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# A free boundary problem for the Fisher-KPP equation with a given moving boundary

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## 1 Introduction and Main Results

In this article, based on a recent work [12], we consider the following free boundary problem of the Fisher-KPP equation:

$$\begin{cases} u_t = u_{xx} + u(1-u), & t>0,\ ct<x<h(t), \\ u(t,ct) = u(t,h(t)) = 0, & t>0, \\ h'(t) = -\mu u_x(t,h(t)), & t>0, \\ h(0)=h_0,\ u(0,x)=u_0(x), & 0\le x\le h_0, \end{cases} \tag{1}$$

where $c$, $\mu$ and $h_0$ are given positive constants, $x = h(t)$ is the moving boundary to be determined together with $u(t,x)$. Initial function $u_0$ belongs to $\mathscr{X}(h_0)$ for given $h_0 > 0$, where

$$\mathscr{X}(h_0) := \left\{ \phi \in C^2[0,h_0] : \begin{array}{l} \phi(0)=\phi(h_0)=0, \\ \phi'(h_0)<0,\ \phi(x)>0 \text{ in } (0,h_0) \end{array} \right\}.$$

This model may be used to describe the spreading of a new or invasive species with population density $u(t,x)$ over one dimensional habitat $(ct, h(t))$. The free boundary $x = h(t)$ represents the spreading front. The behavior of the free boundary is determined by the Stefan-like condition which implies that the population pressure at the free boundary is driving force of the spreading front. In this model, we impose zero Dirichlet boundary condition at left moving boundary $x = ct$. This means that the left boundary of the habitat is a very hostile environment for the species and that the habitat is eroded away by the left moving boundary at constant speed $c$.

Recently, problem (1) with $c = 0$ was studied in pioneer paper [4](in which Neumann boundary condition is imposed at left fixed boundary $x = 0$), [9] and [10]. The authors showed that (1) has a unique solution which is defined for all $t > 0$ and one of the following situation happens:

- (vanishing) $\lim_{t\to\infty} h(t) = h_\infty < \infty$ and $\lim_{t\to\infty} \|u(t,\cdot)\|_{C[0,h(t)]} = 0$
- (spreading) $\lim_{t\to\infty} h(t) = \infty$ as $t \to \infty$ and

$$\lim_{t\to\infty} u(t,x) = \begin{cases} 1 & \text{Neumann condition case} \\ v(x) & \text{Dirichlet condition case} \end{cases} \quad \text{locally uniformly on } [0,\infty)$$

where $v(x)$ is a unique positive solution of

$$\begin{cases} v'' + v(1-v) = 0, & x>0, \\ v(0)=0, v(\infty)=1. \end{cases}$$

See also [5] for the double fronts free boundary problem with monostable, bistable or combustion type nonlinearity. Moreover, in the case of spreading, it is shown in [4, 5] that there exists $c^* = c^*(\mu) > 0$ such that $\lim_{t\to\infty}(h(t)/t) = c^*$. In this sense, $c^*$ is called the asymptotic spreading speed of corresponding free boundary problems. In [5], the authors showed that $c^*$ is determined by the unique solution pair $(c, q) = (c^*, q^*)$ of the following problem

$$\begin{cases} q'' + cq + q(1-q) = 0, \quad z \in (-\infty, 0), \\ q(0) = 0, \ q(-\infty) = 1, \ q'(0) = -c/\mu, \ q(z) > 0 \quad z \in (-\infty, 0). \end{cases} \tag{2}$$

Using a simple variation of the techniques in [4], we can see that for any $h_0 > 0$ and $u_0 \in \mathscr{X}(h_0)$, (1) has a unique solution defined on some maximal time interval $(0, T_{\max})$ with maximal existence time $T_{\max} \in (0, \infty]$. The main purpose of this paper is to study the behavior of solutions to (1). When $T_{\max} = \infty$, the solution is global and so we can study its asymptotic behavior. On the other hand, in this problem, $T_{\max}$ may be a finite number for the reason that $h(t) - ct \to 0$ as $t \nearrow T_{\max}$, that is the habitat of the species may shrink to a single point. Such a phenomenon is observed first in free boundary problems considered by [2, 3]. We concern with the following questions:

(Q1) When the situation that $T_{\max} < \infty$ and $h(t) - ct \to 0$ as $t \nearrow T_{\max}$ occur?

(Q2) Can the situation that $T_{\max} = \infty$ and $h(t) - ct \to 0$ as $t \to \infty$ occur?

(Q3) When $T_{\max} < \infty$ and $h(t) - ct \to 0$ as $t \nearrow T_{\max}$, how about the behavior of $u$ as $t \nearrow T_{\max}$ is ?

(Q4) When $T_{\max} = \infty$, reveal all possible long-time dynamical behavior of the solutions.

Now we state our main theorems. First theorem is a trichotomy result for the case $0 < c < c^*$.

**Theorem A.** *Suppose that $0 < c < c^*$ and $(u, h)$ is the unique solution of (1) on a time interval $(0, T_{\max})$ with maximal existence time $T_{\max}$. Then exactly one of the following situations happens:*

(1) **Vanishing:** $T_{\max} < \infty$, $\lim_{t\nearrow T_{\max}}(h(t) - ct) = 0$,

$$\lim_{t\nearrow T_{\max}} \left\{ \max_{x\in[ct,h(t)]} u(t,x) \right\} = 0.$$

(2) **Spreading:** $T_{\max} = \infty$, $\lim_{t\to\infty}(h(t)/t) = c^*$ *and for any small* $\varepsilon > 0$

$$\lim_{t\to\infty} \left\{ \max_{x\in[(c+\varepsilon)t,(c^*-\varepsilon)t]} |u(t,x) - 1| \right\} = 0.$$

(3) **Transition:** $T_{\max} = \infty$, $\lim_{t\to\infty}(h(t) - ct) = L_c$ *and*

$$\lim_{t\to\infty}\left\{\max_{x\in[ct,h(t)]}|u(t,x) - \mathcal{V}_c(x - h(t) + L_c)|\right\} = 0,$$

*where* $L_c > 0$ *are determined by a unique solution pair* $(L, \mathcal{V}) = (L_c, \mathcal{V}_c)$ *to the problem*

$$\begin{cases} \mathcal{V}'' + c\mathcal{V}' + \mathcal{V}(1-\mathcal{V}) = 0, \ \mathcal{V}(z) > 0 \ \text{ for } \ z \in (0, L), \\ \mathcal{V}(0) = \mathcal{V}(L) = 0, \ -\mu\mathcal{V}'(L) = c. \end{cases} \tag{3}$$

If the initial function $u_0$ in (1) has the form $u_0 = \sigma\phi$ $(\sigma > 0)$ with some fixed $\phi \in \mathscr{X}(h_0)$, we can obtain the following sharp threshold result.

**Theorem B.** *Suppose that the initial function* $u_0$ *in* (1) *has the form* $u_0 = \sigma\phi$ *with some fixed* $\phi \in \mathscr{X}(h_0)$. *Then there exists* $\overline{\sigma} \in (0, \infty]$ *such that vanishing happens when* $0 < \sigma < \overline{\sigma}$, *spreading happens when* $\sigma > \overline{\sigma}$, *and transition happens when* $\sigma = \overline{\sigma}$.

When $c \geq c^*$, vanishing always happens.

**Theorem C.** *Assume that* $c^* \leq c$ *and* $(u, h)$ *is the unique solution of* (1) *on a time interval* $(0, T_{\max})$ *with maximal existence time* $T_{\max}$. *Then we have* $T_{\max} < \infty$ *and* $\lim_{t\nearrow T_{\max}}(h(t) - ct) = 0$ *and* $\lim_{t\nearrow T_{\max}} \max_{x\in[ct,h(t)]} u(t,x) = 0$.

Some of the proofs of key steps are inspired by the proof in [2, 3] and [7].

From a mathematical point of view, our main results can be seen as a drastic change of classification of behaviors of solutions, which is caused by the simple replacement of left fixed boundary $x = 0$ by moving boundary $x = ct$ in the problems considered earlier in [4, 10, 9]. The problem (1) with logistic nonlinearity $u(1-u)$ replaced by general monostable, bistable or combustion type nonlinearity will be considered in the forthcoming paper [11].

## 2 Basic Results and Answers for (Q1) to (Q3)

In this section, I will give some basic results and answers for (Q1) to (Q3). The results here are valid for rather general nonlinearity. In this section, we assume that

$$f \in C^1, \ \ f(0) = f(1) = 0, \ \ f'(1) < 0, \ \ f(u) < 0 \ \text{ for } \ u > 1 \tag{4}$$

and consider

$$\begin{cases} u_t = u_{xx} + f(u), & t > 0, \ ct < x < h(t), \\ u(t, ct) = u(t, h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ h(0) = h_0, \ u(0, x) = u_0(x), & 0 \leq x \leq h_0, \end{cases} \tag{5}$$

instead of (1). See section 2 of [12] for the proofs of the results in this section.

**Proposition 2.1.** *For any $h_0 > 0$, $u_0 \in \mathscr{X}(h_0)$ and $\alpha \in (0,1)$, there exists $T > 0$ such that problem* (5) *admit a unique solution $(u,h)$ defined on $(0,T]$ with*

$$u \in C^{\frac{1+\alpha}{2},1+\alpha}(\overline{D}_T) \cap C^{1+\frac{\alpha}{2},2+\alpha}(D_T),\ \ h \in C^{1+\frac{\alpha}{2}}([0,T]),$$

*where $D_T := \{(t,x) \in \mathbb{R}^2 : t \in (0,T], x \in [ct,h(t)]\}$. Moreover we have*

$$\|u\|_{C^{\frac{1+\alpha}{2},1+\alpha}(D_T)} + \|h\|_{C^{1+\frac{\alpha}{2}}([0,T])} \le C,$$

*where $C$ and $T$ depend only on $c$, $\mu$, $h_0$, $\alpha$ and $\|u_0\|_{C^2[0,h_0]}$.*

**Proposition 2.2.** *Let $(u,h)$ be any solution of* (5) *defined on $(0,T_0]$ with some $T_0 \in (0,\infty)$. Then the solution satisfies*

$$\begin{aligned} &0 < u(t,x) \le C_1 \quad \text{for } \ 0 < t \le T_0,\ ct < x < h(t),\\ &0 < h'(t) \le \mu C_2 \quad \text{for } \ 0 < t \le T_0, \end{aligned}$$

*where $C_1$ and $C_2$ are positive constants independent of $T_0$.*

*Moreover the solution can be extended to some interval $(0,\overline{T})$ with $\overline{T} > T_0$ if $\inf_{t\in(0,T_0)]}[h(t)-ct] > 0$.*

In what follows, we assume that the unique solution $(u,h)$ to (5) is defined on $(0,T_{\max})$ with maximal existence time $T_{\max}$. About the properties of solutions which satisfy $T_{\max} < \infty$, we have the following propositions.

**Proposition 2.3.** *If $\lim_{t\nearrow T_{\max}}[h(t)-ct] = 0$, then we have $\lim_{t\nearrow T_{\max}} \|u(t,\cdot)\|_{C[ct,h(t)]} = 0$.*

**Proposition 2.4.** *If $\lim_{t\nearrow T_{\max}}[h(t)-ct] = 0$, then we have $T_{\max} < \infty$.*

**Proposition 2.5.** *There exists a constant $C_3 = C_3(h_0,c,\mu) > 0$ such that if $\|u_0\|_{C[0,h_0]} \le C_3$, then $T_{\max} < \infty$, $\lim_{t\nearrow T_{\max}}(h(t)-ct) = 0$ and $\lim_{t\nearrow T_{\max}} \|u(t,\cdot)\|_{C[ct,h(t)]} = 0$.*

# 3 Proof of Main Theorems

In this section we will prove Theorem A. It is important to prove the following proposition to prove Theorem A.

**Proposition 3.1.** *Suppose that $c \in (0,c^*)$ and $(u,h)$ is the unique solution of* (1) *defined for all $t > 0$. Then we have that*

- *If $h(t)-ct$ is unbounded, then $\lim_{t\to\infty}[h(t)-ct] = \infty$ and $\lim_{t\to\infty}(h(t)/t) = c^*$ holds. Moreover for any given small $\varepsilon > 0$*

$$\lim_{t\to\infty} \max_{x\in[(c+\varepsilon)t,(c^*-\varepsilon)t]} |u(t,x) - 1| = 0.$$

- *If $h(t) - ct$ is bounded then $\lim_{t\to\infty}[h(t) - ct] = L_c$ and*

$$\lim_{t\to\infty}\left\{\sup_{x\in[ct,h(t)]}|u(t,x) - \mathcal{V}_c(x - h(t) + L_c)|\right\} = 0. \tag{6}$$

*holds, where $(L_c, \mathcal{V}_c)$ is determined by problem* (3).

The proof of this proposition will be achieved by proving several lemmas. Suppose that $c \in (0, c^*)$ and $(u, h)$ is the unique solution of (1) defined for all $t > 0$

**Lemma 3.2** ([12, Lemma 4.2]). *Suppose that $h(t) - ct$ is unbounded, we have $\lim_{t\to\infty}[h(t) - ct] = \infty$.*

To prove this lemma, we investigate the zero number of $u(t,x) - \mathcal{V}_c(x - ct - l)$ for any $l > 0$ and then we can show that for any $l > 0$ there exists $T_l > 0$ such that $h(t) - ct > l$ for $t > T_l$. See also Lemma 4.2 of [7].

By constructing an upper solution of the form

$$\begin{aligned}\overline{h}(t) &:= c^*t + M(e^{-\delta T} - e^{-\delta t}) + H\\ \overline{u}(t,x) &:= (1 + Me^{-\delta t})q^*(x - \overline{h}(t)),\end{aligned}$$

with suitable $M$, $\delta$, $H$ and $T > 0$ as in [6, Lemma 3.2] we can obtain the following lemma.

**Lemma 3.3** ([12, Proposition 2.12]). *There exists $C_0 > 0$ such that $h(t) - c^*t < C_0$ for $t > 0$.*

The next lemma indicates that when $h(t) - ct$ is unbounded, the asymptotic spreading speed $\lim_{t\to\infty}(h(t)/t)$ coincided with the speed of semiwave $c^*$ determined by problem (2). This suggests that when $h(t) - ct$ is unbounded, spreading in the sense of Theorem A only occur.

**Lemma 3.4** ([7, Lemma 4.3]). *If $h(t) - ct$ is unbounded, then we have $\lim_{t\to\infty}(h(t)/t) = c^*$.*

By the same argument in [7, Theorem 3.9](see also [12, Appendix]) we can obtain the following results.

**Proposition 3.5.** *If $H_c(t)$ is unbounded, then $\lim_{t\to\infty}(h(t)/t) = c^*$ and for any given small $\varepsilon > 0$*

$$\lim_{t\to\infty}\max_{x\in[(c+\varepsilon)t,(c^*-\varepsilon)t]}|u(t,x) - 1| = 0.$$

Now we investigate the case where $h(t) - ct$ is bounded.

**Lemma 3.6** ([12, Proposition 4.4]). *If $h(t) - ct$ is bounded, then $\lim_{t\to\infty}[h(t) - ct]$ exists.*

To prove this lemma, the zero number argument as in [11, Lemma 3.7] is used, that is, we prove that for any $b \in (0,\infty)\backslash\{L_c\}$, $H_c(t) - b$ changes its sign at most finitely many times by investigating the zero number of $u(t,x) - \mathcal{V}_c(x - ct - b)$ (see [12, Lemma 4.5]).

**Proposition 3.7** ([12, Lemma 4.6, Theorem 4.10]). *Suppose that $h(t)-ct$ is bounded. Then we have $\lim_{t\to\infty}[h(t)-ct]=L_c$. Moreover we have*

$$\lim_{t\to\infty}\left\{\sup_{x\in[ct,h(t)]}|u(t,x)-\mathcal{V}_c(x-h(t)+L_c)|\right\}=0. \tag{7}$$

*Sketch of Proof of Proposition 3.6.* Let $H_c(t):=h(t)-ct$ and $H_c^*:=\lim_{t\to\infty}H_c(t)$.

**Step 1.** Suppose that $H_c^*<L_c$. Define

$$v(t,z):=u(t,z+ct),\ w(t,y):=u(t,y+h(t)).$$

It is clear that $v$ and $w$ satisfy

$$\begin{cases} v_t=v_{zz}+cv_z+v(1-v), & t>0,\ 0<z<H_c(t),\\ v(t,0)=0, & t>0,\end{cases} \tag{8}$$

$$\begin{cases} w_t=w_{yy}+(c+H_c'(t))w_y+w(1-w), & t>0,\ -H_c(t)<y<0,\\ w(t,-H_c(t))=w(t,0)=0, & t>0,\\ H_c'(t)=-\mu w_y(t,0)-c, & t>0.\end{cases} \tag{9}$$

Now we take any sequence $\{t_n\}\subset\mathbb{R}$ satisfying $\lim_{n\to\infty}t_n=\infty$ and define

$$H_{c,n}(t);=H_c(t+t_n),\ v_n(t,z):=v(t+t_n,z),\ w_n(t,y):=w(t+t_n,y).$$

From (8), (9), we have

$$\begin{cases} \dfrac{\partial v_n}{\partial t}=\dfrac{\partial^2 v_n}{\partial z^2}+c\dfrac{\partial v_n}{\partial z}+v(1-v), & t>0, 0<z<H_{c,n}(t),\\ v_n(t,0)=0, & t>0,\end{cases} \tag{10}$$

$$\begin{cases} \dfrac{\partial w_n}{\partial t}=\dfrac{\partial^2 w_n}{\partial y^2}+(c+H_{c,n}'(t))\dfrac{\partial w_n}{\partial y}+w(1-w), & t>-t_n, -H_{c,n}(t)<y<0,\\ w_n(t,-H_{c,n}(t))=w_n(t,0)=0, & t>-t_n,\\ H_{c,n}'(t)=-\mu\dfrac{\partial w_n}{\partial y}(t,0)-c, & t>-t_n.\end{cases} \tag{11}$$

We first examine (11). By Proposition 2.2, $\|w_n\|_\infty$ and $\|H_{c,n}'\|_\infty$ are bounded, so we can apply the parabolic $L^p$ estimates, the Sobolev embedding theorem and the Schauder estimates to deduce that $\{w_n\}$ is bounded in $C^{1+\frac{\alpha}{2},2+\alpha}([-R,R]\times[-H_c^*+\frac{1}{R},0])$ for any $R>0$ and $0<\alpha<1$. Hence $H_{c,n}'$ is uniformly bounded in $C^\alpha(I)$ for any bounded interval $I\subset\mathbb{R}$, and then by passing to a subsequence, which is still denoted by $\{t_n\}$, we have

$$H_{c,n}'\to\tilde{H}_c\ \text{ in }\ C^{\alpha'}_{\mathrm{loc}}(\mathbb{R})\ \text{ as }\ n\to\infty$$

for some function $\tilde{H}$ and any $\alpha'\in(0,\alpha/2)$. By passing to a further subsequence, we have

$$w_n\to\hat{w}\ \text{ in }\ C^{1+\frac{\alpha'}{2},2+\alpha'}_{\mathrm{loc}}(\mathbb{R}\times(-H_c^*,0])\ \text{ as }\ n\to\infty$$

and $\hat{w}$ satisfies

$$\begin{cases} \hat{w}_t = \hat{w}_{yy} + (\tilde{H}_c + c)\hat{w}_y + \hat{w}(1-\hat{w}), & t \in \mathbb{R}, \ -H_c^* < y < 0, \\ \hat{w}(t,0) = 0, & t \in \mathbb{R}, \\ \tilde{H}_c(t) = -\mu \hat{w}_y(t,0) - c, & t \in \mathbb{R}. \end{cases}$$

Moreover, since $\lim_{t\to\infty} H_c(t)$ exists, we can deduce that $\tilde{H}(t) \equiv 0$ for all $t \in \mathbb{R}$ and that $\hat{w}$ satisfies

$$\begin{cases} \hat{w}_t = \hat{w}_{yy} + c\hat{w}_y + \hat{w}(1-\hat{w}), & t \in \mathbb{R}, -H_c^* < y < 0, \\ \hat{w}(t,0) = 0, & t \in \mathbb{R}, \\ \hat{w}_y(t,0) = -\dfrac{c}{\mu}, & t \in \mathbb{R}. \end{cases}$$

Similarly as for $w_n$, we can show that

$$v_n \to \hat{v} \ \text{ in } \ C^{1+\frac{\alpha'}{2}, 2+\alpha'}(\Omega_0)$$

where $\Omega_0 := \{(t,z) : t \in \mathbb{R}, \ z \in [0, H_c^*)\}$ and $\hat{v}$ satisfies

$$\hat{v}_t = \hat{v}_{zz} + c\hat{v}_z + \hat{v}(1-\hat{v}) \ \text{ in } \ \Omega_0.$$

From the relation $v_n(t,z) = w_n(t, z - H_{c,n}(t))$, we have

$$\hat{v}(t,z) = \hat{w}(t, z - H_c^*) \text{ for } \ 0 < z < H_c^*. \tag{12}$$

Since $\hat{v}(t,0) = 0$, we can easily see that

$$\lim_{y\to -H_c^*} \hat{w}(t,y) = \lim_{y\to -H_c^*} \hat{v}(t, y + H_c^*) = 0.$$

So we have $\hat{w} \in C^{1,2}(\mathbb{R} \times [-H_c^*, 0])$ and

$$\begin{cases} \hat{w}_t = \hat{w}_{yy} + c\hat{w}_y + \hat{w}(1-\hat{w}), & t \in \mathbb{R}, -H_c^* < y < 0, \\ \hat{w}(t, -H_c^*) = \hat{w}(t,0) = 0, & t \in \mathbb{R}, \\ \hat{w}_y(t,0) = -\dfrac{c}{\mu}. \end{cases} \tag{13}$$

By the strong maximum principle, we also have $\hat{w}(t,y) > 0$ for $t \in \mathbb{R}$ and $y \in (-H_c^*, 0)$.

Now we define $\eta(t,y) = \hat{w}(t,y) - \mathcal{V}_c(y + L_c)$. Clearly $\eta$ satisfies

$$\begin{aligned} &\eta_t = \eta_{yy} + c\eta_y + m(t,y)\eta, \ t \in \mathbb{R}, \ y \in [-H_c^*, 0], \\ &\eta(t, -H_c^*) < 0, \ \eta(t,0) = 0 \end{aligned}$$

for some bounded function $m(t,y)$. Therefore we can use the zero number result of Angenent [1] to conclude that, for any $t \in \mathbb{R}$, the number of zeros of $\eta(t,\cdot)$ in $[-H_c^*, 0]$, say $\mathcal{Z}_{[-H_c^*,0]}(t)$, is finite and nonincreasing in $t$, and if $\eta(t_0,\cdot)$ has a degenerate zero in $[-H_c^*,0]$ for some $t_0\in\mathbb{R}$, then for any $s<t_0<t$ we have

$$\mathcal{Z}_{[-H_c^*,0]}(t)\le\mathcal{Z}_{[-H_c^*,0]}(s)-1.$$

Since $\mathcal{Z}_{[-H_c^*,0]}(t)<\infty$, it follows that there may be at most finitely many value of $t$ such that $\eta(t,\cdot)$ has a degenerate zero. However $\eta$ satisfies

$$\eta_y(t,0)=\hat{w}_y(t,0)-V_c'(L_c)=0,$$

so $\eta(t,\cdot)$ has degenerate zero $y=0$ for any $t\in\mathbb{R}$. This is contradiction. Thus we have $L_c\le H_c^*$.

**Step 2.** Suppose that $L_c<H_c^*$. Arguing as in Step 1, we obtain $\hat{w}$ satisfying (13) and $\hat{w}(t,y)>0$ for $t\in\mathbb{R}$ and $y\in(-H_c^*,0)$. Noting that $L_c<H_c^*$, we consider $\eta(t,y)$ on $\{(t,y):t\in\mathbb{R},y\in[-L_c,0]\}$. Then we have $\eta(t,-L_c)>0$ and we can obtain a contradiction by similar zero number argument to Step 1.

**Step 3.** As in Steps 1 and 2, we obtain that for any $\alpha\in(0,1)$, there exist a subsequence of $\{t_n\}$, functions $\hat{w}$ and $\hat{v}$ such that

$$\begin{aligned}&H_{c,n}'\to 0\ \text{ in }\ C^{\alpha}_{\rm loc}(\mathbb{R}),\\ &w_n\to\hat{w}\ \text{ in }\ C^{1+\frac{\alpha}{2},2+\alpha}_{\rm loc}(\mathbb{R}\times(-L_c,0]),\\ &v_n\to\hat{v}\ \text{ in }\ C^{1+\frac{\alpha}{2},2+\alpha}_{\rm loc}(\mathbb{R}\times[0,L_c)),\end{aligned}$$

along the subsequence, and $\hat{v}$ and $\hat{w}$ satisfies

$$\begin{cases}\hat{v}_t=\hat{v}_{zz}+c\hat{v}_z+\hat{v}(1-\hat{v}), & t\in\mathbb{R},\ z\in[0,L_c),\\ \hat{v}(t,0)=0 & t\in\mathbb{R},\end{cases}$$
$$\begin{cases}\hat{w}_t=\hat{w}_{yy}+c\hat{w}_y+\hat{w}(1-\hat{w}), & t\in\mathbb{R},\ y\in(-L_c,0],\\ \hat{w}(t,-L_c)=\hat{w}(t,0)=0, & t\in\mathbb{R},\\ \hat{w}_y(t,0)=-\dfrac{c}{\mu}, & t\in\mathbb{R}.\end{cases}$$

From same zero number argument as in Step 1, we can conclude that $\hat{w}(t,y)\equiv\mathcal{V}_c(y+L_c)$. From (12) with $H_c^*=L_c$, we also have $\hat{v}(t,z)\equiv\mathcal{V}_c(z)$ on $\mathbb{R}\times[0,L_c)$.

Since $(L_c,\mathcal{V}_c)$ is uniquely determined by (3) and thus does not depend on any subsequence of $\{t_n\}$, we can conclude that

$$\lim_{t\to\infty}\left\{\sup_{y\in[-L,0]}|w(t,y)-\mathcal{V}_c(y+L_c)|\right\}=0,\tag{14}$$

$$\lim_{t\to\infty}\left\{\sup_{z\in[0,L]}|v(t,z)-\mathcal{V}_c(z)|\right\}=0\tag{15}$$

holds for any $L\in(0,L_c)$. From (14) and (15), we obtain (7). □

From Lemma 3.2, Lemma 3.4 and Proposition 3.6, the assertions of Proposition 3.1 follows. Now we have completed the proof Theorem A.

For the proof Theorem B, please see section 5 of [12].

Now I will give the sketch of proof of Theorem C.

*Scketch of proof of Theorem C.* From Lemma 3.3, it is easy to see that if $c^* < c$, then $T_{\max}$ must be finite.

Now we assume that $c = c^*$. Suppose that $T_{\max} = \infty$.

**Step 1**: Let $H(t) := h(t) - c^*t$. By investigating the zero number of $\eta(t,z) = u(t, z + c^*t) - q^*(z - b)$ for any $b \in \mathbb{R}$ as in [11, Lemma 3.7], we can show that $H_\infty := \lim_{t\to\infty} H(t)$ exists.

**Step 2**: Take any sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ and let $H_n(t) := H(t + t_n)$, $v_n(t,z) = u(t + t_n, z + c^*(t + t_n))$ and $w_n(t,y) = u(t + t_n, y + H(t + t_n))$. Then by the same argument in the proof of Proposition 3.7 we can obtain that

$$\begin{aligned}
&H_n' \to 0 \ \text{ as } \ n \to \infty \ \text{ in } \ C^{\alpha}_{\mathrm{loc}}(\mathbb{R}),\\
&w_n \to \hat{w} \ \text{ as } \ n \to \infty \ \text{ in } \ C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}(\mathbb{R} \times (-H_\infty, 0]),\\
&v_n \to \hat{v} \ \text{ as } \ n \to \infty \ \text{ in } \ C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}(\mathbb{R} \times [0, H_\infty)),
\end{aligned}$$

along a subsequence of $\{t_n\}$ and then $\hat{w}$ and $\hat{v}$ satisfy

$$\begin{cases}
\hat{w}_t = \hat{w}_{yy} + c^*\hat{w}_y + \hat{w}(1 - \hat{w}), & t \in \mathbb{R}, -H_\infty < z < 0.\\
\hat{w}(t, 0) = 0, & t \in \mathbb{R},\\
\hat{w}_y(t, 0) = -\dfrac{c^*}{\mu}, & t \in \mathbb{R},
\end{cases}$$

and

$$\hat{v}_t = \hat{v}_{zz} + c^*\hat{v}_z + \hat{v}(1 - \hat{v}), \quad t \in \mathbb{R},\ 0 < z < H_\infty.$$

By relation $v_n(t, y + H_n(t)) = w_n(t,y)$, we have $\hat{v}(t, y + H_\infty) = \hat{w}(t,y)$ for $t \in \mathbb{R}$ and $y \in (-H_\infty, 0)$ and

$$\lim_{y\to -H_\infty} \hat{w}(t,y) = \lim_{y\to -H_\infty} \hat{v}(t, y + H_\infty) = 0.$$

Thus $\hat{w} \in C^{1,2}(\mathbb{R} \times [-H_\infty, 0])$ and

$$\begin{cases}
\hat{w}_t = \hat{w}_{yy} + c^*\hat{w}_y + \hat{w}(1 - \hat{w}), & t \in \mathbb{R}, -H_\infty < z < 0.\\
\hat{w}(t, -H_\infty) = \hat{w}(t, 0) = 0, & t \in \mathbb{R},\\
\hat{w}_y(t, 0) = -\dfrac{c^*}{\mu}, & t \in \mathbb{R}.
\end{cases}$$

Define $\tilde{\eta}(t,y) = \hat{w}(t,y) - q^*(y)$. By the same zero number argument as in Step 3 of Proposition 3.7 we can see that $\hat{w}(t,y) \equiv q^*(y)$. This is the contradiction to $w(t, -H_\infty) = 0$. The proof of Theorem C have been completed. □

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