数理解析研究所講究録
第2066巻 2018年 109-117

Well-posedness for Keller-Segel system coupled with the Navier-Stokes fluid †1

Masanari MIURA †2
Graduate school of Mathematics, Kyushu University

1 Introduction.

Keller-Segel systems describe chemotaxis phenomena which are the collective motions of cells. We deal with the following Keller-Segel system coupled with the Navier-Stokes equations in $\mathbb{R}^N$ ($N \geq 2$):

$$
\text{(NCS)} \qquad \begin{cases} \dfrac{\partial n}{\partial t} + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c) - \nabla \cdot (n \nabla v) & \text{in } \mathbb{R}^N \times (0,\infty), \\ \dfrac{\partial c}{\partial t} + u \cdot \nabla c = \Delta c - nc & \text{in } \mathbb{R}^N \times (0,\infty), \\ \dfrac{\partial v}{\partial t} + u \cdot \nabla v = \Delta v - \gamma v + n & \text{in } \mathbb{R}^N \times (0,\infty), \\ \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u = \Delta u - \nabla \pi - nf & \text{in } \mathbb{R}^N \times (0,\infty), \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ n|_{t=0} = n_0, \ c|_{t=0} = c_0, \ v|_{t=0} = v_0, \ u|_{t=0} = u_0 & \text{in } \mathbb{R}^N, \end{cases}
$$

where $n = n(x,t)$, $c = c(x,t)$, $v = v(x,t)$, $u = u(x,t) = (u_1(x,t), u_2(x,t), \cdots, u_N(x,t))$ and $\pi = \pi(x,t)$ denote the unknown density of amoebae, the unknown oxygen concentration, the unknown concentration of chemical attractant, the unknown fluid velocity field and the unknown pressure, respectively, while $n_0 = n_0(x), c_0 = c_0(x), v_0 = v_0(x)$ and $u_0 = u_0(x) = (u_{0,1}(x), u_{0,2}(x), \cdots, u_{0,N}(x))$ denote the given initial data.

In 2005, the following chemotaxis-fluid model had been introduced by Tuval et. al. [7] so that:

$$
\text{(E-1)} \qquad \begin{cases} \dfrac{\partial n}{\partial t} + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c) & \text{in } \mathbb{R}^N \times (0,\infty), \\ \dfrac{\partial c}{\partial t} + u \cdot \nabla c = \Delta c - nf(c) & \text{in } \mathbb{R}^N \times (0,\infty), \\ \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u = \Delta u - \nabla \pi - nf & \text{in } \mathbb{R}^N \times (0,\infty), \\ \operatorname{div}\, u = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ n|_{t=0} = n_0, \ u|_{t=0} = u_0 & \text{in } \mathbb{R}^N. \end{cases}
$$

*This is based on a joint work with Professors H.Kozono (Waseda University) and Y.Sugiyama (Kyushu University).

†E-mail : m-miura@math.kyushu-u.ac.jp

Winkler [8] considered the system (E-1) with four unknowns $\{n, c, u, \pi\}$ under the Navier-Stokes fluid, and proved the global existence of classical solutions in $2D$ bounded domains. It should be noted that minus sign $-nc$ plays a decisive role for showing the global existence. He also constructed a global weak solution in $3D$ bounded domains. Some generalizations with the same sign have been introduced by Chae-Kang-Lee [1] and Tao-Winkler [6].

On the other hand, Lorz [5] treated the case with four unknowns (n, v, u, π) which is so-called a Keller-Segel model under the linear Stokes fluid as follows:

$$\text{(E-2)} \qquad \begin{cases} \dfrac{\partial n}{\partial t} + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla v) & \text{in } \mathbb{R}^N \times (0, \infty), \\ u \cdot \nabla v = \Delta v + n & \text{in } \mathbb{R}^N \times (0, \infty), \\ \dfrac{\partial u}{\partial t} = \Delta u - \nabla \pi - nf & \text{in } \mathbb{R}^N \times (0, \infty), \\ \text{div } u = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ n|_{t=0} = n_0, \ u|_{t=0} = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Lorz [5] showed the existence of global weak solutions for (E-2) with the small initial data in both $2D$ and $3D$ whole spaces. Recently, Kozono, Sugiyama and the author treated the following system (E-3) to construct a blow-up solution. See the forthcoming paper [4] in detail.

$$\text{(E-3)} \qquad \begin{cases} \dfrac{\partial n}{\partial t} + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla v) & \text{in } \mathbb{R}^N \times (0, T), \\ -\Delta v = -\gamma v + n & \text{in } \mathbb{R}^N \times (0, T), \\ \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u = \Delta u - \nabla \pi - |n|^{\alpha} f & \text{in } \mathbb{R}^N \times (0, T), \\ \text{div } u = 0 & \text{in } \mathbb{R}^N \times (0, T), \\ n|_{t=0} = n_0, \ u|_{t=0} = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

The purpose of this paper is to show the existence of global mild solutions with the small initial data in the scaling invariant space. Our method is based on the implicit function theorem which yields necessarily continuous dependence of solutions for the initial data. As a byproduct, we show the asymptotic stability of solutions as the time goes to infinity. Since we may deal with the initial data in the weak L^p-spaces, the existence of selfsimilar solutions provided the initial data are small homogeneous functions.

Let us first introduce the following hypotheses on the initial data:

Assumption. We assume that $N \geq 2$ and $\gamma \geq 0$.

(i) For $N \geq 3$, the initial data $\{n_0, c_0, v_0, u_0\}$ satisfies

$$\begin{aligned} & n_0 \in L_w^{\frac{N}{2}}(\mathbb{R}^N), \quad c_0 \in L^\infty(\mathbb{R}^N) \text{ with } \nabla c_0 \in L_w^N(\mathbb{R}^N), \\ & v_0 \in \mathcal{S}' \text{ with } \nabla v_0 \in L_w^N(\mathbb{R}^N), \quad \text{and } u_0 \in PL_w^N(\mathbb{R}^N), \end{aligned}$$

For $N = 2$, we replace $n_0 \in L_w^1(\mathbb{R}^2)$ by $n_0 \in L^1(\mathbb{R}^2)$.

(ii) The external force f satisfies $f \in L_w^N(\mathbb{R}^N)$. Note that L_w^p denotes the weak L^p space.

Here and in what follows, we denote by $P = \{P_{jk}\}_{j,k=1,\cdots,N}$ the projection operator onto the solenoidal vector fields with the expression

$$P_{jk} = \delta_{jk} + R_j R_k \quad (R_j \equiv \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}} : \text{ Riesz operator})$$

for $j, k = 1, 2, \cdots, N$.

Our definition of mild solutions to (NCS) now reads:

Definition 1. (mild solution) Let $N \geq 2$, and let $\{n_0, c_0, v_0, u_0, f\}$ be as in the Assumption. A pair $\{n, c, v, u\}$ of measurable functions on $\mathbb{R}^N \times (0, \infty)$ is called a mild solution of (NCS) on $(0, \infty)$ if $n, c, v, u \in L^q_{loc}(0, \infty; L^r(\mathbb{R}^N))$ for some $1 \leq q, r \leq \infty$, and if the identities

$$\text{(IE)} \begin{cases} n(t) &= e^{t\Delta} n_0 - \int_0^t e^{(t-\tau)\Delta}(u \cdot \nabla n)(\tau)\, d\tau - \int_0^t \nabla \cdot e^{(t-\tau)\Delta}(n\nabla c + n\nabla v)(\tau)\, d\tau, \\ c(t) &= e^{t\Delta} c_0 - \int_0^t e^{(t-\tau)\Delta}(u \cdot \nabla c + nc)(\tau)\, d\tau, \\ v(t) &= e^{-\gamma t} e^{t\Delta} v_0 - \int_0^t e^{-\gamma(t-\tau)} e^{(t-\tau)\Delta}(u \cdot \nabla v - n)(\tau)\, d\tau, \\ u(t) &= e^{t\Delta} u_0 - \int_0^t e^{(t-\tau)\Delta} P(u \cdot \nabla u + nf)(\tau)\, d\tau \end{cases}$$

hold for $0 < t < \infty$, where $e^{t\Delta}$ denotes the heat semi-group defined by

$$(e^{t\Delta} g)(x) \equiv \int_{\mathbb{R}^N} G(x - y, t) g(y)\, dy$$

with $G(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp(-\frac{|x|^2}{4t})$.

2 Main Results.

Our result on unique global existence of mild solutions reads as follows:

Theorem 1 ([3]). *For $N \geq 3$, suppose that the exponents p, q and r satisfy the following either (i), (ii) or (iii).*

$$\begin{array}{llll} (i) & \frac{N}{2} < q < N, & N < p < \frac{Nq}{N-q}, & N < r < \frac{Nq}{N-q}; \\ (ii) & q = N, & N < p < \infty, & N < r < \infty; \\ (iii) & N < q < 2N, & N < p < \frac{Nq}{q-N}, & q \leq r < \frac{Nq}{q-N}. \end{array}$$

For $N = 2$, we assume that the exponents p, q and r satisfy the above condition (iii) with $N = 2$. There is a constant $\delta = \delta(N, p, q, r)$ with the following property. If the initial data $\{n_0, c_0, v_0, u_0\}$ and the external force f in the Assumption satisfy

$$\|n_0\|_{L^{\frac{N}{2}}_w(\mathbb{R}^N)} + \|c_0\|_{L^\infty(\mathbb{R}^N)} + \|\nabla c_0\|_{L^N_w(\mathbb{R}^N)} + \|\nabla v_0\|_{L^N_w(\mathbb{R}^N)} + \|u_0\|_{L^N_w(\mathbb{R}^N)} + \|f\|_{L^N_w(\mathbb{R}^N)} < \delta \quad \text{for } N \geq 3; \tag{2.1}$$

$$\|n_0\|_{L^1(\mathbb{R}^2)} + \|c_0\|_{L^\infty(\mathbb{R}^2)} + \|\nabla c_0\|_{L^2_w(\mathbb{R}^2)} + \|\nabla v_0\|_{L^2_w(\mathbb{R}^2)} + \|u_0\|_{L^2_w(\mathbb{R}^2)} + \|f\|_{L^2_w(\mathbb{R}^2)} < \delta \quad \text{for } N = 2, \tag{2.2}$$

then there exists a mild solution $\{n, c, v, u\}$ of (NCS) on $(0, \infty)$ with the property that

$$t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{q})}n \in BC_w([0,\infty); L^q(\mathbb{R}^N)), \tag{2.3}$$

$$c \in BC_w([0,\infty); L^\infty(\mathbb{R}^N)), \quad t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{r})}\nabla c \in BC_w([0,\infty); L^r(\mathbb{R}^N)), \tag{2.4}$$

$$t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{r})}\nabla v \in BC_w([0,\infty); L^r(\mathbb{R}^N)), \tag{2.5}$$

$$t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}u \in BC_w([0,\infty); L^p(\mathbb{R}^N)), \tag{2.6}$$

where $BC_w([0,\infty); X)$ denotes the set of bonded weakly-star continuous functions on $(0,\infty)$ with values in the Banach space X.

Such a mild solution $\{n, c, v, u\}$ is unique provided the norms corresponding to the spaces (2.3)–(2.6) are sufficiently small. Moreover, the mild solution $\{n, c, v, u\}$ exhibits the following asymptotic behavior.

$$\|n(t) - e^{t\Delta}n_0\|_{L^q(\mathbb{R}^N)} = O\Big(t^{-\frac{N}{2}(\frac{2}{N}-\frac{1}{q})}\Big), \tag{2.7}$$

$$\|\nabla c(t) - \nabla e^{t\Delta}c_0\|_{L^r(\mathbb{R}^N)} = O\Big(t^{-\frac{N}{2}(\frac{1}{N}-\frac{1}{r})}\Big), \tag{2.8}$$

$$\|\nabla v(t) - \nabla e^{-\gamma t}e^{t\Delta}v_0\|_{L^r(\mathbb{R}^N)} = O\Big(t^{-\frac{N}{2}(\frac{1}{N}-\frac{1}{r})}\Big), \tag{2.9}$$

$$\|u(t) - e^{t\Delta}u_0\|_{L^p(\mathbb{R}^N)} = O\Big(t^{-\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\Big) \tag{2.10}$$

as $t \to \infty$.

Remarks. (i) It is easy to see that if $\{n, c, v, u, \pi\}$ solves (NCS), so does $\{n_\lambda, c_\lambda, v_\lambda, u_\lambda, \pi_\lambda\}$ for all $\lambda > 0$, where $n_\lambda(x,t) \equiv \lambda^2 n(\lambda x, \lambda^2 t), c_\lambda(x,t) \equiv c(\lambda x, \lambda^2 t), v_\lambda(x,t) \equiv v(\lambda x, \lambda^2 t), u_\lambda(x,t) \equiv \lambda u(\lambda x, \lambda^2 t), \pi_\lambda(x,t) \equiv \lambda^2 \pi(\lambda x, \lambda^2 t)$. The spaces (2.3)-(2.6) of solution are related to scaling invariant class which implies that

$$\begin{aligned}
&\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{q})}\|n(t)\|_{L^q(\mathbb{R}^N)} = \sup_{0<t<\infty} t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{q})}\|n_\lambda(t)\|_{L^q(\mathbb{R}^N)},\\
&\sup_{0<t<\infty} \|c(t)\|_{L^\infty(\mathbb{R}^N)} = \sup_{0<t<\infty} \|c_\lambda(t)\|_{L^\infty(\mathbb{R}^N)},\\
&\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{r})}\|\nabla c(t)\|_{L^r(\mathbb{R}^N)} = \sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{r})}\|\nabla c_\lambda(t)\|_{L^r(\mathbb{R}^N)},\\
&\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{r})}\|\nabla v(t)\|_{L^r(\mathbb{R}^N)} = \sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{r})}\|\nabla v_\lambda(t)\|_{L^r(\mathbb{R}^N)},\\
&\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|u(t)\|_{L^p(\mathbb{R}^N)} = \sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|u_\lambda(t)\|_{L^p(\mathbb{R}^N)}
\end{aligned}$$

hold for all $\lambda > 0$.

(ii) The exponents p, q and r determine such a class of functions as in (2.3)–(2.6) to which the mild solution $\{n, c, v, u\}$ belongs. By our theorem, we see that q plays a more decisive role than that of p and r, which seems to be understood that behaviour of the density of n of the amoebae is dominant in comparison with the effect of the imcompressible fluid u.

(iii) Concerning the initial data $\{n_0, c_0, v_0, u_0\}$, our hypothesis coincides with scaling invariant class in the sense that

$$\|n_{0,\lambda}\|_{L_w^{\frac{N}{2}}(\mathbb{R}^N)} = \|n_0\|_{L_w^{\frac{N}{2}}(\mathbb{R}^N)} \quad \text{for } N \geq 3, \quad \|n_{0,\lambda}\|_{L^1(\mathbb{R}^2)} = \|n_0\|_{L^1(\mathbb{R}^2)} \quad \text{for } N = 2,$$

$$\|c_{0,\lambda}\|_{L^\infty(\mathbb{R}^N)} = \|c_0\|_{L^\infty(\mathbb{R}^N)}, \quad \|\nabla c_{0,\lambda}\|_{L^N_w(\mathbb{R}^N)} = \|\nabla c_0\|_{L^N_w(\mathbb{R}^N)},$$
$$\|\nabla v_{0,\lambda}\|_{L^N_w(\mathbb{R}^N)} = \|\nabla v_0\|_{L^N_w(\mathbb{R}^N)}, \quad \|u_{0,\lambda}\|_{L^N_w(\mathbb{R}^N)} = \|u_0\|_{L^N_w(\mathbb{R}^N)}$$

for all $\lambda > 0$, where $n_{0,\lambda}(x) = \lambda^2 n_0(\lambda x)$, $c_{0,\lambda}(x) = c_0(\lambda x)$, $v_{0,\lambda}(x) = v_0(\lambda x)$, $u_{0,\lambda}(x) = \lambda u_0(\lambda x)$.

Next, we shall show the global stability of our mild solution under the initial disturbance and the perturbation of external forces in scaling invariant class.

Theorem 2 ([3]). *Let the exponents p, q and r be as in Theorem 1. Suppose that $\delta = \delta(N,p,q,r)$ is the same constant as in (2.1) and (2.2). For any $\eta > 0$, there is a constant $\delta_1 = \delta_1(N,p,q,r,\eta) > 0$ with the following property: Assume that two initial data $\{n_0, c_0, v_0, u_0\}$ and $\{n_0', c_0', v_0', u_0'\}$ and two external forces f and f' satisfy that*

$$\begin{aligned}(2.11)\qquad & \|n_0\|_{L^{\frac{N}{2}}_w(\mathbb{R}^N)} + \|c_0\|_{L^\infty(\mathbb{R}^N)} + \|\nabla c_0\|_{L^N_w(\mathbb{R}^N)} \\ & + \|\nabla v_0\|_{L^N_w(\mathbb{R}^N)} + \|u_0\|_{L^N_w(\mathbb{R}^N)} + \|f\|_{L^N_w(\mathbb{R}^N)} < \delta,\end{aligned}$$

$$\begin{aligned}(2.12)\qquad & \|n_0'\|_{L^{\frac{N}{2}}_w(\mathbb{R}^N)} + \|c_0'\|_{L^\infty(\mathbb{R}^N)} + \|\nabla c_0'\|_{L^N_w(\mathbb{R}^N)} \\ & + \|\nabla v_0'\|_{L^N_w(\mathbb{R}^N)} + \|u_0'\|_{L^N_w(\mathbb{R}^N)} + \|f'\|_{L^N_w(\mathbb{R}^N)} < \delta\end{aligned}$$

for $N \geq 3$ and that (2.11) and (2.12) with $L^{\frac{N}{2}}_w(\mathbb{R}^N)$ replaced by $L^1(\mathbb{R}^2)$ for $N = 2$. Suppose that $\{n, c, v, u\}$ and $\{n', c', v', u'\}$ are mild solutions of (NCS) on $[0,\infty)$ given by Theorem 1 with $\{n, c, v, u\}|_{t=0} = \{n_0, c_0, v_0, u_0\}$ and $\{n', c', v', u'\}|_{t=0} = \{n_0', c_0', v_0', u_0'\}$ in the class (2.3)-(2.6), respectively. If it holds that

$$\begin{aligned}(2.13)\qquad & \|n_0 - n_0'\|_{L^{\frac{N}{2}}_w(\mathbb{R}^N)} + \|c_0 - c_0'\|_{L^\infty(\mathbb{R}^N)} + \|\nabla c_0 - \nabla c_0'\|_{L^N_w(\mathbb{R}^N)} \\ & + \|\nabla v_0 - \nabla v_0'\|_{L^N_w(\mathbb{R}^N)} + \|u_0 - u_0'\|_{L^N_w(\mathbb{R}^N)} + \|f - f'\|_{L^N_w(\mathbb{R}^N)} < \delta_1 \quad \textit{for } N \geq 3\end{aligned}$$

and that (2.13) with $L^{\frac{N}{2}}_w(\mathbb{R}^N)$ replaced by $L^1(\mathbb{R}^2)$ for $N = 2$, then we have

$$\begin{aligned}(2.14)\qquad & \sup_{0<t<\infty} t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{q})}\|n(t) - n'(t)\|_{L^q(\mathbb{R}^N)} + \sup_{0<t<\infty} \|c(t) - c'(t)\|_{L^\infty(\mathbb{R}^N)} \\ & + \sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{r})}\|\nabla c(t) - \nabla c'(t)\|_{L^r(\mathbb{R}^N)} + \sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{r})}\|\nabla v(t) - \nabla v'(t)\|_{L^r(\mathbb{R}^N)} \\ & + \sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|u(t) - u'(t)\|_{L^p(\mathbb{R}^N)} < \eta \qquad \textit{for } N \geq 2.\end{aligned}$$

As a byproduct of our construction of solutions in the weak L^p-spaces, we have the following existence result on forward self-similar solutions to (NCS).

Corollary 1 ([3]). *(self-similar solution) Let $N \geq 3$ and $\gamma = 0$. Assume that $\{n_0, c_0, v_0, u_0\}$ and f are as in the Assumption. Suppose that n_0, c_0, v_0 and u_0 are homogeneous functions with degree $-2, 0, 0$ and -1, respectively, i.e.,*

$$n_0(\lambda x) = \lambda^{-2} n_0(x), \ c_0(\lambda x) = c_0(x), \ v_0(\lambda x) = v_0(x), \ u_0(\lambda x) = \lambda^{-1} u_0(x)$$

for all $x \in \mathbb{R}^N$ and all $\lambda > 0$. Assume also that $f = f(x)$ is a homogeneous function of $x \in \mathbb{R}^N$ with degree -1, i.e., $f(\lambda x) = \lambda^{-1} f(x)$ for all $x \in \mathbb{R}^N$ and all $\lambda > 0$. If $\{n_0, c_0, v_0, u_0\}$ and f satisfy *the condition (2.1), then the solution $\{n, c, v, u\}$ given by Theorem 1 is a forward self-similar one, i.e., it holds that*

$$n(\lambda x, \lambda^2 t) = \lambda^{-2} n(x,t), c(\lambda x, \lambda^2 t) = c(x,t), v(\lambda x, \lambda^2 t) = v(x,t), u(\lambda x, \lambda^2 t) = \lambda^{-1} u(x,t)$$

for all $x \in \mathbb{R}^N$, $t > 0$ and all $\lambda > 0$.

3 Key lemma.

To solve (IE) for the given initial data $\{n_0, c_0, v_0, u_0, f\}$, we make use of the implicit function theorem. Let us introduce two function spaces X and Y defined by

$$X \equiv \left\{ \{n_0, c_0, v_0, u_0, f\}; n_0 \in L_w^{\frac{N}{2}}, c_0 \in L^\infty, \nabla c_0 \in L_w^N, \nabla v_0 \in L_w^N, u_0 \in L_w^N, f \in L_w^N \right\}$$

with the norm

$$\|\{n_0, c_0, v_0, u_0, f\}\|_X \equiv \|n_0\|_{L_w^{\frac{N}{2}}} + \|c_0\|_{L^\infty} + \|\nabla c_0\|_{L_w^N} + \|\nabla v_0\|_{L_w^N} + \|u_0\|_{L_w^N} + \|f\|_{L_w^N}$$

and

$$\begin{aligned} Y \equiv \Big\{ & \{n, c, v, u\};\ t^{\frac{N}{2}(\frac{2}{N} - \frac{1}{q})} n(\cdot) \in BC_w([0,\infty); L^q),\ c \in L^\infty(0,\infty; L^\infty) \\ & \text{with } t^{\frac{N}{2}(\frac{1}{N} - \frac{1}{r})} \nabla c(\cdot) \in BC_w([0,\infty); L^r),\ t^{\frac{N}{2}(\frac{1}{N} - \frac{1}{r})} \nabla v(\cdot) \in BC_w([0,\infty); L^r), \\ & t^{\frac{N}{2}(\frac{1}{N} - \frac{1}{p})} u(\cdot) \in BC_w([0,\infty); L^p) \Big\} \end{aligned}$$

with the norm

$$\begin{aligned} \|\{n, c, v, u\}\|_Y \equiv & \sup_{0<t<\infty} t^{\frac{N}{2}(\frac{2}{N} - \frac{1}{q})} \|n(t)\|_{L^q} + \sup_{0<t<\infty} \|c(t)\|_{L^\infty} + \sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N} - \frac{1}{r})} \|\nabla c(t)\|_{L^r} \\ & + \sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N} - \frac{1}{r})} \|\nabla v(t)\|_{L^r} + \sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N} - \frac{1}{p})} \|u(t)\|_{L^p}, \end{aligned}$$

respectively. For $N = 2$, we replace $n_0 \in L_w^1$ by $n_0 \in L^1$ in X.

Here and in what follows, we abbreviate $L^p(\mathbb{R}^N)$ and $L_w^p(\mathbb{R}^N)$ to L^p and L_w^p, respectively. It should be noted that L_w^p denotes the weak L^p-space with the norm $\|\cdot\|_{L_w^p}$ defined by

$$\|f\|_{L_w^p} = \sup_{s>0} s\mu\{x \in \mathbb{R}^N; |f(x)| > s\}^{\frac{1}{p}},$$

where μ denotes the Lebesgue measure.

It is easy to see that equipped with the norm $\|\cdot\|_X$ and $\|\cdot\|_Y$, X and Y are Banach spaces. For $\{n_0, c_0, v_0, u_0, f\} \in X$ and $\{n, c, v, u\} \in Y$, we define the map

$$F(n_0, c_0, v_0, u_0, f, n, c, v, u) \equiv \{N, C, V, U\}, \tag{3.1}$$

where

$$\begin{cases} N(t) &= n(t) - e^{t\Delta}n_0 + \int_0^t e^{(t-\tau)\Delta}(u\cdot\nabla n)(\tau)\,d\tau + \int_0^t \nabla\cdot e^{(t-\tau)\Delta}(n\nabla c + n\nabla v)(\tau)\,d\tau, \\ C(t) &= c(t) - e^{t\Delta}c_0 + \int_0^t e^{(t-\tau)\Delta}(u\cdot\nabla c + nc)(\tau)\,d\tau, \\ V(t) &= v(t) - e^{-\gamma t}e^{t\Delta}v_0 + \int_0^t e^{-\gamma(t-\tau)}e^{(t-\tau)\Delta}(u\cdot\nabla v - n)(\tau)\,d\tau, \\ U(t) &= u(t) - e^{t\Delta}u_0 + \int_0^t e^{(t-\tau)\Delta}P(u\cdot\nabla u + nf)(\tau)\,d\tau, \qquad 0 < t < \infty. \end{cases}$$

In addition, for each $\{n,c,v,u\} \in Y$, we define a linear map $L_{\{n,c,v,u\}}(\tilde{n},\tilde{c},\tilde{v},\tilde{u}) = \{\tilde{\mathcal{N}},\tilde{\mathcal{C}},\tilde{\mathcal{V}},\tilde{\mathcal{U}}\}$ on Y by

$$\begin{cases} \tilde{\mathcal{N}}(t) &= \tilde{n}(t) + \int_0^t e^{(t-\tau)\Delta}(u\cdot\nabla\tilde{n})(\tau)\,d\tau + \int_0^t e^{(t-\tau)\Delta}(\tilde{u}\cdot\nabla n)(\tau)\,d\tau \\ &\quad + \int_0^t \nabla\cdot e^{(t-\tau)\Delta}(\tilde{n}\nabla c + n\nabla\tilde{c})(\tau)\,d\tau + \int_0^t \nabla\cdot e^{(t-\tau)\Delta}(\tilde{n}\nabla v + n\nabla\tilde{v})(\tau)\,d\tau, \\ \tilde{\mathcal{C}}(t) &= \tilde{c}(t) + \int_0^t e^{(t-\tau)\Delta}(\tilde{u}\cdot\nabla c + u\cdot\nabla\tilde{c})(\tau)\,d\tau + \int_0^t e^{(t-\tau)\Delta}(\tilde{n}c + n\tilde{c})(\tau)\,d\tau, \\ \tilde{\mathcal{V}}(t) &= \tilde{v}(t) + \int_0^t e^{-\gamma(t-\tau)}e^{(t-\tau)\Delta}(\tilde{u}\cdot\nabla v + u\cdot\nabla\tilde{v})(\tau)\,d\tau - \int_0^t e^{-\gamma(t-\tau)}e^{(t-\tau)\Delta}\tilde{n}(\tau)\,d\tau, \\ \tilde{\mathcal{U}}(t) &= \tilde{u}(t) + \int_0^t e^{(t-\tau)\Delta}P(\tilde{u}\cdot\nabla u + u\cdot\nabla\tilde{u})(\tau)\,d\tau + \int_0^t e^{(t-\tau)\Delta}P(\tilde{n}f)(\tau)\,d\tau. \end{cases}$$

Then, we have the following key lemma.

Lemma 1 *For $N \geq 3$, suppose that the exponents p, q and r satisfy the following either (i), (ii) or (iii).*

$$\begin{array}{llll} (i) & \frac{N}{2} < q < N, & N < p < \frac{Nq}{N-q}, & N < r < \frac{Nq}{N-q}; \\ (ii) & q = N, & N < p < \infty, & N < r < \infty; \\ (iii) & N < q < 2N, & N < p < \frac{Nq}{q-N}, & q \leq r < \frac{Nq}{q-N}. \end{array}$$

For $N = 2$, we assume that the exponents p, q and r satisfy the above condition (iii) with $N = 2$.
(i) The map F defined by (3.1) is a continuous map from $X \times Y$ into Y;

(ii) For each $\{n_0, c_0, v_0, u_0, f\} \in X$, the map $F(n_0, c_0, v_0, u_0, f, \cdot, \cdot, \cdot, \cdot)$ is of class C^1 from Y into itself.

See [3] for the proof.

Remark. It should be noticed that, $L_{\{n,c,v,u\}}$ is the Fréchet derivative of $F(n_0, c_0, v_0, u_0, f, n, c, v, u)$ at $\{n, c, v, u\} \in Y$, for each fixed $\{n_0, c_0, v_0, u_0, f\} \in X$, i.e., it holds that

$$\lim_{\|\{\tilde{n},\tilde{c},\tilde{v},\tilde{u}\}\|_Y\to 0}\Big(\|F(n_0,c_0,v_0,u_0,f,n+\tilde{n},c+\tilde{c},v+\tilde{v},u+\tilde{u})-F(n_0,c_0,v_0,u_0,f,n,c,v,u)$$
$$-L_{\{n,c,v,u\}}(\tilde{n},\tilde{c},\tilde{v},\tilde{u})\|_Y\Big)/\|\{\tilde{n},\tilde{c},\tilde{v},\tilde{u}\}\|_Y=0$$

for each $\{n_0,c_0,v_0,u_0,f\}\in X$ and each $\{n,c,v,u\}\in Y$.

4 Proof of theorems.

We shall show bijectivity of the Fréchet derivative $L_{\{n,c,v,u\}}$ at $\{n,c,v,u\}=\{0,0,0,0\}$. It follows from the proof of Lemma 1, that we have an expression $L_{\{0,0,0,0\}}(\tilde{n},\tilde{c},\tilde{v},\tilde{u})=\{\tilde{\mathcal{N}}_0,\tilde{\mathcal{C}}_0,\tilde{\mathcal{V}}_0,\tilde{\mathcal{U}}_0\}$ as

$$\begin{aligned}\tilde{\mathcal{N}}_0(t) &= \tilde{n}(t),\quad \tilde{\mathcal{C}}_0(t)=\tilde{c}(t),\quad \tilde{\mathcal{V}}_0(t)=\tilde{v}(t)-\int_0^t e^{-\gamma(t-\tau)}e^{(t-\tau)\Delta}\tilde{n}(\tau)\,d\tau,\\ \tilde{\mathcal{U}}_0(t) &= \tilde{u}(t)\end{aligned}$$

for $\{\tilde{n},\tilde{c},\tilde{v},\tilde{u}\}\in Y$. Hence it is easy to see that $\tilde{\mathcal{N}}_0=\tilde{\mathcal{C}}_0=\tilde{\mathcal{V}}_0=\tilde{\mathcal{U}}_0=0$ implies that $\tilde{n}=\tilde{c}=\tilde{v}=\tilde{u}=0$, which yields that $L_{\{0,0,0,0\}}$ is injective.

For every $\{\tilde{\mathcal{N}}_0,\tilde{\mathcal{C}}_0,\tilde{\mathcal{V}}_0,\tilde{\mathcal{U}}_0\}\in Y$, we may take $\{\tilde{n},\tilde{c},\tilde{v},\tilde{u}\}\in Y$ as

$$\begin{aligned}\tilde{n}(t) &= \tilde{\mathcal{N}}_0(t),\quad \tilde{c}(t)=\tilde{\mathcal{C}}_0(t),\quad \tilde{v}(t)=\tilde{\mathcal{V}}_0(t)+\int_0^t e^{-\gamma(t-\tau)}e^{(t-\tau)\Delta}\tilde{\mathcal{N}}_0(\tau)\,d\tau,\\ \tilde{u}(t) &= \tilde{\mathcal{U}}_0(t)\end{aligned}$$

so that it holds

$$L_{\{0,0,0,0\}}(\tilde{n},\tilde{c},\tilde{v},\tilde{u})=\{\tilde{\mathcal{N}}_0,\tilde{\mathcal{C}}_0,\tilde{\mathcal{V}}_0,\tilde{\mathcal{U}}_0\}.$$

This implies that $L_{\{0,0,0,0\}}$ is surjective from Y onto itself.

Now, it follows from the Banach implicit function theorem that there is a C^1-map g

$$\begin{aligned}g:X_\delta &:= \{\{n_0,c_0,v_0,u_0,f\}\in X;\|\{n_0,c_0,v_0,u_0,f\}\|_X<\delta\}\\ &\to Y_\delta:=\{\{n,c,v,u\}\in Y;\|\{n,c,v,u\}\|_Y<\delta\}\end{aligned}$$

for some $\delta=\delta(N,p,q,r)>0$ such that

$$\begin{aligned}g(0,0,0,0,0) &= \{0,0,0,0\},\\ F(n_0,c_0,v_0,u_0,f,g(n_0,c_0,v_0,u_0,f)) &= \{0,0,0,0\}\end{aligned}$$

for all $\{n_0,c_0,v_0,u_0,f\}\in X_\delta$.

It is easy to see that this $g(n_0,c_0,v_0,u_0,f)$ gives the unique solution of (IE) with properties (2.3)–(2.6) provided $\{n_0,c_0,v_0,u_0,f\}$ satisfies (2.1) and (2.2).

The uniqueness of solutions $\{n,c,v,u\}$ of (IE) with the small norms corresponding to the class (2.3)–(2.6) is a consequence of the existence of the C^1-map g from X_δ to Y_δ. See [3] in detail.

References

[1] M. Chae, K. Kang and J. Lee, *Existence of smooth solutions to coupled chemotaxis-fluid equations.* Discrete Contin. Dyn. Syst. **33** (2013), 2271-2297.

[2] E. F. Keller and L. A. Segel, *Initiation of slime mold aggregation viewed as an instability,* J. Theor. Biol. **26** (1970), 399-415.

[3] H. Kozono, M. Miura and Y. Sugiyama, *Existence and uniqueness theorem on mild solutions to the Keller-Segel system coupled with the Navier-Stokes fluid,* J. Funct. Anal. **270** (2016), 1663-1683.

[4] H. Kozono, M. Miura and Y. Sugiyama, *Time global existence and finite time blow-up criterion for solutions to the Keller-Segel system coupled with Navier-Stokes fluid,* submitted.

[5] A. Lorz, *A coupled Keller-Segel-Stokes model: global existence for small initial data and blow-up delay.* Commun. Math. Sci. **10** (2012), 555-574.

[6] Y. Tao and M. Winkler, *Locally bounded global solutions in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion.* Ann. Inst. H. Poincar Anal. Non Linaire 30 (2013), 157-178.

[7] I. Tuval, L. Cisneros, C. Dombrowski, C.W. Wolgemuth, J.O. Kessler and R.E. Goldstein, *Bacterial swimming and oxygen transport near contact lines.* Proc. Natl. Acad. Sci. USA **102** (2005), 2277-2282.

[8] M. Winkler, *Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops.* Comm. Partial Differential Equations **37** (2012), 319-351.

[9] Q. Zhang and X. Zheng, *Global well-posedness for the two-dimensional incompressible chemotaxis-Navier-Stokes equations.* SIAM J. Math. Anal. **46** (2014), 3078-3105.