数理解析研究所講究録
第2066巻 2018年 59-79



# On some flux saturated diffusion equations.

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January 31, 2017

**Abstract**

In this paper we review some aspects of the theory of flux saturated diffusion equations. After derivation of this type of equations and a summary of well-posedness results, the focus will be in some recent results about qualitative properties of solutions, including waiting time phenomena and creation of singularities.

Flux-saturated diffusion equations are a class of parabolic equations of the form

$$u_t = \operatorname{div} \mathbf{a}(u, \nabla u), \tag{0.1}$$

which have a hyperbolic scaling for large values of the modulus of the gradient, in the sense that

$$\frac{1}{\psi_0(\mathbf{v})} \lim_{t\to+\infty} \mathbf{a}(z, t\mathbf{v}) \cdot \mathbf{v} =: \varphi(z) \quad \text{for all } z \geq 0, \tag{0.2}$$

where $\psi_0 : \mathbb{R}^N \mapsto [0, +\infty)$ is a positively 1-homogeneous convex function, with $\psi_0(0) = 0$ and $\psi_0 > 0$ otherwise and $\varphi$ is a locally Lipschitz function with $\varphi(0) = 0$ and $\varphi(z) > 0$ if $z \neq 0$.

We will mainly consider the following three different equations:
The *porous medium relativistic heat equation*,

$$u_t = \nu \operatorname{div} \left( \frac{u^m \nabla u}{\sqrt{u^2 + \nu^2 c^{-2} |\nabla u|^2}} \right), \quad m \in (1, +\infty), \tag{PMRHE}$$

*the speed limited porous medium equation*

$$u_t = \nu \operatorname{div} \left( \frac{u \nabla u^{M-1}}{\sqrt{1 + \nu^2 c^{-2} |\nabla u^{M-1}|^2}} \right), \quad M \in (1, +\infty) \tag{SLPME}$$

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and the *nonlinear diffusion in transparent media*,

$$u_t = c\,\mathrm{div}\left(u^m \frac{\nabla u}{|\nabla u|}\right), \quad m \in \mathbb{R}. \tag{NDTM}$$

where $\nu > 0$ is a kinematic viscosity and $c > 0$ represents a characteristic limiting speed.

In Section 1 we recall two different derivations of these equations: a physical one developed by Ph. Rosenau and a different one which comes from the mass transportation theory, first pointed out by Y. Brennier. In Section 2 we collect some results about Equations (PMRHE) and (SLPME), including existence and uniqueness of entropy solutions, regularity of solutions, propagation of discontinuity fronts and propagation of the support and waiting time phenomena. Finally, in Section 3 we analyze Equation (NDTM). We state the existence and uniqueness of solutions for the Neumann problem and we show some qualitative properties of the solutions with some examples, such as creation of discontinuities in the interior of the support of solutions.

# 1 From classical diffusion to flux limited one

## 1.1 Physical derivation of (PMRHE).

The classical theory of heat conduction is based on Fourier's law ,

$$\mathbf{q}(t,x) = -\nu \nabla u(t,x), \tag{1.1}$$

which relates the heat flux $\mathbf{q}$ to the temperature $u$. Coupled with the conservation of energy,

$$u_t + \mathrm{div}\;\mathbf{q} = 0, \tag{1.2}$$

it yields the classical linear parabolic heat equation

$$u_t = \nu \Delta u. \tag{1.3}$$

As is easily seen, solutions to the Cauchy problem with datum $u_0$ with compact support become everywhere positive for arbitrary small $t > 0$, i.e., the propagation speed for such diffusion based models is infinite. In fact, one can rewrite equation (1.3) as a continuity equation:

$$u_t + \mathrm{div}(uV_u) = 0. \tag{1.4}$$

with the velocity $V_u$ given by

$$V_u = -\nu \frac{\nabla u}{u}. \tag{1.5}$$

According to (1.5) if $|\frac{\nabla u}{u}|$ diverges to $+\infty$ so will $V_u$. From this naive computation, one infers two important properties of the solutions:

(a) The speed of propagation of the support of solutions is infinite.

(b) Discontinuities ("infinite gradients ") are also propagated with infinite speed, which will give as a result that they are instantaneously smoothed.

However, in any realistic diffusion process, information, particles or individuals cannot travel faster than a fixed speed $c > 0$. This is clearly not the case of the heat equation, according to property (a). Therefore, the classical heat flux is not a realistic one as first criticized by A. Einstein in [25]. Even if it is true that the tail of the solutions become very small, there are some biological diffusive processes in which any amount of information produces some activation. Among these processes, one can find transport of morphogens [4] or chemotaxis [11].

Ph. Rosenau ([33]), in order to impose a macroscopic upper bound $c > 0$ on the allowed free speed, modified the velocity as follows:

$$V_u := \frac{\nu \frac{\nabla u}{u}}{\sqrt{1 + \nu^2 c^{-2} |\frac{\nabla u}{u}|^2}} \tag{1.6}$$

Observe that $|V_u| \leq c$. Substituting it in the continuity equation one obtains

$$u_t = \operatorname{div} \left( \frac{\nu u \nabla u}{\sqrt{u^2 + \nu^2 c^{-2} |\nabla u|^2}} \right); \tag{1.7}$$

i.e (PMRHE) with $m = 1$. Rosenau also observed, that in the case of the diffusion of heat in a neutral gas, then both the kinematic speed and the maximal speed $c$ do depend on the solution itself and that they satisfy $\nu \sim \nu_1 u^{\frac{1}{2}}$, $c \sim c_1 u^{\frac{1}{2}}$. In this case, conservation of energy gives Equation (PMRHE) with $m = \frac{3}{2}$. We finally mention that in [24], an alternative derivation of the saturated diffusion equation (1.7) is given.

## 1.2 Mass transport derivation of SLPME.

In this Section, Equation (SLPME) is obtained by Monge-Kantorovich's mass transport theory. Many mass conservative equations can be recast in the formalism of gradient flows with respect to the optimal transportation differential structure. This approach was first used by Jordan, Kinderlehrer and Otto [31] for the linear Fokker-Planck equation and it has been generalized to many well know equations. M. Agueh, in his PhD thesis [1], considered the general continuity equation (1.4) in the case that

$$V_u := -\nabla k^* [\nabla (F'(u))].$$

Here $k^*$ denotes the Legendre transform of the cost function $k : \mathbb{R}^N \to [0, \infty)$, that is,

$$k^*(z) = \sup_{x \in \mathbb{R}^N} \{x \cdot z - k(x)\}.$$

The cost functions considered in [1] are strictly convex and coercive, $0 = k(0) < k(z)$, for $z \neq 0$ and satisfying the growth

$$\beta|z|^q \leq k(z) \leq \alpha(|z|^q + 1), \quad \text{for } z \in \mathbb{R}^N, \text{ where } \alpha, \beta > 0, \ q > 1.$$

In particular, one recovers with this approach:

- *the heat equation* with $k(z) = \dfrac{|z|^2}{2}$, $z \in \mathbb{R}^N$, and $F(x) = \nu x \log(x)$ (the Boltzmann entropy).
- *the Porous Medium equation*

$$u_t = \nu \Delta u^M, \quad M > 1,$$

  where $k(z) = \dfrac{|z|^2}{2}$, $F(x) = \nu \frac{x^M}{M-1}$ (the Tsallis entropy).
- *the parabolic $p$-Laplacian equation*

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right),$$

  where $k(z) = \dfrac{|z|^{p'}}{p'}$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, $p > 1$, and $F(x) = \frac{x^M}{M(M-1)}$, $M = 1 + \frac{p-2}{p-1}$.

Equation (1.4) is interpreted as a "steepest decent" of the internal energy functional

$$\mathcal{P}_a(\mathbb{R}^N) \ni u \mapsto E(u) := \int_{\mathbb{R}^N} F(u(x))\, dx$$

with respect to the Monge-Kantorovich distance between $W_k^h$, where $h > 0$ a given time-step size, $\mathcal{P}_a(\mathbb{R}^N)$ denotes the set of all probability density functions $u : \mathbb{R}^N \to [0, \infty)$ and $W_k^h$ is defined by

$$W_k^h(u_0, u_1) := \left\{ \int_{\mathbb{R}^N \times \mathbb{R}^N} k\left(\frac{x-y}{h}\right) d\gamma(x, y) \ : \ \gamma \in \Gamma(u_0, u_1) \right\},$$

where $\Gamma(u_0, u_1)$ denotes the set of probability measures in $\mathbb{R}^N \times \mathbb{R}^N$, having $u_0$ and $u_1$ as marginals, i.e., for any Borel set $A \subset \mathbb{R}^N$,

$$\gamma(A \times \mathbb{R}^N) = u_0(A) \text{ and } \gamma(\mathbb{R}^N \times A) = u_1(A).$$

Given a mass density $u_{n-1}^h$ at time $t_{n-1} = (n-1)h$, one defines $u_n^h$ at time $t_n = nh$, as the unique minimizer of the following variational problem

$$(P_n^h) : \quad \inf_{u \in \mathcal{P}_a(\mathbb{R}^N)} \left\{ h W_k^h\left(u_{n-1}^h, u\right) + E(u) \right\}. \tag{1.8}$$

The corresponding Euler-Lagrange equation for $(P_n^h)$ is

$$\frac{u_n^h - u_{n-1}^h}{h} = \operatorname{div}\left\{u_n^h \nabla k^* \left[\nabla\left(F'(u_n^h)\right)\right]\right\} + A_n(h) \tag{1.9}$$

for $n \in \mathbb{N}$, with $A_n(h)$ converging to 0 as $h \to 0$. Then, one defines the approximate solution $u^h$ of (1.4), as the time-discrete function

$$\begin{cases} u(t,x) = u_n^h(x) & \text{if } t \in ((n-1)h, nh] \\ u^h(0,x) = u_0, \end{cases}$$

and deduces from (1.9) that $u^h$ satisfies

$$\begin{cases} (u^h)_t = \operatorname{div}\left\{u^h \nabla k^* \left[\nabla\left(F'(u^h)\right)\right]\right\} + A(h) & \text{in } (0,\infty)\times\mathbb{R}^N \\ u^h(0,x) = u_0(x) & x \in \mathbb{R}^N \end{cases} \tag{1.10}$$

weakly. Letting $h \to 0$ in (1.10) one then shows that the sequence $(u^h)_h$ converges to a function $u$, which solves (1.4) in the weak sense.

Y. Brenier in [14], observed that Equation (SLPME) with $M = 1$ can be formally obtained by taking the cost function as

$$k(z) := \begin{cases} c^2\left(1 - \sqrt{1 - \frac{|z|^2}{c^2}}\right) & \text{if } |z| \le c \\ +\infty & \text{if } |z| > c. \end{cases}, \tag{1.11}$$

coupled with the Boltzmann entropy. He also gave this equation its now well known name: the *relativistic heat equation*. If instead of choosing the Boltzmann entropy one chooses the Tsallis entropy then one formally obtains a rescaled version of (SLPME).

The program sketched above can be rigorously done for Equation (SLPME) assuming that the initial datum is strictly positive inside its support. This was done in [32] and it has been generalized to some other *relativistic costs* recently in [13].

# 2 Equations (PMRHE) and (SLPME): Well-posedness results and some qualitative properties.

## 2.1 Notation

For $a, b, \ell \in \mathbb{R}$ we consider the following set of truncations:

$$\mathcal{T}^+ = \{T_{a,b}^\ell : 0 < a < b,\ \ell \le a\}, \quad \text{where} \quad T_{a,b}^\ell(r) = \max\{\min\{b, r\}, a\} - \ell.$$

For a given function $T = T^{\ell}_{a,b} \in \mathcal{T}^+$, we denote with the superscript 0 its translation of a height $\ell$: that is, we let $T^0 := T + \ell = T^0_{a,b} \in \mathcal{T}^+$. For $f \in L^1_{loc}(\mathbb{R})$ we let

$$J_f(r) := \int_0^r f(s)\,\mathrm{d}s.$$

We use standard notations and concepts for $BV$ functions as in [2]; in particular, for $u \in BV(\mathbb{R}^N)$, $\nabla u \mathcal{L}^N$, resp. $D^s u$, denote the the absolutely continuous, resp. singular, parts of $Du$ with respect to the Lebesgue measure $\mathcal{L}^N$, $J_u$ denotes its jump set and we assume that $u^+(x) > u^-(x)$ for $x \in J_u$

## 2.2 Notion of solution. Well posedness

In this section we recall the notion of entropy solution to the Cauchy problem for the general Equation (0.1) first introduced in [5] and later extended in [7, 18, 20]. The flux $\mathbf{a}$ is assumed to satisfy the following

**Assumption 2.1.** Let $Q = (0, \infty) \times \mathbb{R}^N$. The function $\mathbf{a} : \overline{Q} \to \mathbb{R}^N$ is such that:
$(i)$ (*Lagrangian*) there exists $f \in C(\overline{Q})$ such that $\nabla_{\mathbf{v}} f = \mathbf{a} \in C(\overline{Q})$, $f(z, \cdot)$ is convex, $f(z, 0) = 0$ for all $z \in [0, \infty)$, and

$$C_0(z)|\mathbf{v}| - D_0(z) \le f(z, \mathbf{v}) \le M_0(z)(1 + |\mathbf{v}|) \quad \text{for all } (z, \mathbf{v}) \in Q$$

for continuous functions $0 \le M_0, C_0 \in C([0, \infty))$ and $0 \le D_0 \in C((0, \infty))$, with $C_0(z) > 0$ for $z > 0$;
$(ii)$ (*flux*) $D_{\mathbf{v}}\mathbf{a} \in C(\overline{Q})$; $\mathbf{a}(z, 0) = \mathbf{a}(0, \mathbf{v}) = 0$ and $h(z, \mathbf{v}) := \mathbf{a}(z, \mathbf{v}) \cdot \mathbf{v} = h(z, -\mathbf{v})$ for all $(z, \mathbf{v}) \in \overline{Q}$; for any $R > 0$ there exists $M_R > 0$ such that

$$|\mathbf{a}(z, \mathbf{v}) - \mathbf{a}(\hat{z}, \mathbf{v})| \le M_R |z - \hat{z}| \quad \text{for all } z, \hat{z} \in [0, R] \text{ and all } \mathbf{v} \in \mathbb{R}^N; \tag{2.1}$$

$(iii)$ (*recession functions*) the *recession functions* $f^0$ and $h^0$, defined by

$$f^0(z, \mathbf{v}) = \lim_{t \to +\infty} \frac{1}{t} f(z, t\mathbf{v}), \quad h^0(z, \mathbf{v}) = \lim_{t \to +\infty} \frac{1}{t} h(z, t\mathbf{v}),$$

exist in $\overline{Q}$; furthermore, a function $\varphi \in \mathrm{Lip}_{loc}([0, \infty))$ with $\varphi(0) = 0$ and $\varphi > 0$ in $(0, \infty)$ and a 1-homogeneous convex function $\psi_0 : \mathbb{R}^N \mapsto \mathbb{R}$ with $\psi_0(0) = 0$ and $\psi_0(\mathbf{v}) > 0$ for $\mathbf{v} \ne 0$ exist such that

$$f^0(z, \mathbf{v}) = h^0(z, \mathbf{v}) = \varphi(z)\psi_0(\mathbf{v}) \quad \text{for all } (z, \mathbf{v}) \in \overline{Q} \tag{2.2}$$

and

$$|\mathbf{a}(z, \mathbf{w}) \cdot \mathbf{v}| \le \varphi(z)\psi_0(\mathbf{v}) \text{ for all } (z, \mathbf{v}) \in Q,\ \mathbf{w} \in \mathbb{R}^N. \tag{2.3}$$

We note that in Equation PMRHE (resp. SLPME), $\varphi(z) = z^m$ and $\psi^0(\xi) = |\xi|$ (resp. $\varphi(z) = z$).

In the concept of solution there is an entropy inequality which follows from formally testing (0.1) by $\phi S(u)T(u)$ with $S, T \in \mathcal{T}^+$ and $0 \le \phi$. In particular, when constructing a solution as limit of solutions to suitable approximating problems, one needs to argue by lower semi-continuity on terms of the form

$$S(u)a(u, \nabla u) \cdot \nabla T(u) = S(T^0(u))h(T^0(u), \nabla T^0(u))$$

(see the discussion in [5, §2.2 and 3.2]). This leads to the following entropy inequality:

$$\begin{aligned} &\int_0^{+\infty} \langle h_S(u, DT(u)) + h_T(u, DS(u)), \phi\rangle \,\mathrm{d}t \\ &\quad \le \int_0^{+\infty} \int_{\mathbb{R}^N} \left(J_{TS}(u)\phi_t - T(u)S(u)\mathbf{a}(u, \nabla u) \cdot \nabla\phi\right) \,\mathrm{d}x\,\mathrm{d}t\,, \end{aligned} \tag{2.4}$$

where $h_S(u, DT(u))$ is the Radon measure defined by

$$\begin{aligned} \langle h_S(u, DT(u)), \phi\rangle &:= \int_{\mathbb{R}^N} \phi S(T^0(u))h(T^0(u), \nabla T^0(u))\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^N} \phi\psi_0\left(\frac{DT^0(u)}{|DT^0(u)|}\right) \mathrm{d}|D^s J_{S\varphi}(T^0(u))| \quad \text{for all } \phi \in C_c(\mathbb{R}^N) \end{aligned} \tag{2.5}$$

and $\varphi$,$\psi_0$ are defined through (2.2). This motivates the following definition:

**Definition 2.2.** Let a such that Assumption 2.1 holds and let $0 \le u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. $0 \le u \in C([0, +\infty); L^1(\mathbb{R}^N)) \cap L^\infty((0, \infty) \times \mathbb{R}^N)$ is an entropy solution to the Cauchy problem for (0.1) with initial datum $u_0$ if $u(0) = u_0$ and:

(*i*) $T^a_{a,b}(u) \in L^1_{loc}((0, +\infty); BV(\mathbb{R}^N))$ for all $0 < a < b$;

(*ii*) $u_t = \operatorname{div}(\mathbf{a}(u, \nabla u))$ in the sense of distributions;

(*iii*) inequality (3.12) holds for any $S, T \in \mathcal{T}^+$ and $0 \le \phi \in C_c^\infty((0, +\infty) \times \mathbb{R}^N)$.

The following well posedness result is contained in, or follows easily from, earlier results in [5], resp. [20].

**Theorem 2.3.** Let Assumption 2.1 be satisfied. Then, for any initial datum $0 \le u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ thre exists a unique entropy solution $u$ to (0.1) in $(0, T) \times \mathbb{R}^N$ for any $T > 0$. Moreover, if $u, \overline{u}$ are the entropy solutions to (0.1) corresponding to $0 \le u_0, \overline{u}_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then

$$\|(u(t) - \overline{u}(t))^+\|_1 \le \|(u_0 - \overline{u}_0)^+\|_1 \quad \text{for all } t \ge 0.$$

Moreover, in case that $u_0 \in BV(\Omega)$, then the contraction principle above yields that $u(t) \in BV(\Omega)$ for all $t > 0$.

## 2.3 Regularity results. Asymptotic regimes

In general, one cannot expect solutions to (PMRHE) and (SLPME) to be regular, not even continuous. In spite of the fact that the equations are of parabolic type (and therefore a regularizing effect is expected), these equations possess a hyperbolic character for large gradients. In fact, if the kinematic viscosity $\nu \to \infty$, then at least formally

$$\nu u^m \frac{\nabla u}{\sqrt{u^2 + \nu^2 c^{-2}|\nabla u|^2}} \sim c u^m \frac{\nabla u}{|\nabla u|},$$

while

$$\nu u \frac{\nabla u^{M-1}}{\sqrt{1 + \nu^2 c^{-2}|\nabla u^{M-1}|^2}} \sim c u \frac{\nabla u}{|\nabla u|}.$$

Then, one can infer that, at large gradients (or at a jump discontinuity point) solutions to these equations behave as solutions to (NDTM). In one dimension, and supposing that near a jump point solutions are monotone non increasing, then close to this jump point, solutions are expected to behave as solutions to the corresponding Burger's equation:

$$u_t \sim -c(u^m)_x\,, \quad \text{for (PMRHE),} \tag{2.6}$$

$$u_t \sim -c(u)_x\,, \quad \text{for (SLPME).} \tag{2.7}$$

On the other hand, if the speed of propagation $c \to \infty$ then formally solutions converge to solutions to the classical Porous Medium Equation:

$$u_t \sim \frac{\nu}{m}\,\mathrm{div}(\nabla u^m) \qquad u_t \sim \frac{\nu M}{M-1}\,\mathrm{div}(\nabla u^M)\,.$$

From this heuristic analysis, it is easily seen that these equations posses a parabolic regime for small gradients and a hyperbolic one for large gradients. Moreover, in some particular cases, this analysis can be rigorously performed :

**Theorem 2.4.** If $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then ([20]) solutions to (PMRHE) and (SLPME) converge as $c \to +\infty$ to solutions to the porous medium equation. Solutions to (1.7) converge as $\nu \to +\infty$ to solutions to (NDTM) with $m = 1$ ([8]).

These phenomena have been first observed numerically in [9] and [17]. In figures 1 and 2 (obtained in [17]) we observe that jump discontinuities can appear at the boundary of the support and that they are spread through the evolution and that there is a regularizing effect on small gradients (figure 1). We also observe that jump discontinuities can appear also at the interior of the support (figure 2).

The first regularity result was obtained in [6] for Equation 1.7, but it can be easily deduced for (PMRHE) in case $m > 1$ by the homogeneity of order $m$ of the operator (see [12]).

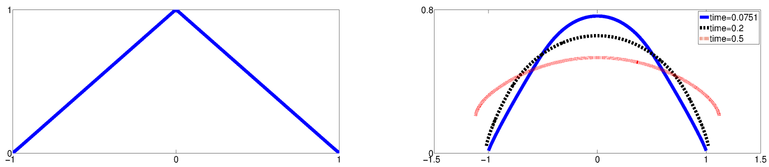


Figure 1: Left: initial datum. Right: solution to PMRHE, $m = 2$

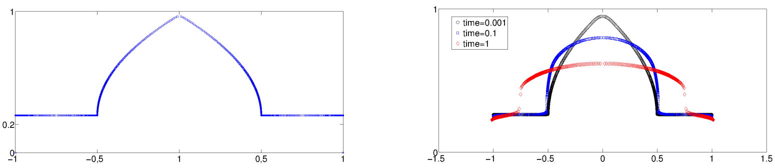


Figure 2: Left: initial datum. Right: solution to PMRHE, $m = 4$.

**Proposition 2.5.** Let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be such that $u_0(x) \geq \alpha > 0$ when $x \in \Omega$ and $u_0 = 0$ outside $\Omega$. Assume that $u_0 \in W^{2,1}(\Omega)$ and $\nabla u_0 \in L^\infty(\Omega)$. Let $u(t)$ be the entropy solution of (PMRHE) with $u(0) = u_0$. Then for any $t > 0$, $u_t(t)$ is a Radon measure in $\mathbb{R}^N$.

Next result is the main one in [6]:

**Proposition 2.6.** Assume that $\Omega \subseteq \mathbb{R}^N$ is a $C^{1,1}$ open bounded convex set. Let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $u_0(x) \geq \alpha > 0$ when $x \in \Omega$ and $u_0 = 0$ outside $\Omega$. Assume that $u_0$ is log-concave in $\overline{\Omega}$. Let $u(t,x)$ be the entropy solution of (1.7) with $u(0,x) = u_0(x)$. Then $u(t)$ is log-concave in $\Omega(t)$ and $u$ is smooth in $\Omega^T$, i.e. $C^{1,\alpha/2}$ in the time variable and $C^{2,\alpha}$ in space.

In the one dimensional setting, and for Equation 1.7, more can be obtained by studying the equation in Lagrangian coordinates. Let $u$ be the entropy solution to 1.7 corresponding to the initial datum $u_0$ with mass equal to one. Let

$$a(t) := \min \operatorname{supp} u(t) \qquad b(t) := \max \operatorname{supp} u(t).$$

Then, the inverse distribution function $\Phi$ defined by

$$\int_{a(t)}^{\Phi(t,\eta)} u(t,x)\,, dx = \eta \qquad \text{for } 0 \leq \eta \leq 1$$

is a weak diffeomorphism between $[a(t), b(t)]$ and $[0,1]$. Letting $v(t,\eta) := u(t, \Phi(t,\eta))$, formally $v$ is a large solution to a dual problem. More explicitly,

$$\begin{cases} v_t = \left( \dfrac{\nu v v_\eta}{\sqrt{v^4 + \nu^2 c^{-2} (v_\eta)^2}} \right)_\eta & \text{in } (0,T) \times (0,1) \\ v = +\infty & \text{on } (0,T) \times \{0,1\} \end{cases} \tag{2.8}$$

In [17] (see [16] for an improvement), this equation is solved and sufficient regularity of solutions is obtained to invert this change of variables and to show next result:

**Theorem 2.7.** Let $0 \leq u_0 \in L^\infty(\mathbb{R}^N)$ be such that either there exists $\alpha > 0$ with $u_0 \geq \alpha$ in $(a(0), b(0))$ and $u = 0$ in $\mathbb{R}^N \setminus \{[a(0), b(0)]\}$ or $u_0 \in W^{1,\infty}_{\text{loc}}(a, b)$ and $u_0(x) \to 0$ as $x \to a, b$. Then, the entropy solution to (1.7) satisifies $u(t) \in BV(\mathbb{R})$, $u(t) \in W^{1,1}(a-t, b+t)$ for almost any $t \in (0, T)$, and $u(t)$ is smooth inside its support.

The regularity in time obtained in Proposition 2.5 together with the fact that $u \in BV(\mathbb{R}^N)$ (true if $u_0 \in BV(\mathbb{R}^N)$ is enough to derive a Rankine-Hugoniot condition for the velocity of a discontinuity jump. In fact, in this case, it is easily seen that $u \in BV_{loc}((0,T) \times \mathbb{R}^N)$ and then, one can consider $\nu = (\nu_x, \nu_t)$; i.e the normal to a jump set (space-time). The speed of the discontinuity set is defined as

$$\mathrm{v}(t,x) := \frac{\nu_t(t,x)}{|\nu_x(t,x)|}, \quad \mathcal{H}^{N-1} - \text{a.e. on } J_u(t,x).$$

**Proposition 2.8.** Let $u \in BV((0,T) \times \mathbb{R}^N)$ be the entropy solution of (PMRHE) (resp. (SLPME)) with $0 \leq u(0) = u_0 \in L^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then, the speed of the discontinuity set is given by

$$\mathrm{v} = \frac{(u^m(t))^+ - (u^m(t))^-}{u(t)^+ - u(t)^-} \qquad (resp. \mathrm{v} = 1).$$

## 2.4 Propagation of the support. Waiting-time phenomena

The finite speed of propagation property is proved in [27, Theorem 1.2] for a class of equations of the form (0.1) which includes (PMRHE) and (SLPME) (see [27, Assumption 1.1]).

**Theorem 2.9.** If $0 \leq u_0 \in L^\infty(\mathbb{R}^N)$ has compact support, then the entropy solution $u$ to the Cauchy problem for (0.1) with initial datum $u_0$ is such that

$$\operatorname{supp}(u(t)) \subseteq \operatorname{supp}(u_0) + \overline{B(0, Vt)} \quad \text{for all } t > 0 \tag{2.9}$$

where

$$V := \operatorname*{ess\,sup}_{z \in (0, \|u_0\|_\infty)} \varphi'(z) \tag{2.10}$$

From the result above and from the the formal asymptotics as $\nu \to \infty$ discussed in Section 2.3, one can infer that the qualitative behavior of solutions of Equations (PMRHE) and (SLPME) is quite different. This has been highlighted also by numerical simulations as in [15, 9, 17]. For instance, (2.6) suggests that (PMRHE) may yield to the formation of jump discontinuities if $m > 1$, whereas (SLPME) may not. Moreover, Proposition 2.8 suggests that the speed of propagation of the support (in case the datum is continuous) is formally given by $mu^{m-1} = 0$ for (PMRHE) and by 1 for (SLPME). For this reason, in the former case the formation of a discontinuity is expected to be not only sufficient ([19]), but also necessary for the support to expand.

The aforementioned difference manifests itself also in the *waiting time phenomenon*, a positive time before which the solutions' support does not expand around a point $x_0 \in \mathbb{R}^N$. This phenomenon is known to occur for various classes of degenerate parabolic equations, such as the classical porous medium equation (see [34]). Concerning (PMRHE) and (SLPME), after numerical and formal arguments in [9, 17], rigorous sufficient conditions for a positive waiting time have been recently given in [27].

**Theorem 2.10.** A positive constant $C$, depending only on $N$ and $m$ (resp. $M$), exists such that if

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |x-x_0|^{-\frac{1}{m-1}} u_0(x) = L < +\infty \qquad \text{if } u \text{ solves (PMRHE), or} \tag{2.11}$$

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |x-x_0|^{-\frac{2}{M-1}} u_0(x) = L < +\infty \qquad \text{if } u \text{ solves (SLPME),} \tag{2.12}$$

then the entropy solution to the Cauchy problem for (PMRHE), resp. (SLPME), is such that

$$u(t,x_0) = 0 \quad \text{for all} \quad t \le T_\ell := \begin{cases} CL^{1-m} & \text{if } u \text{ solves (PMRHE)} \\ CL^{1-M} & \text{if } u \text{ solves (SLPME)} \end{cases} \tag{2.13}$$

This result provides a *lower bound* $T_\ell$ on the waiting time. We have recently obtained the corresponding upper bound result in [28].

**Theorem 2.11.** Let $0 \le u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. Let $u$ be the solution to the Cauchy problem for (PMRHE) (resp. (SLPME)) with initial datum $u_0$ and let

$$t_* = \sup\left\{t \ge 0 : \ x_0 \in \overline{\mathbb{R}^N \setminus \operatorname{supp}(u(\tau))} \quad \text{for all } \tau \in [0,t]\right\}.$$

If $v_0 \in \mathbb{S}^{N-1}$ exists such that

$$\lim_{\rho\to 0^+} \operatorname*{ess\,inf}_{x\in B(x_0+\rho v_0,\rho)} u_0(x)|x-x_0|^{-\frac{1}{m-1}} \ge L \in (0,+\infty] \quad \text{if } u \text{ solves (PMRHE),} \tag{2.14}$$

or

$$\lim_{\rho\to 0^+} \operatorname*{ess\,inf}_{x\in B(x_0+\rho v_0,\rho)} u_0(x)|x-x_0|^{-\frac{2}{M-1}} \ge L \in (0,+\infty] \quad \text{if } u \text{ solves (SLPME),} \tag{2.15}$$

then a positive constant $C$, depending on $m$ (resp. $M$) and $N$, exists such that

$$t_* \le T_u := \begin{cases} CL^{1-m} & \text{if } u \text{ solves (PMRHE)} \\ CL^{1-M} & \text{if } u \text{ solves (SLPME).} \end{cases} \tag{2.16}$$

In particular, $t_* = 0$ if $L = +\infty$.

The results in Theorem 2.11 are sharp. Indeed, comparing Theorem 2.11 with (2.11)-(2.12) one sees that the growth exponents in (2.14)-(2.15) are optimal. Note that the growth exponent $2/(M-1)$ coincides with that of the limiting porous medium equation, whereas $1/(m-1)$ does not. In addition, comparing Theorem 2.11 with (2.13), we see that the *upper bound* $T_u$ on the waiting time given in (2.16) is also optimal, in terms of scaling with respect to $L$.

This result is proved by comparison with suitable subsolutions. The first result that we obtained is that there is a comparison principle between entropy subsolutions (defined below) and solutions.

**Definition 2.12.** Let $\tau > 0$ and let a such that Assumption 2.1 holds. A nonnegative function $u \in C([0,\tau); L^1(\mathbb{R}^N)) \cap L^\infty([0,\tau] \times \mathbb{R}^N)$ is an entropy subsolution to equation (0.1) in $(0,\tau) \times \mathbb{R}^N$ if:

$(i)$ $T^a_{a,b}(u) \in L^1_{loc}((0,\tau); BV(\mathbb{R}^N))$ for all $0 < a < b$;

$(ii)$ inequality (3.12) holds for any $S, T \in \mathcal{T}^+$ and any nonnegative $\phi \in C_c^\infty((0,\tau) \times \mathbb{R}^N)$.

**Theorem 2.13.** Let $\tau > 0$ and let a such that Assumption 2.1 holds. Let $u$ be an entropy solution to the Cauchy problem for (0.1) with initial datum $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and $\underline{u}$ be an entropy subsolution to equation (0.1) in $(0,\tau)$. If $\underline{u}(0) \leq u_0$, then $\underline{u}(t) \leq u(t)$ for all $t \in (0,\tau)$.

We now explain the construction of subsolutions for (PMRHE). As we have previously observed, in this case, it is natural to look for subsolutions with a jump discontinuity at the boundary of their support. Imposing a vertical contact angle on the boundary, we look at subsolutions of the form:

$$u(t,x) = \frac{1}{A(t)}\left(1 + \sqrt{r(t)^2 - |x|^2}\right)\chi_{Q_0}(t,x). \tag{2.17}$$

Coupled with Rankine-Hugoniot condition as given by Proposition 2.8 and with a homogeneity condition in the interior of the support, yields that 2.17 with the choice of

$$\begin{aligned} A(t) &= [(m-1)(1+\gamma t)]^{\frac{1}{m-1}}\,, && (2.18)\\ r(t) &= r_0 + \frac{1}{\gamma(m-1)}\log(1+\gamma t)\,, && (2.19)\\ Q_0 &= \{(t,x) : \; t \in (0,T),\; x \in B(0,r(t))\} && (2.20) \end{aligned}$$

is a candidate to be a subsolution. In fact, the following result confirms it.

**Proposition 2.14.** Let $N \geq 1$, $m > 1$, $T > 0$ and $r_1 > 0$. Then there exist a value $\gamma_0 \geq 1$ such that the function $u$ defined by (2.17)-(2.20) is a subsolution to (PMRHE) for any $\gamma \geq \gamma_0$ and any $r_0 \in \left[\frac{r_1}{2}, r_1\right]$.

The case for Equation (SLPME) is easier in the sense that subsolutions are continuous but computations are much more involved.

**Theorem 2.15.** If $b > 0$, $\ell > 1$ $K > 0$, and $w > 0$ are such that

$$\frac{2N(M-1)}{b} \leq w \leq \frac{1}{\sqrt{1 + \frac{b^2}{4}(\ell - 1 + \frac{\ell}{K})^2}}, \tag{2.21}$$

holds, then for any $s > 0$ and any $\xi \in \mathbb{R}^N$ the function

$$\underline{u}(t,x) = b^{\frac{1}{1-M}} \left( \frac{\ell}{s} - \frac{1}{s + wt} \right)^{\frac{1}{1-M}} \left( 1 - \frac{|\xi - x|^2}{(s + wt)^2} \right)_+^{\frac{1}{M-1}}, \tag{2.22}$$

is a subsolution to (SLPME) in $(0, \frac{s}{wK}) \times \mathbb{R}^N$.

In Figures 3 and 4 we plot the initial datum with the critical growth and the subsolutions at some time $t \geq 0$.

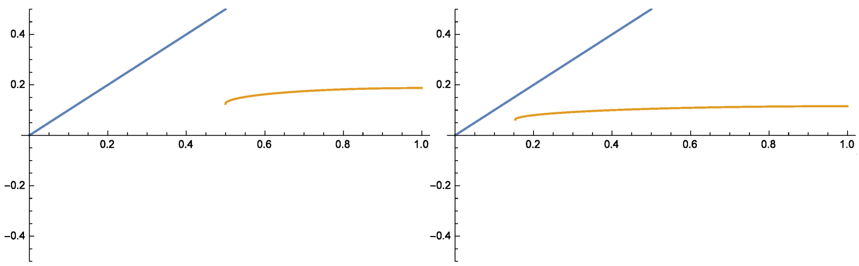


Figure 3: PMRHE, $m = 2$. Left: initial datum (blue) and subsolution at time $t = 0$ (orange). Right: initial datum (blue) and subsolution at time $t = 2$ (orange).

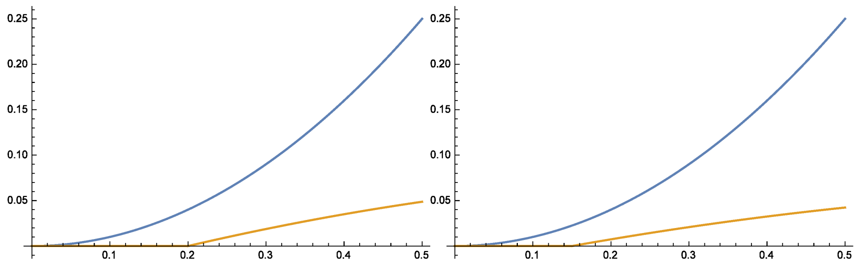


Figure 4: SLPME , $m = 2$. Left: initial datum (blue) and subsolution at time $t = 0$ (orange). Right: initial datum (blue) and subsolution at time $t = 1$ (orange).

# 3 Nonlinear diffusion in transparent media.

The mechanism and the dynamics of shock formation for solutions to (PMRHE) is not yet fully understood. Since, as explained in Section 2.3, (NDTM) and (PMRHE) formally coincide where $|\nabla u| \gg 1$, in particular at a discontinuity front, (NDTM) may be seen as a prototype equation for investigating such phenomena. More generally, in flux-saturated diffusion equations such as (PMRHE), one expects to see strong interplays between hyperbolic and parabolic mechanisms: the scaling invariance of (NDTM) with respect to $x$ should make these interplays more transparent and easier to study qualitatively.

In [29], we study this equation in a bounded domain $\Omega$ with Lipschitz continuous boundary $\partial\Omega$, coupled with homogeneous Neumann boundary conditions or with nonhomogenous Dirichlet ones and with $m \in \mathbb{R}$. For simplicity of the exposition, we will focus on the degenerate case ($m > 1$) and homogeneous Neumann boundary conditions. We observe that, in the particular case that $m = 0$, Equation (NDTM) is the well know Total Variation Flow. As we will see later, the case $m \neq 0$ is totally different from the usual Total Variation as seen from the qualitative properties of their solutions. The first step to obtain well posedness of the problem

$$\begin{cases} u_t = \operatorname{div}\left(u^m \dfrac{\nabla u}{|\nabla u|}\right) & \text{in } (0,T) \times \Omega \\ u^m \dfrac{\nabla u}{|\nabla u|} \cdot \nu = 0 & \text{on } \partial\Omega \\ u(0,x) = u_0 & \text{in } (0,T) \end{cases} \tag{3.1}$$

is to study the associated elliptic problem, for $0 \leq f \in L^\infty(\Omega)$:

$$\begin{cases} u - f = \operatorname{div}\left(u^m \dfrac{\nabla u}{|\nabla u|}\right) & \text{in } (0,T) \times \Omega \\ u^m \dfrac{\nabla u}{|\nabla u|} \cdot \nu = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

In order to introduce the concept of solution for this problem, we need several previous notations and results:

### 3.0.1 Divergence-measure vector-fields

Let

$$X_{\mathcal{M}}(\Omega) = \left\{ \mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N) \ : \ \operatorname{div} \mathbf{z} \in \mathcal{M}(\Omega) \right\}.$$

In [10, Theorem 1.2] (see also [3, 23]), the weak trace on $\partial\Omega$ of the normal component of $\mathbf{z} \in X_{\mathcal{M}}(\Omega)$ is defined as a linear operator $[\cdot, \nu] : X_{\mathcal{M}}(\Omega) \to L^\infty(\partial\Omega)$ such that $\| [\mathbf{z}, \nu] \|_{L^\infty(\partial\Omega)} \leq \|\mathbf{z}\|_\infty$ for all $\mathbf{z} \in X_{\mathcal{M}}(\Omega)$ and $[\mathbf{z}, \nu]$ coincides with the point-wise trace of the normal component if $\mathbf{z}$ is smooth:

$$[\mathbf{z}, \nu](x) = \mathbf{z}(x) \cdot \nu(x) \quad \text{for all } x \in \partial\Omega \ \text{ if } \mathbf{z} \in C^1(\overline{\Omega}, \mathbb{R}^m).$$

It follows from [23, Proposition 3.1] that $\operatorname{div} \mathbf{z}$ is absolutely continuous with respect to $\mathcal{H}^{N-1}$. Therefore, given $\mathbf{z} \in X_{\mathcal{M}}(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$, the functional $(\mathbf{z}, Du) \in \mathcal{D}'(\Omega)$ given by

$$\langle (\mathbf{z}, Du), \varphi \rangle := -\int_\Omega u^* \varphi \, \mathrm{d}(\operatorname{div} \mathbf{z}) - \int_\Omega u \, \mathbf{z} \nabla \varphi \, \mathrm{d}x \tag{3.3}$$

is well defined, and the following holds (see [19]).

**Lemma 3.1.** Let $\mathbf{z} \in X_{\mathcal{M}}(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$. Then the functional $(\mathbf{z}, Du) \in \mathcal{D}'(\Omega)$ defined by (3.3) is a Radon measure which is absolutely continuous with respect to $|Du|$. Furthermore

$$\int_\Omega u^* \, \mathrm{d}(\operatorname{div} \mathbf{z}) + (\mathbf{z}, Du)(\Omega) = \int_{\partial\Omega} [\mathbf{z}, \nu] u \, \mathrm{d}\mathcal{H}^{m-1}, \tag{3.4}$$

$$\operatorname{div}(u\mathbf{z}) = u^* \operatorname{div} \mathbf{z} + (\mathbf{z}, Du) \quad \text{as measures.} \tag{3.5}$$

The notion of solution is then the following one ([29]).

**Definition 3.2.** A function $u : \Omega \to [0, +\infty)$ is a solution of problem (3.2) with data $0 \le f \in L^\infty(\Omega)$ if $u \in TBV^+(\Omega) \cap L^\infty(\Omega)$ and there exist $\mathbf{w} \in L^\infty(\Omega; \mathbb{R}^N)$ such that $\|\mathbf{w}\|_\infty \le 1$ and $\mathbf{z} := u^m \mathbf{w}$ satisfies

$$u - f = \operatorname{div} \mathbf{z} \quad \text{in } \mathcal{D}'(\Omega) \tag{3.6}$$

$$\left| D \frac{(T(u))^{m+1}}{m+1} \right| \le (\mathbf{z}, DT(u)) \quad \text{as measures for any } T \in \mathcal{T}^+, \tag{3.7}$$

and

$$[\mathbf{z}, \nu] = 0 \tag{3.8}$$

By approximating the problem with

$$\begin{cases} u - f = \operatorname{div}\left( (\varepsilon + |u|)^m \frac{\nabla u}{|\nabla u|_\varepsilon} + \varepsilon \nabla u \right) & \text{in } \Omega \\ \left( (\varepsilon + |u|)^m \frac{\nabla u}{|\nabla u|_\varepsilon} + \varepsilon \nabla u \right) \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

suitable a-priori estimates and lower semicontinuity results permit us to obtain the following result.

**Theorem 3.3.** There exists a unique solution $u$ of (3.2) with data $f$ in the sense of Definition 3.2. Furthermore $\mathcal{H}^{N-1}(J_u) = 0$ and it holds

$$(\mathbf{w}, DT_a^b(u)) = |DT_a^b(u)| \quad \text{for a.e. } 0 < a < b \le +\infty, \tag{3.10a}$$

and

$$(\mathbf{z}, DT_a^b(u)) = \left| D \frac{(T_a^b(u))^{m+1}}{m+1} \right| \quad \text{for a.e. } 0 < a < b \le +\infty. \tag{3.10b}$$

We note that the property that the jump set is empty (in a measure theoretically sense) in the elliptic case is a property which also holds for the corresponding elliptic resolvent equations for (PMRHE) and (SLPME).

Next step is to associate an accretive operator in $L^1(\Omega)$ to the problem 3.1.

**Definition 3.4.** $(u,v) \in B$ if and only if $0 \leq u \in TBV^+(\Omega) \cap L^\infty(\Omega)$, $0 \leq v \in L^\infty(\Omega)$ and $u$ is an entropy solution of problem (3.2).

**Proposition 3.5.** $B$ is an accretive operator in $L^1(\Omega)$ with $D(B)$ dense in $L^1(\Omega)^+$, satisfying the comparison principle and the range condition

$$L^\infty(\Omega)^+ \subset R(I+B).$$

We denote by $\mathcal{B}$ the closure in $L^1(\Omega)$ of the operator $B$. Then, it follows that $\mathcal{B}$ is accretive in $L^1(\Omega)$, it satisfies the comparison principle, and verifies the range condition $\overline{D(\mathcal{B})}^{L^1(\Omega)} = L^1(\Omega)^+ \subset R(I + \lambda\mathcal{B})$ for all $\lambda > 0$. Therefore, according to Crandall-Liggett's Theorem (c.f., e.g., [22]), for any $0 \leq u_0 \in L^1(\Omega)$ there exists a unique mild solution $u \in C([0,T]; L^1(\Omega))$ of the abstract Cauchy problem

$$u'(t) + \mathcal{B}u(t) \ni 0, \quad u(0) = u_0. \tag{3.11}$$

Moreover, $u(t) = S(t)u_0$ for all $t \geq 0$, where $(\mathcal{S}(t))_{t\geq 0}$ is the semigroup in $L^1(\Omega)^+$ generated by Crandall-Liggett's exponential formula, i.e.,

$$S(t)u_0 = \lim_{n\to\infty}\left(I + \frac{t}{n}\mathcal{B}\right)^{-n} u_0.$$

We point out that, contrary to the case of Total Variation, the operator is not completely accretive and then it is not a contraction in the $L^\infty$ norm. If this were the case, then the fact that the jump set is a null set, would be transferred to the parabolic case. We will see in Example 3.8 below that this is not the case.

The definition of solution for the parabolic problem is the following one.

**Definition 3.6.** Let $0 \leq u_0 \in L^1(\Omega)$. $0 \leq u \in C([0,+\infty); L^1(\Omega)) \cap L^\infty((0,\infty)\times\Omega)$ is an entropy solution to the Cauchy problem for (3.1) with initial datum $u_0$ if $u(0) = u_0$ and there exists $\mathbf{w} \in L^\infty((0,T)\times\Omega$ with $\|\mathbf{w}\|_\infty \leq 1$ such that $\mathbf{z}(t) := u^m\mathbf{w}(t) \in X_\mu(\Omega)$ for all $t \in [0,T]$, and

(*i*) $T^a_{a,b}(u) \in L^1_{loc}((0,+\infty); BV(\Omega))$ for all $0 < a < b$;

(*ii*) $u_t = \operatorname{div}(\mathbf{z})$ in the sense of distributions;

(*iii*) $[\mathbf{z}(t),\nu] = 0$ $\mathcal{H}^{N-1}-$ a.e. on $\partial\Omega$ for all $t \in [0,T]$ and

$(iv)$ the entropy inequality

$$\int_0^{+\infty} \langle h_S(u, DT(u)) + h_T(u, DS(u)), \phi\rangle \, \mathrm{d}t \\ \leq \int_0^{+\infty} \int_\Omega (J_{TS}(u)\phi_t - T(u)S(u)\mathbf{z}\cdot\nabla\phi) \, \mathrm{d}x \, \mathrm{d}t\,, \qquad (3.12)$$

holds for any $S, T \in \mathcal{T}^+$ and any nonnegative $\phi \in C_c^\infty((0,+\infty)\times\Omega)$. Here $h_S(u, DT(u))$ is the Radon measure defined by

$$\langle h_S(u, DT(u)), \phi\rangle := \int_\Omega \phi S(T^0(u)) u^m |\nabla T^0(u)| \, \mathrm{d}x \\ + \int_\Omega \phi \, \mathrm{d}|D^s J_{S\varphi}(T^0(u))| \quad \text{for all } \phi \in C_c(\Omega) \qquad (3.13)$$

The existence part of next result is proved by showing that the semigroup solution is in fact an entropy solution. The contraction principle follows from a Kruzhkov's doubling variables technique (in space-time) (see [30]).

**Theorem 3.7.** For any initial datum $0 \leq u_0 \in L^1(\Omega)$ there exists a unique entropy solution $u$ to (3.1) in $(0,T)\times\Omega$ for any $T > 0$. Moreover, if $u, \overline{u}$ are the entropy solutions to (3.1) corresponding to $0 \leq u_0, \overline{u}_0 \in L^1(\Omega)\cap L^\infty(\mathbb{R}^N)$, then

$$\|(u(t) - \overline{u}(t))^+\|_1 \leq \|(u_0 - \overline{u}_0)^+\|_1 \quad \text{for all } t \geq 0.$$

We finish with two illustrating examples. In the first one we see that even the source $f$ has jump discontinuities, solutions to (3.2) do not, while this property is lost in the parabolic case (see Figure 5).

**Example 3.8.** Let $\Omega := [-2,2]$, $m = 1$ and $u_0 = f = 3\chi_{[-\frac{1}{2},\frac{1}{2}]}$. Then, the solution to (3.2) is exactly

$$u(x) = e^{-\frac{1}{2}}\chi_{[-2,-1]\cup[1,2]} + e^{\frac{1}{2}-|x|}\chi_{[-1,-\frac{1}{2}]\cup[\frac{1}{2},1]} + \chi[-\frac{1}{2},\frac{1}{2}].$$

For a proof, it suffices to take

$$\mathbf{w}(x) = (2-|x|)\chi_{[-2,-1]\cup[1,2]} - \text{signx}\chi_{[-1,-\frac{1}{2}]\cup[\frac{1}{2},1]} - 2x\chi_{[-\frac{1}{2},\frac{1}{2}]}$$

The solution to (3.1) is instead

$$u(t,x) = \frac{3}{1+2t}\chi_{[-\frac{1}{2}-t,\frac{1}{2}+t]}.$$

In this case one can take

$$\mathbf{w}(t,x) := \frac{-2x}{1+2t}\chi_{[-\frac{1}{2}-t,\frac{1}{2}+t]}.$$

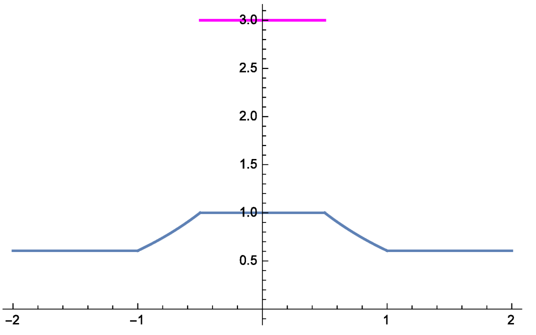


Figure 5: Regularity. Magenta: initial datum, blue: solution to (3.2).

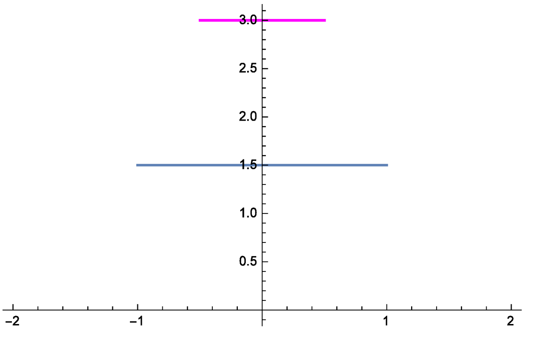


Figure 6: Regularity. Magenta: initial datum, blue: solution to (3.1)

Contrary to the case of the Total Variation Flow (i.e. $m = 0$), new discontinuities can appear during the evolution (see [21] for regularity results for the Total Variation Flow).

**Example 3.9.** Creation of discontinuities. Let $m = 2$, $\Omega = [-10, 10]$ and

$$u_0 = \chi_{[-3,-2]\cup[2,3]} + (3 - |x|)\chi_{[-2,-1]\cup[1,2]} + 2\chi_{[-1,1]}.$$

The solution (up to time $t^*$ in which it becomes a characteristic function in a time dependent interval) is given by

$$u(t,x) = \begin{cases} \chi_{A(t)} + \dfrac{3-|x|}{1-2t}\chi_{B(t)} + \dfrac{3-\sqrt{16t+1}}{1-2t}\chi_{C(t)} & \text{if } t \le \frac{1}{2} \\ \chi_{D(t)} + g(t)\chi_{E(t)} & \text{if } \frac{1}{2} \le t \le t^* \end{cases}$$

with $A(t) := [-3-t, -2-2t] \cup [2+2t, 3+t]$, $B(t) := [-2-2t, -\sqrt{16t+1}] \cup [\sqrt{16t+1}, 2+2t]$ and $C(t) := [-\sqrt{16t+1}, \sqrt{16t+1}]$,

$$E(t) := \left[ \frac{64\left(1 + W\left(\frac{e^{-1/4}}{4}\left(\frac{t-\frac{1}{2}}{4} - 1\right)\right)\right)}{\left(\frac{t-\frac{1}{2}}{4} - 1\right)W\left(\frac{e^{-1/4}}{4}\left(\frac{t-\frac{1}{2}}{4} - 1\right)\right)}, \frac{-64\left(1 + W\left(\frac{e^{-1/4}}{4}\left(\frac{t-\frac{1}{2}}{4} - 1\right)\right)\right)}{\left(\frac{t-\frac{1}{2}}{4} - 1\right)W\left(\frac{e^{-1/4}}{4}\left(\frac{t-\frac{1}{2}}{4} - 1\right)\right)} \right]$$

$$D(t) := [-3 - t, , 3 + t] \setminus E(t),$$
$$g(t) := \frac{1}{1 + W\left(\frac{e^{-1/4}}{4}\left(\frac{t-\frac{1}{2}}{4} - 1\right)\right)}$$

with $W$ being the Lambert $W$ function and $t^*$ being the first time in which $D(t^*) = \emptyset$.

We observe that a discontinuity is created at time $t = \frac{1}{2}$ and that then the evolution follows by Rankine- Hugoniot condition and conservation of mass property. Here we prefer not to give an explicit form of the vector field $\mathbf{w}$.

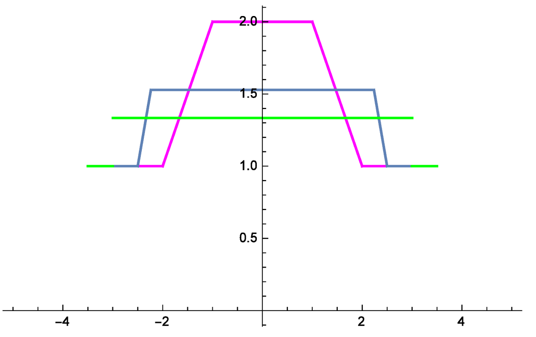


Figure 7: Creation of a discontinuity in 3.1, $m = 2$. Magenta: initial data, blue: solution at time $t = \frac{1}{4}$, green: solution at time $t = \frac{1}{2}$.

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