mathcal{J}(y_t - y_{n_j})\rangle + b_j(t) \\
&\le (1-t)^2(\|Sx_t - Sy_{n_j}\| + \|Sy_{n_j} - y_{n_j}\|)^2 \\
&\quad + 2t\langle fx_t - x_t, \mathcal{J}(y_t - y_{n_j})\rangle + 2t\|x_t - y_{n_j}\|\|y_t - y_{n_j}\| \\
&\le (1-t)^2\|x_t - y_{n_j}\|^2 + a_j(t) + b_j(t) \\
&\quad + 2t\langle fx_t - x_t, \mathcal{J}(y_t - y_{n_j})\rangle + 2t\|x_t - y_{n_j}\|^2 + 2t\|x_t - y_{n_j}\|\|y_t - x_t\|.
\end{aligned}$$

The last inequality implies

$$\langle (I-f)x_t, \mathcal{J}(y_t - y_{n_j})\rangle \le \frac{t}{2}\|x_t - y_{n_j}\|^2 + \frac{1}{2t}(a_j(t) + b_j(t)) + \|x_t - y_t\|\|x_t - y_{n_j}\|.$$

It follows that

$$\limsup_{j\to\infty}\langle (I-f)x_t, \mathcal{J}(y_t - y_{n_j})\rangle \le \frac{t}{2}M^2 + \|x_t - y_t\|M, \tag{3.9}$$

where $M = \sup\{\|x_t - y_n\| : n \ge 0 \text{ and } t \in (0,1)\}$. Recalling (5) in Proposition 3.1, taking the lim sup as $t \to 0$ in (3.9), and noticing the fact that the two limits are interchangeable due to the fact that J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak* topology of E^*, we have

$$\limsup_{j\to\infty}\langle (I-f)q, \mathcal{J}(q - y_{n_j})\rangle \le 0.$$

Step 8. We show that $\lim_{n\to\infty}\|x_n - q\| = 0$. By using (3.3), we have

$$\|x_{n+1} - q\| \le \|y_n - q\| = \|\alpha_n(fx_n - q) + (1-\alpha_n)(Sx_n - q)\|.$$

Applying Lemma 2.1, we obtain

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \|y_n - q\|^2 \\
&\le (1-\alpha_n)^2\|Sx_n - q\|^2 + 2\alpha_n\langle fx_n - q, \mathcal{J}(y_n - q)\rangle \\
&\le (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle fx_n - fq, \mathcal{J}(y_n - q)\rangle \\
&\quad + 2\alpha_n\langle fq - q, \mathcal{J}(y_n - q)\rangle \\
&\le (1-\alpha_n)^2\|x_n - q\|^2 + 2k\alpha_n\|x_n - q\|\|y_n - q\| \\
&\quad + 2\alpha_n\langle fq - q, \mathcal{J}(y_n - q)\rangle \\
&\le (1-\alpha_n)^2\|x_n - q\|^2 + 2k\alpha_n\|x_n - q\|^2 \\
&\quad + 2k\alpha_n\|x_n - q\|\|y_n - x_n\| + 2\alpha_n\langle fq - q, \mathcal{J}(y_n - q)\rangle.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le (1 - 2(1-k)\alpha_n + \alpha_n^2)\|x_n - q\|^2 \\
&\quad + 2k\alpha_n\|x_n - q\|\|y_n - x_n\| + 2\alpha_n\langle fq - q, \mathcal{J}(y_n - q)\rangle \\
&\le (1 - 2(1-k)\alpha_n)\|x_n - q\|^2 + \alpha_n^2 L^2 \\
&\quad + 2kL\alpha_n\|y_n - x_n\| + 2\alpha_n\langle (I-f)q, \mathcal{J}(q - y_n)\rangle,
\end{aligned} \tag{3.10}$$

where $L = \sup\{\|x_n - q\| : n \geq 0\}$. Put

$$\lambda_n = 2(1-k)\alpha_n \quad \text{and}$$

$$\delta_n = \frac{\alpha_n L^2}{2(1-k)} + \frac{kL}{(1-k)}\|y_n - x_n\| + \frac{1}{1-k}\langle (I-f)q, \mathcal{J}(q-y_n)\rangle.$$

From (C1), (C2), Step 4 and Step 7, it follows that have $\lambda_n \to 0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Since (3.10) reduces to

$$\|x_{n+1} - q\|^2 \leq (1-\lambda_n)\|x_n - q\|^2 + \lambda_n \delta_n,$$

from Lemma 2.4 with $\gamma_n = 0$, we conclude that $\lim_{n\to\infty} \|x_n - q\| = 0$. By Step 4, we also have $\lim_{n\to\infty} y_n = q$.

Case 2. Assume that $\{\|x_n - q\|\}$ is not a monotone sequence. Then, we can define a sequence of integers $\{\tau(n)\}$ for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n,\ \|x_k - q\| < \|x_{k+1} - q\|\}.$$

Clearly, $\{\tau(n)\}$ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\|x_{\tau(n)} - q\| \leq \|x_{\tau(n)+1} - q\|$$

for all $n \geq n_0$. In this case, by using the same argument as in Step 2 – Step 8 with $\{x_{\tau(n)}\}$, $\{y_{\tau(n)}\}$, $\{J_{r_{\tau(n)}} y_{\tau(n)}\}$, $\{J_r y_{\tau(n)}\}$, $\{Sx_{\tau(n)}\}$, $\{Sy_{\tau(n)}\}$, and $\{fx_{\tau(n)}\}$, we obtain the following:

Step 2′ $\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0$;
Step 3′ $\lim_{n\to\infty} \|y_{\tau(n)} - J_{r_{\tau(n)}} y_{\tau(n)}\| = 0$.
Step 4′ $\lim_{n\to\infty} \|x_{\tau(n)} - y_{\tau(n)}\| = 0$.
Step 5′ $\lim_{n\to\infty} \|y_{\tau(n)} - Sy_{\tau(n)}\| = 0$.
Step 6′ $\lim_{n\to\infty} \|y_{\tau(n)} - J_r y_{\tau(n)}\| = 0$ for $r > 0$.
Step 7′ $\limsup_{n\to\infty} \langle (I-f)q, \mathcal{J}(q - y_{\tau(n)})\rangle \leq 0$.
Step 8′ $\lim_{n\to\infty} \|x_{\tau(n)} - q\| = 0$ and $\lim_{n\to\infty} \|x_{\tau(n)+1} - q\| = 0$.

Thus, from Lemma 2.5, we have

$$\|x_n - q\| \leq \|x_{\tau(n)+1} - q\|.$$

Therefore, $\lim_{n\to\infty} \|x_n - q\| = 0$. This completes the proof. □

Corollary 3.5. *Let E be a uniformly convex and uniformly smooth Banach space. Let C, A, J_{r_n}, S, and f be as in Theorem 3.4. Let $\{\alpha_n\} \in (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy the conditions* (C1), (C2), (C3) and (C4) *in Theorem 3.4. Let $x_0 = x \in C$ be chosen arbitrarily, and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = J_{r_n}(\alpha_n f x_n + (1-\alpha_n)Sx_n), \quad \forall n \geq 0.$$

Let $\{y_n\}$ be a sequence defined by $y_n = \alpha_n f x_n + (1-\alpha_n)Sx_n$. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in A^{-1}0 \cap F(S)$, where q is the unique solution of the variational inequality (3.2).

Corollary 3.6. *Let E, C, A, J_{r_n}, S, and f be as in Theorem 3.4. Let $\{\alpha_n\} \in (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy the conditions* (C1), (C2), (C3) and (C4) *in Theorem 3.4. Let $x_0 = x \in C$ be chosen arbitrarily, and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = J_{r_n}(\alpha_n f x_n + (1-\alpha_n)Sx_n + e_n), \quad \forall n \geq 0,$$

where $\{e_n\} \subset E$ satisfies $\sum_{n=0}^{\infty} \|e_n\| < \infty$ or $\lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$. Let $\{y_n\}$ be a sequence defined by $y_n = \alpha_n f x_n + (1-\alpha_n)Sx_n + e_n$. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in A^{-1}0 \cap F(S)$, where q is the unique solution of the variational inequality (3.2).

Remark 3.7. (1) We point out that our iterative algorithms (3.1) and (3.3) for finding common point in the set of zeros of an accretive operator and the set of fixed points of a nonexpansive mapping are new ones different from those in the literature (see [6] and others in References). Thus Theorem 3.2 and Theorem 3.4 develop, and complement the recent corresponding results studied by many authors in this direction.

(2) If we take $fx = u$, $\forall x \in C$, as a constant function and $Sx = x$, $\forall x \in C$, as the identity mapping in Corollary 3.6, then the result extends corresponding results of Xu [24] and Song and Yang [23] in Hilbert spaces to a Banach space setting.

(3) The control condition (C3) in Theorem 3.4 can be replaced by the condition $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; or the condition $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$, which are not comparable ([7]).

(4) The results in this paper apply to all L^p spaces, $1 < p < \infty$.

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Department of Mathematics, Dong-A University, Busan 49315, Korea
E-mail address: jungjs@dau.ac.kr