数理解析研究所講究録
第2065巻 2018年 94-99

Rotation invariant norms on $\mathbb{R}^2$ and geometric constants

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1 introduction

In this paper, we mainly consider the James constants of rotation invariant norms on $\mathbb{R}^2$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$, and let $\theta\in(0,2\pi)$. Then $\|\cdot\|$ is said to be *θ-rotation invariant* if the θ-rotation matrix

$$R(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

is an isometry on $(\mathbb{R}^2,\|\cdot\|)$. All norms on $\mathbb{R}^2$ are clearly π-rotation invariant, and the Euclidean norm is θ-rotation invariant for each $\theta\in(0,2\pi)$. For the study of James constant, the class of $\pi/2$-rotation invariant norms are very important; and it is rich since every symmetric absolute norms on $\mathbb{R}^2$, that is, a norm satisfying $\|(|a|,|b|)\|=\|(a,b)\|=\|(b,a)\|$ for each (a,b), is $\pi/2$-rotation invariant.

Now let X be a Banach space. Then S_X denotes the unit sphere of X. The James constant $J(X)$ of X was introduced in 1990 by Gao and Lau [3] as a measure of the squareness of the unit ball. Namely, we define $J(X)$ by

$$J(X)=\sup\{\min\{\|x+y\|,\|x-y\|\}:x,y\in S_X\}.$$

It is known that $J(X)$ has the following properties:

(i) $\sqrt{2}\le J(X)\le 2$ ([3]).

(ii) If H is a Hilbert space, then $J(H)=\sqrt{2}$.

(iii) If $\dim X\ge 3$, then $J(X)=\sqrt{2}$ implies that X is a Hilbert space ([5]); and hence $J(X)=\sqrt{2}$ if and only if X is a Hilbert space provided that $\dim X\ge 3$.

(iv) $J(X)<2$ if and only if X is uniformly non-square, that is, there exists a $\delta>0$ such that $\min\{\|x+y\|,\|x-y\|\}<2(1-\delta)$ whenever $x,y\in S_X$.

We here emphasize that (iii) does not hold in the two-dimensional case. Indeed, if we consider the norm on $\mathbb{R}^2$ whose unit sphere is a regular octagon, then its James constant is $\sqrt{2}$ though it is clearly not a Hilbert space. Actually, this fact comes from the following more general result by Gao and Lau [3, Proposition 2.8].

Proposition 1.1 (Gao and Lau [3]). *Let $\|\cdot\|$ be a $\pi/4$-rotation invariant norm on $\mathbb{R}^2$. Then $J((\mathbb{R}^2,\|\cdot\|))=\sqrt{2}$.*

Thus, in fact, there are many non-Hilbert two-dimensional spaces with James constant $\sqrt{2}$.

The purpose of this paper is to consider the converse to the above proposition. We present a partial converse by using the notion of rotation invariant norms on $\mathbb{R}^2$. To do this, the calculation formula for the James constants of $\pi/2$-rotation rotation invariant norms on $\mathbb{R}^2$ is given. Moreover, we also give construction of $\pi/4$-rotation invariant norms using certain convex functions on the unit interval.

2 James constants of $\pi/2$-rotation invariant norms

We start this section with the following result of Komuro, Saito and Mitani [4] which provides an important characterization of $\pi/2$-rotation invariant norms on $\mathbb{R}^2$. Recall that an element x of a normed space is said to be *isosceles orthogonal* to another element y, denoted by $x \perp_I y$, if $\|x+y\| = \|x-y\|$.

Theorem 2.1. *Let $\|\cdot\| \in N_2$. Then the following are equivalent.*

(i) $\|\cdot\|$ *is $\pi/2$-rotation invariant.*

(ii) $x \perp_I y$ *if and only if* $\langle x, y\rangle = 0$ *whenever* $\|x\| = \|y\| = 1$.

In which cases, $x \perp_I R(\pi/2)x$ for each x.

Now let Ψ_2 be the set of all convex functions ψ on $[0,1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$. For each $\psi \in \Psi_2$, the formula

$$\|(a,b)\|_\psi = \begin{cases} (|a|+|b|)\psi\left(\dfrac{|b|}{|a|+|b|}\right) & ((a,b) \neq (0,0)) \\ 0 & ((a,b) = (0,0)) \end{cases}$$

defines an absolute normalized norm on $\mathbb{R}^2$, that is, $\|(a,b)\|_\psi = \|(|a|,|b|)\|_\psi$ for each (a,b) and $\|(1,0)\|_\psi = \|(0,1)\|_\psi$. Let AN_2 be the set of all absolute normalized norms on $\mathbb{R}^2$. Then, the correspondence $\psi \mapsto \|\cdot\|_\psi$ gives a bijection from Ψ_2 onto AN_2; see [2, 9].

Using convex functions in Ψ_2, we can construct more general two-dimensional normed spaces. Namely, for each pair $\varphi, \psi \in \Psi_2$, let $\|\cdot\|_{\varphi,\psi}$ be the norm on $\mathbb{R}^2$ given by

$$\|(a,b)\|_{\varphi,\psi} = \begin{cases} \|(a,b)\|_\varphi & (ab \geq 0) \\ \|(a,b)\|_\varphi & (ab \geq 0) \end{cases}.$$

The space $(\mathbb{R}^2, \|\cdot\|_{\varphi,\psi})$ is called a Day-James space, and is denoted by $\ell^2_{\varphi,\psi}$; see [8].

In fact, the class of Day-James spaces are essential among all two-dimensional real normed spaces in the following sense.

Proposition 2.2 (Alonso [1]). *Every two-dimensional real normed space is isometrically isomorphic to some Day-James space.*

For each $\psi \in \Psi_2$, let $\widetilde{\psi}(t) = \psi(1-t)$ for each $t \in [0,1]$. Then $\widetilde{\psi} \in \Psi_2$.

The following proposition shows that the Day-James space $\ell^2_{\psi,\widetilde{\psi}}$ is always $\pi/2$-rotation invariant for any $\psi \in \Psi_2$.

Proposition 2.3. *Let $\psi \in \Psi_2$. Then $\|\cdot\|_{\psi,\tilde{\psi}}$ is $\pi/2$-rotation invariant.*

We have a refinement of Proposition 2.2 for the case of $\pi/2$-rotation invariant norms on $\mathbb{R}^2$.

Theorem 2.4 ([6]). *Let $\|\cdot\|$ be a $\pi/2$-rotation invariant norm on $\mathbb{R}^2$. Then there exists a $\psi \in \Psi_2$ satisfying the following conditions.*

(i) $(\mathbb{R}^2, \|\cdot\|)$ *is isometrically isomorphic to* $\ell^2_{\psi,\tilde{\psi}}$.

(ii) $\|\cdot\| \in N_2(\theta)$ *if and only if* $\|\cdot\|_{\psi,\tilde{\psi}} \in N_2(\theta)$ *for each* $\theta \in [0, 2\pi]$.

Hence, the Day-James space of the form $\ell^2_{\psi,\tilde{\psi}}$ are essential among all $\pi/2$-rotation invariant normed spaces.

We now consider the class of two-dimensional normed spaces with James constant $\sqrt{2}$. Let X be a normed space. For each $x \in S_X$, let

$$\beta(x) = \sup\{\min\{\|x+y\|, \|x-y\|\} : y \in S_X\}.$$

Then $J(X) = \sup\{\beta(x) : x \in S_X\}$. By [3, Lemma 2.2], if $y \in S_X$ and $x \perp_I y$ then $\|x+y\| = \|x-y\| = \beta(x)$.

The following is the $\pi/2$-rotation invariant analogue of [7, Theorem 1].

Proposition 2.5. *Let $\psi \in \Psi_2$. Then*

$$J(\ell^2_{\psi,\tilde{\psi}}) = \max\left\{\max_{0\le t\le 1/2} \frac{2-2t}{\psi(t)}\psi\left(\frac{1}{2-2t}\right), \max_{1/2\le t\le 1} \frac{2t}{\psi(t)}\psi\left(\frac{2t-1}{2t}\right)\right\}.$$

In what follows, for each ψ, let

$$f_\psi(t) = \frac{2-2t}{\psi(t)}\psi\left(\frac{1}{2-2t}\right) \quad \text{and} \quad g_\psi(t) = \frac{2t}{\psi(t)}\psi\left(\frac{2t-1}{2t}\right),$$

respectively. The following lemma is simple, but important.

Lemma 2.6. *Let $\psi \in \Psi_2$. Then*

$$f_\psi(t) g_\psi\left(\frac{1}{2-2t}\right) = 2$$

for each $t \in [0, 1/2]$.

Combining the preceding lemma with Proposition 2.5, we have the following improvement.

Theorem 2.7. *Let $\psi \in \Psi_2$. Then*

$$J(\ell^2_{\psi,\tilde{\psi}}) = \max\left\{\max_{0\le t\le 1/2} f_\psi(t), \frac{2}{\min_{0\le t\le 1/2} f_\psi(t)}\right\}.$$

Using this formula, we can characterize the $\pi/2$-rotation invariant norms with James constant $\sqrt{2}$ as follows.

Theorem 2.8. *Let $\psi \in \Psi_2$. Then the following are equivalent.*

(i) $J(\ell^2_{\psi,\widetilde{\psi}}) = \sqrt{2}$.

(ii) $f_\psi(t) = \sqrt{2}$ *for each* $t \in [0,1/2]$ *and* $g_\psi(t) = \sqrt{2}$ *for each* $t \in [1/2,1]$.

(iii) $f_\psi(t) = \sqrt{2}$ *for each* $t \in [0,1/2]$.

(iv) $g_\psi(t) = \sqrt{2}$ *for each* $t \in [1/2,1]$.

(v) $\|\cdot\|_{\psi,\widetilde{\psi}} \in N_2(\pi/4)$.

As a consequence of Theorems 2.4 and 2.8, we have the following partial converse to Proposition 1.1, which is our aim in this section.

Theorem 2.9. *Let* $\|\cdot\| \in N_2(\pi/2)$. *Then* $J((\mathbb{R}^2, \|\cdot\|)) = \sqrt{2}$ *if and only if* $\|\cdot\| \in N_2(\pi/4)$.

3 Construction of $\pi/4$-rotation invariant norms

As it was shown in the preceding section, the equality $J((\mathbb{R}^2, \|\cdot\|)) = \sqrt{2}$ characterizes the $\pi/4$-rotation invariant norms among the $\pi/2$-rotation invariant norms. The aim of this section is to study the structure of $\pi/4$-rotation invariant norms via some properties of certain convex functions on the unit interval.

Let $DJ = \{\|\cdot\|_{\varphi,\psi} : \varphi, \psi \in \Psi_2\}$, and let $DJ(\theta)$ be the set of all elements in DJ which are θ-rotation invariant. Then the set DJ is obviously in a one-to-one correspondence with $\Psi_2 \times \Psi_2$. Moreover, if $\|\cdot\|_{\varphi,\psi} \in DJ(\pi/2)$ then $\psi = \widetilde{\varphi}$. This and Proposition 2.3 together show that the set $DJ(\pi/2)$ is in a one-to-one correspondence with the set $\{(\psi, \widetilde{\psi}) : \psi \in \Psi_2\}$ which can be identified with Ψ_2. These observations are summarized as follows.

Proposition 3.1. *The map* $\Psi_2 \ni \psi \mapsto \|\cdot\|_{\psi,\widetilde{\psi}} \in DJ(\pi/2)$ *is bijective.*

Now let

$$\Gamma = \left\{\psi \in \Psi_2 : \max\left\{1 - \left(1 - \frac{1}{\sqrt{2}}\right)t, \frac{1}{\sqrt{2}} + \left(1 - \frac{1}{\sqrt{2}}\right)t\right\} \leq \psi(t)\right\}.$$

Then we have the following.

Proposition 3.2. *Let* $\psi \in \Psi_2$. *If* $\|\cdot\|_{\psi,\widetilde{\psi}} \in DJ(\pi/4)$, *then*

$$\psi^\flat(t) = (1 + (\sqrt{2} - 1)t)\psi\left(\frac{t}{\sqrt{2} + (2 - \sqrt{2})t}\right)$$

defines an element of Γ.

Remark 3.3. The preceding proposition is an extension of [5, Lemma 3.4] which states that $\|\cdot\|_\psi \in AN_2 \cap N_2(\pi/4)$ implies $\psi^\flat \in \Gamma^S = \Gamma \cap \Psi_2^S$.

The following can be viewed as the "converse" of Proposition 3.2.

Proposition 3.4. *Let $\psi \in \Gamma$. Then*

$$\psi^\sharp(t) = \begin{cases} (1-(2-\sqrt{2})t)\psi\left(\dfrac{\sqrt{2}t}{1-(2-\sqrt{2})t}\right) & (t \in [0,1/2]), \\ (\sqrt{2}-1)(1+\sqrt{2}t)\psi\left(\dfrac{2t-1}{(\sqrt{2}-1)(1+\sqrt{2}t)}\right) & (t \in [1/2,1]) \end{cases}$$

defines an element of Ψ_2 such that $\|\cdot\|_{\psi^\sharp,\widetilde{\psi^\sharp}} \in DJ(\pi/4)$.

We now present the main theorem in this section which extends [5, Theorem 3.8] to $\pi/4$-rotation invariant Day-James norms.

Theorem 3.5. *The following hold.*

(i) $(\psi^\flat)^\sharp = \psi$ *for each* $\psi \in \Psi_2$ *with* $\|\cdot\|_{\psi,\widetilde{\psi}} \in DJ(\pi/4)$.

(ii) $(\psi^\sharp)^\flat = \psi$ *for each* $\psi \in \Gamma$.

(iii) *The map* $\Gamma \ni \psi \mapsto \|\cdot\|_{\psi^\sharp,\widetilde{\psi^\sharp}} \in DJ(\pi/4)$ *is bijective.*

The preceding theorem, together with Propositions 3.2 and 3.4, provide a specific way to construct all $\pi/4$-rotation invariant norms on $\mathbb{R}^2$.

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