数理解析研究所講究録
第2066巻 2018年 46-58

Fractional Cahn-Hilliard equation

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Abstract

In this note, we review a recent joint work with G. Schimperna and A. Segatti (University of Pavia, Italy) published in the paper [1], where a fractional variant of the Cahn-Hilliard equation is introduced and existence and uniqueness of solutions are proved. Moreover, some limiting problems related to fractional porous medium/fast diffusion and Allen-Cahn equations are discussed.

This note is based on a joint work [1] with Giulio Schimperna and Antonio Segatti (University of Pavia, Italy).

1 Introduction

The Cahn-Hilliard equation is a well-studied *phase separation* (or *spinodal decomposition*) model and it was originally proposed by J.W. Cahn and J.E. Hilliard [3] for describing a flat interface between two coexisting phases. More precisely, the standard Cahn-Hilliard equation reads,

$$\partial_t c = D\Delta\mu, \quad \mu = -\Delta c + W'(c), \tag{1}$$

where $c = c(x,t)$ and $\mu = \mu(x,t)$ stand for a phase parameter (or concentration of fluid/material) and a chemical potential, respectively, $\partial_t = \partial/\partial t$, Δ is the classical Laplace operator, W' is the derivative of a double-well potential, e.g., $W'(c) = c^3 - c$, and D is a diffusion coefficient. It is well known that the total mass $\int_\Omega c(x,t)\,\mathrm{d}x$ is conserved under the evolution of $c(x,t)$ by (1) along with the homogeneous Neumann boundary condition for μ (cf. as for the Allen-Cahn equation, the total mass is not conservative). On the other hand, the Cahn-Hilliard equation is also studied under Dirichlet boundary conditions (see, e.g., [2] and [9]), where total mass is not conserved.

In [1], a variant of the Cahn-Hilliard equation is proposed by replacing the classical Laplacian with the so-called *fractional Laplacian*, which is originally defined as a pseudodifferential operator (or Fourier multiplier operator),

$$(-\Delta)^s u(x) := \mathcal{F}^{-1}\left[|\xi|^{2s}\widehat{u}(\xi)\right](x), \quad 0 < s < 1, \tag{2}$$

where $\widehat{u}(\xi)$ is the Fourier transform of $u : \mathbb{R}^N \to \mathbb{R}$ and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Moreover, $(-\Delta)^s$ is formulated in terms of a singular integral and such an integral formulation is more convenient to consider boundary value problems. One of important features of the fractional Laplacian (given by (2)) lies on a probabilistic interpretation which characterizes $(-\Delta)^s$ as a generator of a jump process such as Lévy flight. In the context of phase separation model, the fractional Laplacian enables us to describe long interaction processes. Indeed, in a microscopic view, it is characterized as a random motion of particles with jump, and in a macroscopic view, it is formulated as a nonlocal operator defined by a singular integral. In this note, we particularly focus on the Dirichlet problem on a bounded domain $\Omega \subset \mathbb{R}^N$ as a typical setting of open systems. In contrast with the classical Laplacian, which is characterized by a Brownian motion, one should take account of the outside of domain and impose the following Dirichlet condition:

$$u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega \tag{3}$$

instead of a usual one, i.e., $u = 0$ on $\partial\Omega$, since in a microscopic view, a particle may not stop on the boundary and may directly jump to the outside of the system (moreover, it is also related to the fact that the trace operator is not generally well defined in H^s for $s \leq 1/2$). Here we emphasize that the fractional Laplacian given by (2) equipped with (3) does not coincide with a fractional power A^s of the Dirichlet Laplacian $A = (-\Delta)^{\mathrm{D}}_{\Omega}$ (see Remark 4.3 of §4 below).

On the other hand, to the best of the author's knowledge, the corresponding Neumann problem has not yet been well studied; indeed, it is more unclear which boundary conditions for the fractional Laplacian given by (2) is proper. One may also consider the fractional power A^s of the Neumann Laplacian $A = (-\Delta)^{\mathrm{N}}_{\Omega}$; however, it may have a different representation from (2) (cf. Neumann type boundary problems are recently studied exactly for the fractional Laplacian (2) in the frame of Hörmander's theory of μ-transmission pseudodifferential operators [6, 7]. Moreover, in [5] a nonlocal normal derivative operator is introduced to consider a Neumann-type problem for $(-\Delta)^s$ (with a probabilistic interpretation). However, in this note, we shall not touch this direction).

Therefore, we formulate a *fractional Cahn-Hilliard equation* equipped with the Dirichlet condition (3) as follows:

$$\partial_t u + (-\Delta)^s w = 0 \quad \text{in } \Omega \times (0, +\infty), \tag{4}$$

$$w = (-\Delta)^\sigma u + W'(u) \quad \text{in } \Omega \times (0, +\infty), \tag{5}$$

$$u(x, 0) = u_0(x) \quad \text{in } \Omega, \tag{6}$$

$$u = w = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \tag{7}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $(-\Delta)^s$ and $(-\Delta)^\sigma$ are fractional Laplace operators with $0 < s, \sigma < 1$ and u_0 is a given initial data. For simplicity, we shall treat only a double-well potential of power type,

$$W(u) := \frac{1}{p}|u|^p - \frac{1}{2}u^2, \quad 1 < p < \infty, \quad p \neq 2.$$

We note that two relations (4) and (5) hold only on Ω but not on the outside of Ω. On the other hand, the fractional Laplacian is a nonlocal operator and it also depends on the values of operands (unknowns) on the outside of Ω. Hence one cannot combine two equations (at least in a strong formulation) into a single one, say

$$\partial_t u + (-\Delta)^s \left((-\Delta)^\sigma u + W'(u)\right) = 0 \quad \text{in } \Omega \times (0, +\infty),$$

in contrast with the classical Cahn-Hilliard equation (1). Furthermore, we also consider the case $s \neq \sigma$; indeed, such a situation must occur when we shall consider limiting problems of solutions for (4)–(7) as s or σ goes to zero. Then a new difficulty arises from the difference of two fractional powers (see §3).

In the next section, we shall review equivalent formulations of the fractional Laplacian (2), in particular, we focus on variational formulations of (2) in terms of singular integrals, which are well adapted to the Dirichlet problem. In Section 3, we shall discuss existence and uniqueness of solutions to (4)–(7). Section 4 is devoted to handling limiting problems of solutions to (4)–(7) as either s or σ approaches to zero. Then we shall exhibit relations of the fractional CH equation to fractional Porous Medium/Fast Diffusion and Allen-Cahn equations.

2 Variational view of fractional Laplacians

In this section we summarize variational reformulations of the fractional Laplace operator $(-\Delta)^s$ defined by (2) and introduce a weak formulation for $(-\Delta)^s$ equipped with the Dirichlet condition (3). As for further details, we refer the reader to Hitchhicker's guide [10].

For $s \in (0,1)$ and $u \in \mathcal{S}(\mathbb{R}^N)$, where $\mathcal{S}(\mathbb{R}^N)$ is the Schwartz class of rapidly decaying functions at infinity, the fractional Laplace operator $(-\Delta)^s$ is reformulated by

$$(-\Delta)^s u(x) = C(N,s)\, p.v. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, \mathrm{d}y, \tag{8}$$

where *p.v.* means the *Cauchy principal value*, that is,

$$p.v. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, \mathrm{d}y = \lim_{\varepsilon \to 0_+} \int_{|x-y|>\varepsilon} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, \mathrm{d}y$$

and $C(N,s)$ is a constant given by

$$C(N,s) := \left(\int_{\mathbb{R}^N} \frac{1 - \cos(\zeta_1)}{|\zeta|^{N+2s}}\, \mathrm{d}\zeta \right)^{-1} = \frac{4^s \Gamma(\frac{N}{2} + s)}{\pi^{N/2} |\Gamma(-s)|}.$$

Here $\Gamma(\cdot)$ stands for the Gamma function. Moreover, it is known that

$$\lim_{s \to 0_+} \frac{C(N,s)}{s(1-s)} = \frac{2}{\omega_{N-1}}, \tag{9}$$

where ω_{N-1} denotes the $(N-1)$-dimensional measure of the unit sphere $\mathbb{S}^{N-1}$. This fact will be used in §4 (see Lemma 4.2). Furthermore, one can equivalently rewritten (8) as

$$(-\Delta)^s u(x) = -\frac{1}{2}C(N,s)\int_{\mathbb{R}^N} \frac{u(x+h)+u(x-h)-2u(x)}{|h|^{N+2s}}\,\mathrm{d}h$$

(see also [8] for ten equivalent definitions of fractional Laplacian).

For smooth functions u, v, one can observe, at least formally, that

$$\begin{aligned}\int_{\mathbb{R}^N} (-\Delta)^s u(x) v(x)\,\mathrm{d}x &= C(N,s)\iint_{\mathbb{R}^{2N}} \frac{u(x)-u(y)}{|x-y|^{N+2s}} v(x)\,\mathrm{d}x\mathrm{d}y\\ &= \frac{C(N,s)}{2}\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\end{aligned}$$

(see, e.g., [1] for a rigorous argument). Hence we regard the relation above as a weak formulation of (2), and moreover, we note that the right-hand side of the weak form is well defined at least for

$$\begin{aligned}u, v \in H^s(\mathbb{R}^N) &:= \left\{ z \in L^2(\mathbb{R}^N) \colon \frac{|z(x)-z(y)|}{|x-y|^{\frac{N}{2}+s}} \in L^2(\mathbb{R}^N_x \times \mathbb{R}^N_y) \right\}\\ &= \left\{ z \in L^2(\mathbb{R}^N) \colon (1+|\xi|^2)^{s/2}\widehat{z}(\xi) \in L^2(\mathbb{R}^N_\xi) \right\}\end{aligned}$$

(see, e.g., [10, Proposition 3.4]), where $(-\Delta)^{s/2}u$ belongs to $L^2(\mathbb{R}^N)$, and hence, we have

$$\begin{aligned}\frac{C(N,s)}{2}\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}^N_\xi} |\xi|^s\widehat{u}(\xi)\overline{|\xi|^s\widehat{v}(\xi)}\,\mathrm{d}\xi\\ &= \left((-\Delta)^{s/2}u, (-\Delta)^{s/2}v\right)_{L^2(\mathbb{R}^N)}.\end{aligned}$$

Now, let us combine the weak form above with the Dirichlet boundary condition (3). To this end, we set

$$\mathcal{X}_{s,0} := \left\{ v \in H^s(\mathbb{R}^N) \colon v = 0 \ \text{ in } \mathbb{R}^N \setminus \Omega \right\}$$

and define a map $\mathfrak{A}_s$ from $\mathcal{X}_{s,0}$ to the dual space $\mathcal{X}_{s,0}'$ (of $\mathcal{X}_{s,0}$) as a weak form of $(-\Delta)^s$ by

$$\langle \mathfrak{A}_s u, v\rangle_{\mathcal{X}_{s,0}} := \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y$$

for $u, v \in \mathcal{X}_{s,0}$ (this weak formulation was proposed by Servadei and Valdinoci [13]). Then we note that $\mathfrak{A}_s$ has a variational structure,

$$\mathfrak{A}_s v = J_s'(v),$$

where J_s' is the Fréchet derivative of the (smooth convex) functional $J_s : \mathcal{X}_{s,0} \to \mathbb{R}$ given by

$$J_s(v) := \frac{C(N,s)}{4}\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \quad \text{for } v \in \mathcal{X}_{s,0}.$$

Here we remark that J_s corresponds to the Dirichlet integral for the classical Laplacian. Then one can find out a variational structure in the right-hand side of (5),

$$w = \mathbb{E}'_\sigma(u),$$

where $\mathbb{E}_\sigma$ is the Fréchet derivative of the functional $\mathbb{E}_\sigma : \mathcal{E}_\sigma \to \mathbb{R}$ defined on the Banach space $\mathcal{E}_\sigma := \mathcal{X}_{\sigma,0} \cap L^p_0(\mathbb{R}^N)$ by

$$\mathbb{E}_\sigma(v) := J_\sigma(v) + \int_\Omega g(v)\,\mathrm{d}x \quad \text{for } v \in \mathcal{E}_\sigma.$$

Here $L^p_0(\mathbb{R}^N) := \{v \in L^p(\mathbb{R}^N) \colon v = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$, which can be identified with $L^p(\Omega)$.

In order to introduce a weak formulation of (4)–(7), we introduce the following Gel'fand triplet:

$$\mathcal{X}_{\sigma,0} \hookrightarrow L^2_0(\mathbb{R}^N) \simeq L^2_0(\mathbb{R}^N)' \hookrightarrow \mathcal{X}'_{\sigma,0}$$

by identifying the Hilbert space $L^2_0(\mathbb{R}^N)$ with its dual space $L^2_0(\mathbb{R}^N)'$. Then (4)–(7) is reduced to the following Cauchy problem:

$$\partial_t u(t) + \mathfrak{A}_s w(t) = 0 \text{ in } \mathcal{X}'_{s,0}, \quad w(t) = \mathbb{E}'_\sigma(u(t)) \text{ in } \mathcal{E}'_\sigma, \quad 0 < t < \infty,$$
$$u|_{t=0} = u_0.$$

3 Existence and uniqueness of solutions

We are concerned with *weak solutions* of (4)–(7) in the following sense:

Definition 3.1. *A pair (u, w) is called a* weak solution *to the Cauchy-Dirichlet problem* (4)-(7) *for the fractional Cahn-Hilliard system if, for all $T > 0$, it holds that*

$$u \in C_w([0,T]; \mathcal{E}_\sigma) \cap C([0,T]; L^2_0(\mathbb{R}^N)) \cap W^{1,2}(0,T; \mathcal{X}'_{s,0}), \tag{10}$$
$$w \in L^2(0,T; \mathcal{X}_{s,0}), \tag{11}$$

where C_w means the class of weakly continuous functions; moreover, the couple (u, w) satisfies the following weak formulation of (4)-(5):

$$\partial_t u + \mathfrak{A}_s w = 0 \qquad \text{in } \mathcal{X}'_{s,0} \text{ a.e. in } (0,T), \tag{12}$$
$$w = \mathfrak{A}_\sigma u + B(u) - u \quad \text{in } \mathcal{E}'_\sigma \text{ a.e. in } (0,T), \tag{13}$$

where $B(u)$ denotes the bounded linear functional on $L^p(\mathbb{R}^N)$ defined by

$$\beta(u) := |u|^{p-2}u, \quad \langle B(u), v\rangle_{L^p(\mathbb{R}^N)} = \int_\Omega \beta(u(x))v(x)\,\mathrm{d}x \quad \text{for } v \in L^p(\mathbb{R}^N),$$

and finally, the initial condition (6) *holds in the following sense:*

$$u(t) \to u_0 \quad \text{strongly in } L^2_0(\mathbb{R}^N) \text{ and weakly in } \mathcal{E}_\sigma \text{ as } t \searrow 0. \tag{14}$$

Our result reads,

Theorem 3.1 (Existence and uniqueness of solutions, [1]). *Let $s, \sigma \in (0,1)$, $p \in (1,\infty) \setminus \{2\}$ and $u_0 \in \mathcal{E}_\sigma$. Then, the fractional Cahn-Hilliard system (4)-(7) admits a unique weak solution (u, w) in the sense of Definition 3.1, which additionally satisfies*

$$\|\beta(u(\cdot,t))\|^2_{L^2_0(\mathbb{R}^N)} \le 2\left(\|w(t)\|^2_{L^2_0(\mathbb{R}^N)} + \|u(t)\|^2_{L^2_0(\mathbb{R}^N)}\right) \quad \text{for a.e. } t \in (0,T), \tag{15}$$

$$\beta(u) \in L^2(0,T; L^2_0(\mathbb{R}^N)). \tag{16}$$

Moreover, $u(t) := u(\cdot,t)$ and $\mathbb{E}_\sigma(u(t))$ are right-continuous on $[0,T)$ in the strong topology of $\mathcal{E}_\sigma$ and in $\mathbb{R}$, respectively, and the following energy inequality holds true:

$$\|w(t)\|^2_{\mathcal{X}_{s,0}} + \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_\sigma(u(t)) \le 0 \quad \text{for a.e. } t \in (0,T). \tag{17}$$

In particular, if $\sigma \ge s$, then $u \in C([0,T]; \mathcal{E}_\sigma)$ and $\mathbb{E}_\sigma(u(t))$ is absolutely continuous on $[0,T]$; moreover, the inequality (17) can be replaced by an equality, namely we have

$$\|w(t)\|^2_{\mathcal{X}_{s,0}} + \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_\sigma(u(t)) = 0 \quad \text{for a.e. } t \in (0,T).$$

The method of proof relies on the semi-discretization of the equations as well as a variational argument to construct discretized solutions. More precisely, we introduce the following discretized problems:

$$\frac{u_n - u_{n-1}}{\tau} + \mathfrak{A}_s w_n = 0 \qquad \text{in } \mathcal{X}'_{s,0}, \tag{18}$$

$$w_n = \mathfrak{A}_\sigma u_n + B(u_n) - u_{n-1} \quad \text{in } \mathcal{E}'_\sigma \tag{19}$$

for $n = 1, 2, \dots, m$ and $\tau := T/m > 0$. Then each discretized solution u_n can be obtained as a minimizer of the functional,

$$F_n(u) := \frac{\tau}{2}\left\|\frac{u - u_{n-1}}{\tau}\right\|^2_{\mathcal{X}'_{s,0}} + \frac{C(N,\sigma)}{4}\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\sigma}}\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \int_\Omega \hat{\beta}(u)\,\mathrm{d}x - \int_\Omega u_{n-1}u\,\mathrm{d}x \quad \text{for } u \in \mathcal{E}_\sigma,$$

where $\hat{\beta}$ denotes the primitive function of β, that is, $\hat{\beta}(u) = \frac{1}{p}|u|^p$. Since F_n is strictly convex, coercive and of class C^1 in $\mathcal{E}_\sigma$, one can obtain a unique minimizer $u_n \in \mathcal{E}_\sigma$ of F_n. Moreover, w_n is also obtained by (19). After deriving a priori estimates for piecewise linear and constant interpolants of discretized solutions, one proceeds to a limiting procedure, and then, by Minty's trick, one can identify the limit of the nonlinear term. We refer the reader to [1] for more details. Instead, let us here give a remark on a difficulty arising from the difference of fractional powers.

In order to establish energy inequalities, one may use chain-rules, which first ensure the differentiability of the composition $t \mapsto \mathbb{E}_\sigma(u(t))$ and then lead us to obtain energy inequalities. However, we here encounter a difficulty which stems from the difference of fractional powers s, σ. According to Definition 3.1, one has the regularity $u \in W^{1,2}(0,T; \mathcal{X}'_s) \cap C_w([0,T]; \mathcal{E}_\sigma)$. However, it is insufficient in view of a standard chain-rule, where regularity of u_t and $\mathbb{E}'_\sigma(u)$ is supposed to satisfy a "duality" in a proper sense. More precisely, in this case, we have only $u_t \in L^2(0,T;\mathcal{X}'_s)$ and $\mathbb{E}'_\sigma(u) \in L^2(0,T;\mathcal{E}'_\sigma)$, although one may expect that a duality pairing of u_t and $\mathbb{E}'_\sigma(u)$ is well defined. Here we can use the equation (5) to improve the regularity of u; indeed $w = \mathbb{E}'_\sigma(u)$ and w belongs to $L^2(0,T;\mathcal{X}_{s,0})$. Then the regularity of u_t and $\mathbb{E}'_\sigma(u)$ recovers the duality. On the other hand, $\mathbb{E}_\sigma$ may not be smooth (more precisely, not of class C^1) on $\mathcal{X}_{s,0}$ when $s < \sigma$ or $p > 2N/(N-2s)_+$. Hence (standard) chain-rules are still unavailable. Recalling chain-rules for non-smooth functionals, we observe that convexity may compensate non-smoothness of functionals. Hence rewrite (5) as

$$J'_\sigma(u) = w + u \in L^2(0,T;\mathcal{X}_{s,0}) + L^2(0,T;\mathcal{E}_\sigma),$$

where we recall that J_σ is convex. However, in order to retain a duality between $u_t \in L^2(0,T;\mathcal{X}'_s)$ and $J'_\sigma(u)$, we need an additional assumption, say $\sigma \geq s$ (then, $J'_\sigma(u) = w + u \in L^2(0,T;\mathcal{X}_{s,0})$, and hence, one can exploit a variant of chain-rules for non-smooth but convex functionals). On the other hand, we shall also focus on the case $s < \sigma$ particularly for a limit problem as $s \to 0_+$ (see Sect. 4).

In case $s < \sigma$, we derive approximate energy inequalities and pass to the limit as $m \to \infty$ to obtain integral forms of energy inequalities. Then we derive further regularity of solutions, e.g., differentiability and right-continuity of the energy $t \mapsto \mathbb{E}_\sigma(u(t))$, from the integral forms. Finally, differential forms of energy inequalities follow. In case $s \geq \sigma$, as mentioned above, one can obtain regularity and energy *identities* in an easier way,

$$\langle u_t(t), J_\sigma(u(t))\rangle_{\mathcal{X}_{s,0}} = \langle u_t(t), w(t) + u(t)\rangle_{\mathcal{X}_{s,0}} = \frac{\mathrm{d}}{\mathrm{d}t} J_\sigma(u(t)),$$

and hence,

$$\langle u_t(t), w(t)\rangle_{\mathcal{X}_{s,0}} = \frac{\mathrm{d}}{\mathrm{d}t} J_\sigma(u(t)) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\Omega)} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_\sigma(u(t)).$$

4 Limit problems as either s or σ goes to zero

This section is devoted to discussing asymptotic behavior of solutions for (4)–(7) as either s or σ goes to zero. Main results of this section are stated as follows:

Theorem 4.1 (Limiting behaviors as $\sigma \to 0$ and $s \to 0$, [1]). *The following* (i) *and* (ii) *hold*:

(i) **(From Cahn-Hilliard to Porous Medium)** *Let* $p \in (2,\infty)$, $s \in (0,1)$ *and let* $\{\sigma_k\} \subset (0,1)$ *be such that* $\sigma_k \searrow 0$ *as* $k \nearrow +\infty$. *Let* $\{u_{0,k}\}$ *and* $u_0 \in \mathcal{X}'_{s,0}$ *satisfy*

$$\sup_{k\in\mathbb{N}} \mathbb{E}_{\sigma_k}(u_{0,k}) < +\infty, \quad u_{0,k} \to u_0 \ \textit{in}\ \mathcal{X}'_{s,0}.$$

Let (u_k, w_k) *be a sequence of unique weak solutions to* (4)-(7) *with* $\sigma = \sigma_k$ *and initial datum* $u_{0,k}$. *Then, there exist a (non-relabeled) subsequence of* $\{k\}$ *and* *a pair of limit functions* (u, w) *such that*

$$\begin{aligned} u_k \to u \quad & \text{weakly star in } L^\infty(0,T; L^p_0(\mathbb{R}^N)), \\ & \text{strongly in } L^p(0,T; L^p_0(\mathbb{R}^N)) \cap C([0,T]; \mathcal{X}'_{s,0}), \\ & \text{weakly in } W^{1,2}(0,T; \mathcal{X}'_{s,0}), \\ w_k \to w \quad & \text{weakly in } L^2(0,T; \mathcal{X}_{s,0}). \end{aligned}$$

Moreover, $u \in C_w([0,T]; L^p_0(\mathbb{R}^N)) \cap W^{1,2}(0,T; \mathcal{X}'_{s,0})$, $\beta(u) = w \in L^2(0,T; \mathcal{X}_{s,0})$. *Furthermore,* u *is a (weak) solution to the fractional porous medium equation*

$$\partial_t u + \mathfrak{A}_s \beta(u) = 0 \quad \text{in } \mathcal{X}'_{s,0}, \quad \text{a.e. in } (0,T), \quad u|_{t=0} = u_0.$$

(ii) **(From Cahn-Hilliard to Allen-Cahn)** *Let* $p \in (1,\infty) \setminus \{2\}$, $\sigma \in (0,1)$ *and let* $\{s_k\} \subset (0,1)$ *be s.t.* $s_k \searrow 0$ *as* $k \nearrow +\infty$. *Let* $u_{0,k} \in \mathcal{X}_{\sigma,0}$ *and* $u_0 \in L^2_0(\mathbb{R}^N)$ *satisfy*

$$\sup_k \left(\mathbb{E}_\sigma(u_{0,k}) + \|u_{0,k}\|^2_{\mathcal{X}'_{s_k,0}} \right) < \infty, \quad u_{0,k} \to u_0 \quad \text{strongly in } L^2_0(\mathbb{R}^N).$$

Let (u_k, w_k) *be a sequence of unique weak solutions to* (4)-(7) *with* $s = s_k$ *and initial datum* $u_{0,k}$. *Then, there exist a (non-relabeled) subsequence of* $\{k\}$ *and a pair of limit functions* (u, w) *such that*

$$\begin{aligned} u_k \to u \quad & \text{weakly star in } L^\infty(0,T; \mathcal{E}_\sigma), \\ & \text{strongly in } C([0,T]; L^2_0(\mathbb{R}^N)), \\ & \text{weakly in } W^{1,2}(0,T; \mathcal{E}'_\sigma), \\ w_k \to w \quad & \text{weakly in } L^2(0,T; L^2_0(\mathbb{R}^N)), \\ (-\Delta)^{s_k} w_k \to w \quad & \text{weakly in } L^2(0,T; \mathcal{E}'_\sigma). \end{aligned}$$

Moreover, $u \in C_w([0,T]; \mathcal{E}_\sigma) \cap W^{1,2}(0,T; \mathcal{E}'_\sigma)$, $w \in L^2(0,T; L^2_0(\mathbb{R}^N))$, $w = (-\Delta)^\sigma u + |u|^{p-2}u - u$, *and* u *is a (weak) solution to the fractional Allen-Cahn equation*

$$\partial_t u + \mathfrak{A}_\sigma u + |u|^{p-2}u - u = 0 \quad \text{in } \mathcal{E}'_\sigma, \quad \text{a.e. in } (0,T), \quad u|_{t=0} = u_0.$$

Proofs of these limiting issues are based on convergence properties of graphs of fractional Laplace operators in a proper topology as their fractional powers approach to zero (see §5.2 of [1]) as well as uniform estimates (see §5.1 of [1]) along with standard compactness arguments, e.g. Ascoli's lemma. Particularly for a proof of (ii), to derive a uniform estimate for $\mathfrak{A}_s w_s$ (here w_s is the chemical potential of (4)–(7)) as $s \to 0_+$, we use the following:

Lemma 4.2 ([1]). *Set* $X = \mathcal{X}_{r,0}$ *or* $X = \mathcal{E}_r := \mathcal{X}_{r,0} \cap L^p(\mathbb{R}^N)$ *for a fixed constant* $r \in (s,1)$ *or* $X = H^1_0(\Omega)$. *Then* X *is continuously embedded in* $\mathcal{X}_{s,0}$ *uniformly for* $s \to 0$. *More precisely, there exists a constant* $C_0 > 0$ *independent of* $s \to 0$ *such that*

$$\|v\|_{\mathcal{X}_{s,0}} \le C_0 \|v\|_X \quad \text{for all } v \in X. \tag{20}$$

Furthermore, a dual embedding $\mathcal{X}_{s,0}' \hookrightarrow X'$ *also follows such that*

$$\|v\|_{X'} \le C_0 \|v\|_{\mathcal{X}_{s,0}'} \quad \textit{for all} \ \ v \in \mathcal{X}_{s,0}'. \tag{21}$$

Proof. Indeed, as in [10, Proof of Proposition 2.1], one can verify that, for all $v \in X$,

$$\begin{aligned}
&\frac{C(s)}{2} \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&= \frac{C(s)}{2} \int_{\mathbb{R}^N} \int_{\{x \in \mathbb{R}^N : \, |x-y|>1\}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&\quad + \frac{C(s)}{2} \int_{\mathbb{R}^N} \int_{\{x \in \mathbb{R}^N : \, |x-y|\le 1\}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&\le \frac{C(s)}{s} \omega_{N-1} \|v\|_{L^2(\Omega)}^2 + \begin{cases} \frac{C(s)}{C(r)} \|v\|_{\mathcal{X}_{r,0}}^2 & \text{for } \ X = \mathcal{X}_{r,0} \text{ or } \mathcal{E}_r, \\ \frac{\omega_{N-1}}{4} \frac{C(s)}{1-s} \|\nabla v\|_{L^2(\Omega)}^2 & \text{for } \ X = H_0^1(\Omega). \end{cases}
\end{aligned}$$

Finally, exploit the asymptotics (9) of $C(r)$ as $r \searrow 0$ to obtain (20). □

Indeed, applying this lemma with $r = s$ and $X = \mathcal{E}_\sigma$ (for $\sigma > s$), one can derive

$$\int_0^T \|\mathfrak{A}_s w_s(t)\|_{\mathcal{E}_\sigma'}^2 \, dt \le C$$

from a uniform estimate for w_s in $L^2(0,T;\mathcal{X}_{s,0})$ (equivalently, for $\mathfrak{A}_s w_s$ in $L^2(0,T;\mathcal{X}_{s,0}')$). On the other hand, as for (i), by using (5) together with uniform estimates (at least, in $L^2(0,T;L_0^2(\mathbb{R}^N))$) for u_σ, w_σ and $\beta(u_\sigma)$, where (u_σ, w_σ) is the unique solution of (4)–(7), we can directly verify

$$\int_0^T \|\mathfrak{A}_\sigma u_\sigma(t)\|_{L_0^2(\mathbb{R}^N)}^2 \, dt \le C.$$

One of most delicate points of proof lies on the identification for the limit of the nonlinear term $\beta(u_\sigma) = |u_\sigma|^{p-2} u_\sigma$ as $\sigma \to 0$ (for the case of (i)). Indeed, in this case, one has less information on the compactness of (u_σ), for we obtain only a uniform (in time and σ) L^p (in space) estimate for u_σ. Then one has only, up to a subsequence (still denoted by σ for simplicity),

$$\begin{aligned}
u_\sigma \to u \quad & \text{weakly star in } L^\infty(0,T;L_0^p(\mathbb{R}^N)), \\
& \text{weakly in } W^{1,2}(0,T;\mathcal{X}_{s,0}'), \\
& \text{strongly in } C([0,T];\mathcal{X}_{s,0}'), \\
w_\sigma \to w \quad & \text{weakly in } L^2(0,T;\mathcal{X}_{s,0}), \\
\mathfrak{A}_\sigma u_\sigma \to u \quad & \text{weakly in } L^2(0,T;L_0^2(\mathbb{R}^N)),
\end{aligned}$$

which also implies

$$\beta(u_\sigma) = |u_\sigma|^{p-2} u_\sigma \to \bar{\beta} \quad \text{weakly star in } L^\infty(0,T;L_0^{p'}(\mathbb{R}^N))$$

and $w = \bar{\beta}$ by (5). So in order to identify the limit $\bar{\beta}$ of $\beta(u_\sigma) = |u_\sigma|^{p-2}u_\sigma$, we exploit Minty's trick again and observe that

$$\limsup_{\sigma \to 0_+} \int_0^T \int_{\mathbb{R}^N} \beta(u_\sigma) u_\sigma \, \mathrm{d}x \, \mathrm{d}t \overset{(5)}{\leq} \limsup_{\sigma \to 0_+} \int_0^T \left(\|u_\sigma(t)\|^2_{L^2(\mathbb{R}^N)} - \|u_\sigma(t)\|^2_{\mathcal{X}_{\sigma,0}} \right) \mathrm{d}t + \int_0^T \int_{\mathbb{R}^N} wu \, \mathrm{d}x \, \mathrm{d}t.$$

Hence one can complete the argument, if the following relation is guaranteed:

$$\limsup_{\sigma \to 0_+} \int_0^T \left(\|u_\sigma(t)\|^2_{L^2(\mathbb{R}^N)} - \|u_\sigma(t)\|^2_{\mathcal{X}_{\sigma,0}} \right) \mathrm{d}t \leq \limsup_{\sigma \to 0_+} \left(\frac{1}{\lambda_1(\sigma)} - 1 \right) \|u_\sigma(t)\|^2_{\mathcal{X}_{\sigma,0}} \leq 0,$$

where $\lambda_1(\sigma)$ stands for the principal eigenvalue of $\mathfrak{A}_\sigma$. Since $\|u_\sigma\|_{\mathcal{X}_{\sigma,0}}$ is uniformly bounded, it suffices to show

$$\lambda_1(\sigma) \to 1 \quad \text{as } \sigma \to 0_+. \tag{22}$$

Asymptotic behavior of eigenvalues for the fractional Laplace operator $(-\Delta)^s$ as the fractional power s goes to zero has been studied so far, and it has already been known (see [4]) that, for any $j \geq 1$,

$$c\lambda_j(\Omega)^\sigma \leq \lambda_j(\sigma) \leq \lambda_j(\Omega)^\sigma,$$

where $\lambda_j(\Omega)$ is the j-th eigenvalue of the Dirichlet Laplacian $-\Delta_\Omega^{\mathrm{D}}$ on Ω, for some constant $0 < c \leq 1$.

Remark 4.3 (Difference between $(-\Delta)^s$ and a fractional power $(-\Delta_\Omega^{\mathrm{D}})^s$). In [14], Servadei and Valdinoci exhibit an explicit difference between the fractional Laplacian $(-\Delta)^s$ and the fractional power A^s of the self-adjoint operator $A = -\Delta_\Omega^{\mathrm{D}}$, which is the Laplacian posed on Ω equipped with the homogeneous Dirichlet condition. More precisely, they prove that

$$\lambda_1(s) < \lambda_1(\Omega)^s,$$

where $\lambda_1(\Omega)$ is the principal eigenvalue of A and λ_1^s is that of the fractional power A^s. Hence $(-\Delta)^s$ never coincides with A^s, provided that $\Omega \neq \mathbb{R}^N$ (cf. both two operators coincide each other when $\Omega = \mathbb{R}^N$). Moreover, the Dirichlet problem on the unit ball $B \subset \mathbb{R}^N$,

$$(-\Delta)^s u = 1 \ \text{ in } B, \quad u = 0 \ \text{ in } \mathbb{R}^N \setminus B,$$

has an explicit solution,

$$u(x) = \left(1 - |x|^2\right)^s \chi_B(x),$$

where χ_B denotes the characteristic function supported over B. Then one can check that $u \in C^s(\overline{B})$ and it is the optimal (Hölder) regularity of u up to the boundary (see also [11, 12]). This fact seems quite different from the boundary regularity of solutions for the classical Laplacian (i.e., $s = 1$) due to the Schauder theory (more precisely, u belongs to $C^m(\overline{B})$ for any $m \in \mathbb{N}$).

Hence it follows readily that $\limsup_{\sigma\to 0_+} \lambda_1(\sigma) \leq 1$. Here we are further required to check $\liminf_{\sigma\to 0_+} \lambda_1(\sigma) \geq 1$. In [1], (22) is verified by applying an argument developed in [15]. More precisely, let e_σ be a normalized principal eigenfunction of $\mathfrak{A}_\sigma$. Then

$$\lambda_1(\sigma) = \lambda_1(\sigma)\|e_\sigma\|^2_{L^2(\mathbb{R}^N)} = \langle \mathfrak{A}_\sigma e_\sigma, e_\sigma\rangle_{\mathcal{X}_{\sigma,0}} = \int_{\mathbb{R}^N_\xi} |\xi|^{2\sigma}|\widehat{e}_\sigma(\xi)|^2 \,\mathrm{d}\xi.$$

On the other hand, we also note by Hölder's inequality that

$$|\widehat{e}_\sigma(\xi)|^2 = \left|\frac{1}{(2\pi)^N}\int_\Omega e^{-\imath\xi\cdot x}u(x)\,\mathrm{d}x\right|^2 \leq \frac{|\Omega|}{(2\pi)^N}\|e_\sigma\|^2_{L^2_0(\mathbb{R}^N)} = \frac{|\Omega|}{(2\pi)^N}.$$

Here we prepare the following lemma:

Lemma 4.4. *Given $f \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}|x|^\alpha|f(x)|\,\mathrm{d}x < +\infty$, where $\alpha > 0$, there holds*

$$\|f\|_{L^1(\mathbb{R}^N)} \leq \kappa(N,\alpha)\big\||x|^\alpha f\big\|^{\frac{N}{N+\alpha}}_{L^1(\mathbb{R}^N)}\|f\|^{1-\frac{N}{N+\alpha}}_{L^\infty(\mathbb{R}^N)}, \tag{23}$$

where

$$\kappa(N,\alpha) = \alpha^{-\frac{\alpha}{\alpha+N}}(\alpha+N)N^{-\frac{N}{N+\alpha}}d^{\frac{\alpha}{N+\alpha}}$$

and $d = d(N)$ is the volume of the unit ball in $\mathbb{R}^N$.

Applying the lemma above with $f = |\widehat{e}_\sigma(\xi)|^2$ and $\alpha = 2\sigma$, one can deduce that

$$1 = \|\widehat{e}_\sigma\|^2_{L^2(\mathbb{R}^N)} \leq \kappa(N,2\sigma)\lambda_1(\sigma)^{\frac{N}{N+2\sigma}}\left(\frac{|\Omega|}{(2\pi)^N}\right)^{\frac{2\sigma}{N+2\sigma}},$$

which implies

$$\lambda_1(\sigma) \geq \kappa(N,2\sigma)^{-\frac{N+2\sigma}{N}}\left(\frac{(2\pi)^N}{|\Omega|}\right)^{\frac{2\sigma}{N}}.$$

Thus we conclude that

$$\liminf_{\sigma\to 0_+}\lambda_1(\sigma) \geq 1,$$

which along with the limsup inequality yields

$$\lim_{\sigma\to 0_+}\lambda_1(\sigma) = 1.$$

Finally, we give a remark on the fast diffusion case, namely, $1 < p < 2$. In this case, the energy functional E_σ may lose the coercivity as $\sigma \to 0$, and therefore, no estimate follows for u_σ. Here in order to avoid the lack of coercivity, we modify the double-well potential $W(u)$ appeared in (5) as follows:

$$w = (-\Delta)^\sigma u + \underbrace{|u|^{p-2}u - \lambda_1(\sigma)u}_{=\,W'(u)}. \tag{24}$$

Then the corresponding energy can recover the coercivity in $L^p(\mathbb{R}^N)$ (however, it is still noncoercive in $\mathcal{X}_{\sigma,0}$ or $\mathcal{E}_\sigma$). Thus we obtain

Theorem 4.5 (From Cahn-Hilliard to Fast Diffusion, [1]**).** *Let* $s \in (0,1)$, $p \in (2_*, 2)$, $2_* := 2N/(N+2s)$, *and let* $\{\sigma_k\}$ *be as before. Moreover, let* $\{u_{0,k}\}$ *be a sequence of initial data and* $u_0 \in \mathcal{X}'_{s,0}$ *such that*

$$\sup_{k\in\mathbb{N}} \tilde{\mathbb{E}}_{\sigma_k}(u_{0,k}) < +\infty, \quad u_{0,k} \to u_0 \quad \text{in } \mathcal{X}'_{s,0}. \tag{25}$$

Let (u_k, w_k) *be a sequence of unique weak solutions of* (4)–(7), *with* $\sigma = \sigma_k$, $W'(v) = \beta(v) - \lambda_1(\sigma_k)v$ *and initial data* $u_{0,k}$. *Then, there exist a (non-relabeled) subsequence of* $\{k\}$ *and a pair of limit functions* (u, w) *satisfying the same convergence and regularity as in* (i) *of Theorem 4.1. Moreover,* u *is a (weak) solution to the fractional fast-diffusion equation*

$$\partial_t u + \mathfrak{A}_s \beta(u) = 0 \quad \text{in } \mathcal{X}'_{s,0}, \quad \text{a.e. in } (0,T), \quad u|_{t=0} = u_0.$$

References

[1] G. Akagi, G. Schimperna, A. Segatti, Fractional Cahn-Hilliard, Allen-Cahn and porous medium equations, Journal of Differential Equations **261** (2016), no.6, 2935–2985.

[2] P.W. Bates, J. Han, The Dirichlet boundary problem for a nonlocal Cahn-Hilliard equation, J. Math. Anal. Appl. **311** (2005), 289–312.

[3] J.W. Cahn, J.E. Hilliard, Free energy of a nonuniform system. I. Interfacial free energy, J. Chem. Phys **28** (1958) 258-267.

[4] Z.-Q. Chen, R. Song, Two-sided eigenvalue estimates for subordinate processes in domains, J. Funct. Anal. **226** (2005) 90–113.

[5] S. Dipierro, X. Ros-Oton, E. Valdinoci, Nonlocal problems with Neumann boundary conditions, `arXiv:1407.3313v3` [math.AP]

[6] G. Grubb, Local and nonlocal boundary conditions for μ-transmission and fractional elliptic pseudodifferential operators, Anal. PDE **7** (2014), 1649–1682.

[7] G. Grubb, Fractional Laplacians on domains, a development of Hörmander's theory of μ-transmission pseudodifferential operators, Adv. Math. **268** (2015), 478–528.

[8] M. Kwaśnicki, Ten equivalent definitions of the fractional Laplace operator, `arXiv:1507.07356v2` [math.AP]

[9] Y. Li, D. Jeong, J. Shin, J. Kim, A conservative numerical method for the Cahn-Hilliard equation with Dirichlet boundary conditions in complex domains, Comput. Math. Appl. **65** (2013), 102–115.

[10] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math., **136** (2012), 521–573.

[11] X. Ros-Oton, J. Serra, The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, J. Math. Pures Appl. (9) **101** (2014), 275-302.

[12] X. Ros-Oton, J. Serra, The extremal solution for the fractional Laplacian, Calc. Var. Partial Differential Equations **50** (2014), 723–750.

[13] R. Servadei, E. Valdinoci, Variational methods for non-local operators of elliptic type, Discrete Contin. Dyn. Syst., **33** (2013), 2105–2137.

[14] R. Servadei and E. Valdinoci, On the spectrum of two different fractional operators, Proc. Roy. Soc. Edinburgh Sect. A, **144** (2014), 831–855.

[15] S.Y. Yolcu and T. Yolcu, Estimates for the sums of eigenvalues of the fractional Laplacian on a bounded domain, Commun. Contemp. Math., **15** (2013), 1250048, 15 pp.

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