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Solvability of complex Ginzburg-Landau equations with non-dissipative terms

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Abstract

In this paper, we consider the following complex Ginzburg-Landau equation.

$$\text{(CGL)} \qquad \begin{cases} u_t - (\lambda + i\alpha)\Delta u - (\kappa + i\beta)|u|^{q-2}u - \gamma u = f & (t,x) \in [0,T] \times \Omega, \\ u(t,x) = 0 & (t,x) \in [0,T] \times \partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. Parameters λ, κ are positive, while $\alpha, \beta, \gamma \in \mathbb{R}$ are real parameters and $i = \sqrt{-1}$ is the imaginary unit.

We assume that q is Sobolev sub-critical, i.e., $2 < q < +\infty$ when $N = 1, 2$ and $2 < q < \frac{2N}{N-2}$ when $N \geq 3$. We study the local well-posedness of (CGL) and the global continuation of local solutions for small data.

1 Introduction

We are concerned with the Cauchy problem the following complex Ginzburg-Landau equation.

$$\text{(CGL)} \qquad \begin{cases} u_t - (\lambda + i\alpha)\Delta u - (\kappa + i\beta)|u|^{q-2}u - \gamma u = f & (t,x) \in [0,T] \times \Omega, \\ u(t,x) = 0 & (t,x) \in [0,T] \times \partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. The unknown function $u : [0,T] \times \Omega \to \mathbb{C}$ of time variable t and space variable x represents an order parameter, which indicates the phase of dissipative structures and takes complex values. In our equation, $\lambda, \kappa > 0$ and $\alpha, \beta, \gamma \in \mathbb{R}$ and $i = \sqrt{-1}$ is the imaginary unit.

The complex Ginzburg-Landau equation is originally introduced in 1950 by Ginzburg and Landau [6] to present a mathematical model for superconductivity. This equation is now known as a general equation which describes the various phenomena of dissipative structures around a critical point (see [5], [9]).

We deal with the initial value problem under the homogeneous Dirichlet boundary condition with a given external force $f : [0,T] \to \mathbb{C}$, where $T > 0$ is a given positive number.

From the mathematical point of view, we can regard (CGL) as an intermediate equation between two typical nonlinear equations. The real part of (CGL) can be regarded as a nonlinear heat equation and sometimes called as a real Ginzburg-Landau equation, while the imaginary part of (CGL) can be regarded as a nonlinear Schrödinger equation. Hence, we can expect both parabolic and Schrödinger like features for (CGL).

There are different approaches to (CGL) in accordance with the sign of κ.

When κ is negative, the nonlinear term $-\kappa|u|^{q-2}u$ play as the dissipation. So we can deduce a priori estimates without Sobolev's imbedding theorem under the appropriate assumption on parameters $\lambda, \kappa, \alpha, \beta$.

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From this point of view, global solutions for (CGL) is constructed without upper restriction for q provided that $\lambda, \kappa, \alpha, \beta$ are restricted to the so-called CGL region by Okazawa and Yokota [13] for bounded domains, and by K., Ôtani and Shimizu [7] for general domains.

On the other hand, for positive κ the nonlinear term in turn facilitates the increment of a solution. Because of this fact, we have some difficulties in establishing a priori estimates and we also expect some solutions might blow-up in finite time.

When the initial data u_0 is taken from H_0^1 and q is Sobolev subcritical, the following three results on local well-posedness are known. Cazenave, Dickstein and Weissler [4] proved the existence of a local solution in the whole space for the case where $\frac{\alpha}{\lambda} = \frac{\beta}{\kappa}$, $\gamma = 0$ and $f \equiv 0$. They also proved some blow-up results. Secondly, Cazenave, Dias and Figueira [3] obtained results similar to those in [4] for the case where $\gamma \neq 0$. On the other hand, for an initial data u_0 taken from L^p and $2 < q < 2 + \frac{2N}{p}$, Shimotsuma, Yokota and Yoshii [15] showed the existence of a local solution for various kinds of domains for $f \equiv 0$. They also proved the global continuation of solution with small initial data. These approaches relies on the theory of semi-groups in complex Banach spaces.

In this paper, we introduce new approach for (CGL) in non-dissipative case based on the theory of parabolic equations with perturbations in real Hilbert space, which is successfully admissible for dissipative case. We follows an abstract theory developed by Ôtani [10].

2 Preliminaries

We first introduce product function spaces made up of usual Lebesugue and Sobolev spaces over the real field using the following identification:

$$\mathbb{C} \ni u_1 + iu_2 \to U = (u_1, u_2) \in \mathbb{R}^2.$$

These spaces are also Banach or Hilbert spaces with respect to these norms or inner products.

$\mathbb{L}^r(\Omega) := \mathrm{L}^r(\Omega) \times \mathrm{L}^r(\Omega) \ni U = (u_1, u_2), V = (v_1, v_2),$
norm: $|U|_{\mathbb{L}^r}^r = |u_1|_{\mathrm{L}^r}^r + |u_2|_{\mathrm{L}^r}^r,$
inner product($r = 2$): $(U, V)_{\mathbb{L}^2} = (u_1, v_1)_{\mathrm{L}^2} + (u_2, v_2)_{\mathrm{L}^2}.$
$\mathbb{H}_0^1(\Omega) := \mathrm{H}_0^1(\Omega) \times \mathrm{H}_0^1(\Omega) \ni U = (u_1, u_2), V = (v_1, v_2),$
inner product: $(U, V)_{\mathbb{H}^1} = (u_1, v_1)_{\mathrm{H}^1} + (u_2, v_2)_{\mathrm{H}^1}.$

In addition $\mathcal{H}^S$ denotes the function space with values in $\mathbb{L}^2(\Omega)$ from $[0, S]$ with $S > 0$, which is a Hilbert space with the following inner product.

$\mathcal{H}^S := L^2(0, S; \mathbb{L}^2(\Omega)) \ni U(t), V(t)$, inner product: $(U, V)_{\mathcal{H}^S} = \int_0^S (U, V)_{\mathbb{L}^2}^2 dt,$
norm: $||U||_{\mathcal{H}^S}^2 = (U, U)_{\mathcal{H}^S}.$

Instead of the imaginary unit i in complex field $\mathbb{C}$, we introduce the matrix I defined by

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.1}$$

which plays the same role as the imaginary unit with respect to the inner product, e.g.,

$$\Re(u, v)_{\mathbb{C}} = \Re u\bar{v} = (U, V)_{\mathbb{R}^2},$$
$$\Im(u, v)_{\mathbb{C}} = \Im u\bar{v} = (U, IV)_{\mathbb{R}^2}.$$

To apply the theory of parabolic equations, we write down each term in (CGL) in terms of subdifferentials of some functionals.

Let H be a Hilbert space and denote by $\Phi(\mathrm{H})$ the set of all lower semi-continuous convex function ϕ from H into $(-\infty, +\infty]$ such that the effective domain of ϕ given by $\mathrm{D}(\phi) := \{u \in \mathrm{H} \mid \phi(u) < +\infty\}$ is not empty. Then for $\phi \in \Phi(\mathrm{H})$, the subdifferential of ϕ at $u \in \mathrm{D}(\phi)$ is defined by

$$\partial\phi(u) := \{f \in \mathrm{H} \mid (f, v - u)_{\mathrm{H}} \leq \phi(v) - \phi(u) \text{ for all } v \in \mathrm{H}\}. \tag{2.2}$$

Then $\partial\phi$ becomes a possibly multivalued maximal monotone operator with domain $\mathrm{D}(\partial\phi) = \{u \in \mathrm{H} \mid \partial\phi(u) \neq \emptyset\}$. However for the arguments in what follows, we have only to consider the case where $\partial\phi$ is single valued.

So we define functionals on Hilbert space $\mathbb{L}^2(\Omega)$.

$$\varphi(U) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla U(x)|^2 dx = \dfrac{1}{2}|\nabla U|^2_{\mathbb{L}^2} & U \in \mathbb{H}^1_0(\Omega), \\ +\infty & \text{else.} \end{cases} \tag{2.3}$$

$$\psi_r(U) := \begin{cases} \dfrac{1}{r}\displaystyle\int_\Omega |U(x)|^r dx = \dfrac{1}{r}|U|^r_{\mathbb{L}^r} & U \in \mathbb{L}^r(\Omega), \\ +\infty & \text{else.} \end{cases} \tag{2.4}$$

Since these functionals are proper ($\not\equiv +\infty$), convex and lower semi-continuous, the subdifferentials of these are given as follows.

$$\partial\varphi(U) = -\Delta U, \qquad \mathrm{D}(\partial\varphi) = \left\{U \in \mathbb{L}^2(\Omega) \mid U \in \mathbb{H}^2(\Omega) \cap \mathbb{H}^1_0(\Omega)\right\}. \tag{2.5}$$

$$\partial\psi_r(U) = |U|^{r-2}U, \qquad \mathrm{D}(\partial\psi_r) = \left\{U \in \mathbb{L}^2(\Omega) \mid \ |U|^{r-2}U \in \mathbb{L}^2(\Omega)\right\}. \tag{2.6}$$

By the maximal monotonicity of subdifferential operators, we can consider their Yosida approximations, or Yosida regularizations of functionals. Here we fix notations for resolvent operators and Yosida approximations, and collect their properties for later use.

Let ϕ be a proper convex lower semi-continuous functional on a Hilbert Space H. Since the subdifferential $\partial\phi$ of ϕ is maximal monotone in H, we can define its resolvent $J^\phi_\mu := (1 + \mu\partial\phi)^{-1} : \mathrm{H} \to \mathrm{D}(\partial\phi)$ for $\mu > 0$ and the Yosida approximation of $\partial\phi$ is given by $\partial\phi_\mu := \partial\phi J^\phi_\mu$. It is known that the Yosida approximation of $\partial\phi$ corresponds to the subdifferential of the Moreau-Yosida regularization ϕ_μ of ϕ, which is a Fréchet differentiable function given by

$$\phi_\mu(u) = \inf_{v\in\mathrm{H}} \left\{\frac{1}{2\mu}|u - v|^2_{\mathrm{H}} + \phi(v)\right\} = \frac{\mu}{2}|(\partial\phi)_\mu(u)|_{\mathrm{H}} + \phi(J^\phi_\mu(u)), \tag{2.7}$$

and the following inequality holds (see [10], [1], [2]):

$$|\partial\phi_\mu(u)|_{\mathrm{H}} \leq |\partial\phi(u)|_{\mathrm{H}} \quad \text{for every } u \in \mathrm{D}(\partial\phi). \tag{2.8}$$

Using these notations and I defined by (2.1), we can rewrite our partial differential equation (CGL) in an evolution equation in $\mathbb{L}^2(\Omega)$.

$$\frac{dU}{dt}(t) + (\lambda + \alpha I)\partial\varphi(U) - (\kappa + \beta I)\partial\psi_q(U) - \gamma\partial\psi_2(U) = F(t). \tag{ACGL$_-$}$$

Here we collect the properties of the matrix I:

(2.9)	an imaginary unit	: $I^2 = -1$,
(2.10)	isometricity	: $\|U\|_{\mathbb{L}^2} = \|IU\|_{\mathbb{L}^2}$,
(2.11)	skew-symmetricity	: $(U, IV)_{\mathbb{L}^2} = -(IU, V)_{\mathbb{L}^2}$,
(2.12)	commutativity	: $I\partial\varphi(U) = \partial\varphi(IU), I\partial\psi_r(U) = \partial\psi_r(IU)$,
(2.13)	orthogonarity in $\mathbb{R}^2$	: $(U \cdot IU)_{\mathbb{R}^2} = 0$
(2.14)	orthogonarity in $\mathbb{L}^2(\Omega)$ 1	: $(U, IU)_{\mathbb{L}^2} = (U, I\partial\varphi(U))_{\mathbb{L}^2} = (U, I\partial\psi_r(U))_{\mathbb{L}^2} = 0$,
(2.15)	orthogonarity in $\mathbb{L}^2(\Omega)$ 2	: $(\partial\varphi_\mu(U), IU)_{\mathbb{L}^2} = 0 = (\partial\varphi_\mu(U), I\partial\varphi(U))_{\mathbb{L}^2}$,
(2.16)	orthogonarity in $\mathbb{L}^2(\Omega)$ 3	: $(\partial\psi_{r,\mu}(U), IU)_{\mathbb{L}^2} = 0 = (\partial\psi_{r,\mu}(U), I\partial\psi(U))_{\mathbb{L}^2}$,
(2.17)	Bessel's inequality	: $(U, V)_{\mathbb{L}^2}^2 + (U, IV)_{\mathbb{L}^2}^2 \leq \|U\|_{\mathbb{L}^2}^2 \|V\|_{\mathbb{L}^2}^2$,

where $\partial\varphi_\mu = (\partial\varphi)_\mu = \partial\varphi(1+\mu\partial\varphi)^{-1}$ and $\partial\psi_{r,\mu} = (\partial\psi_r)_\mu = \partial\psi_r(1+\mu\partial\psi_r)^{-1}$ denotes the Yosida approximations of $\partial\varphi$ and $\partial\psi_r$. Though these properties can be proved by direct calculations, we only show the proofs for (2.15) and (2.16).

Proof of (2.15).
Let $V := (1+\mu\partial\varphi)^{-1}U$. Then by (2.11), (2.14) and self-adjointness of $\partial\varphi$, we have the first identity:

$$(I\partial\varphi_\mu(U), U)_{\mathbb{L}^2} = (I\partial\varphi(V), (1+\mu\partial\varphi)V)_{\mathbb{L}^2} = (I\partial\varphi(V), V)_{\mathbb{L}^2} = 0.$$

By virtue of (2.11) and (2.14), we get

$$\begin{aligned}(I\partial\varphi_\mu(U), \partial\varphi(U))_{\mathbb{L}^2} &= (I\partial\varphi(V), \partial\varphi(U))_{\mathbb{L}^2} = \frac{1}{\mu}(I(U-V), \partial\varphi(U))_{\mathbb{L}^2} \\ &= -\frac{1}{\mu}(IV, \partial\varphi(U))_{\mathbb{L}^2} = -\frac{1}{\mu}(I\partial\varphi(V), U)_{\mathbb{L}^2} \\ &= -\frac{1}{\mu}(I\partial\varphi(V), V+\mu\partial\varphi(V))_{\mathbb{L}^2} = -\frac{1}{\mu}(I\partial\varphi(V), V)_{\mathbb{L}^2} = 0.\end{aligned}$$

□

Proof of (2.16).
Let $V := (1+\mu\partial\psi_q)^{-1}U$. By (2.11) and (2.14), we obtain

$$(I\partial\psi_{q,\mu}(U), U)_{\mathbb{L}^2} = (I\partial\psi_q(V), V+\mu\partial\psi_q(V))_{\mathbb{L}^2} = 0.$$

As for the second identity, we obtain by (2.13),

$$\begin{aligned}&(I\partial\psi_{r,\mu}(U), \partial\psi_r(U))_{\mathbb{L}^2} \\ &= (I\partial\psi_r(V), \partial\psi_r(U))_{\mathbb{L}^2} = \frac{1}{\mu}(I(U-V), \partial\psi_r(U))_{\mathbb{L}^2} \\ &= \frac{1}{\mu}(IV, \partial\psi_r(U))_{\mathbb{L}^2} = \frac{1}{\mu}\int_\Omega (IV \cdot |V+\mu\partial\psi_r V|_{\mathbb{R}^2}^{r-2}(V+\mu\partial\psi_r V))_{\mathbb{R}^2} \\ &= \frac{1}{\mu}\int_\Omega (IV \cdot |V+\mu\partial\psi_r V|_{\mathbb{R}^2}^{r-2}(V+\mu|V|_{\mathbb{R}^2}^{r-2}V))_{\mathbb{R}^2} = 0,\end{aligned}$$

where we use temporal notation $|\cdot|_{\mathbb{R}^2}$ for the length of vectors in $\mathbb{R}^2$. □

Under these preparations, we state local well-posedness for (ACGL$_-$) in bounded domains.

Theorem 1 Local well-posedness in bounded domains.
Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of C^2-regular class, $F \in \mathcal{H}^T$ and $2 < q < 2^*$ (subcritical),

$$2^* = \begin{cases} +\infty & (N = 1, 2), \\ \frac{2N}{N-2} & (N \geq 3). \end{cases}$$

Then for all $U_0 \in \mathbb{H}_0^1(\Omega) = \mathrm{D}(\varphi)$, there exists $0 < T_0 \leq T$ and the unique function $U(t) \in \mathrm{C}([0, T_0]; \mathbb{L}^2(\Omega))$ satisfying:

(i) $U \in \mathrm{W}^{1,2}(0, T_0; \mathbb{L}^2(\Omega))$,

(ii) $U(t) \in \mathrm{D}(\partial\varphi) \subset \mathrm{D}(\partial\psi_q)$ for a.e. $t \in [0, T_0]$ and satisfies (ACGL$_-$) for a.e. $t \in [0, T_0]$,

(iii) $\partial\varphi(\cdot), \partial\psi_q(\cdot) \in \mathcal{H}^{T_0}$.

Furthermore the following alternative on the maximal existence time of the solution holds:

Theorem 2 Alternative.
Let T_0 be the maximal existence time of the solution to (ACGL$_-$) obtained in Theorem 1. Then the following alternative on T_0 holds:

- $T_0 = T$, **or**
- $T_0 < T$ and $\lim_{t \uparrow T_0} \varphi(U(t)) = +\infty$.

In order to formulate the existence of small global solutions, we need to use the first eigenvalue λ_1 of $-\Delta$ with homogeneous Dirichlet boundary condition defined by

$$\psi_2(U) \leq \lambda_1^{-1} \varphi(U), \quad \forall U \in \mathbb{H}_0^1(\Omega). \tag{2.18}$$

For $F \in \mathrm{L}^2(0, T; \mathbb{L}^2(\Omega))$, let $\tilde{F}$ be the extention by zero of F over $(T, +\infty)$. We set the notation for scaling the external force F in terms of $\tilde{F}$

$$|||F|||^2 := \sup \left\{ \int_s^{s+1} |\tilde{F}(t)|_{\mathbb{L}^2}^2 dt \mid 0 \leq s < +\infty \right\}.$$

Theorem 3 Existence of small global solutions.
Let all the assumptions in Theorem 1 be satisfied and let $\gamma < \lambda\lambda_1$. Then there exists a sufficiently small number r independent of T such that for all $U_0 \in D(\varphi)$ and $F \in L^2(0, T; \mathbb{L}^2(\Omega))$ with $\varphi(U_0) \leq r^2$ and $|||F||| \leq r$, every local solution given in Theorem 1 can be continued globally up to $[0, T]$.

3 Solvability of Auxiliary Equation

In this section, we consider the following auxiliary equation:

$$(\mathrm{AE}^h) \qquad \begin{cases} \frac{dU_h}{dt}(t) + (\lambda + \alpha I)\partial\varphi(U_h) - (\kappa + \beta I)h(t) - \gamma U_h = F(t) & t \in [0, S], \\ U_h(0) = U_0, \end{cases}$$

which is (ACGL$_-$) with $\partial\psi_q(U)$ replaced by $h(\cdot) \in \mathcal{H}^S$, for arbitrary $0 < S \leq T$.

For this auxiliary equation (AE^h), we can show the global well-posedness:

Proposition 4.
Let $\Omega \subset \mathbb{R}^N$ be bounded or unbounded domain of C^2-regular class, $F \in \mathcal{H}^T$ and $h \in \mathcal{H}^S$, $0 < S \leq T$. For all $U_0 \in \mathbb{H}_0^1(\Omega) = D(\varphi)$, there exists the unique global solution $U(t) \in \mathrm{C}([0, S]; \mathbb{L}^2(\Omega))$ satisfying:

(i) $U \in \mathrm{W}^{1,2}(0,S;\mathbb{L}^2(\Omega))$,

(ii) $U(t) \in \mathrm{D}(\partial\varphi)$ for a.e. $t \in [0,S]$ and satisfies (AE^h) for a.e. $t \in [0,S]$,

(iii) $\partial\varphi(\cdot) \in \mathcal{H}^S$.

First we consider the following approximate equation:

$$(\mathrm{AE}^h_\mu) \qquad \begin{cases} \dfrac{dU_\mu}{dt}(t) + \lambda\partial\varphi(U_\mu) + \alpha I\partial\varphi_\mu(U_\mu) - (\kappa+\beta I)h(t) - \gamma U_\mu = F(t), & t \in [0,S], \\ U_\mu(0) = U_0, \end{cases}$$

which is (AE^h) with $I\partial\phi$ replaced by $I\partial\phi_\mu$. By the standard theory of subdifferential operators, we can easily obtain the unique global solution for (AE^h_μ) satisfying whole properties stated in Proposition 4, since the Yosida approximation $\partial\varphi_\mu$ is Lipschitz continuous.

Here we establish some a priori estimates for the solution U_μ of (AE^h_μ).

Lemma 5 First Energy Estimate.
Let U_μ be the solution of (AE^h_μ). Then there exists C_1 depending only on $\lambda, \kappa, \beta, \gamma$, $|U_0|_{\mathbb{L}^2}, \|h\|_{\mathcal{H}^T}$ and $\|F\|_{\mathcal{H}^S}$ such that

$$\sup_{t\in[0,S]} |U_\mu(t)|^2_{\mathbb{L}^2} + \int_0^S \varphi(U_\mu(t))dt \le C_1. \tag{3.1}$$

Proof.
Multiplying (AE_{h_μ}) by U_μ and by (2.15), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|U_\mu|^2_{\mathbb{L}^2} + 2\lambda\varphi(U_\mu) \\ &= \gamma|U_\mu|^2_{\mathbb{L}^2} + ((\kappa-\beta I)h + F, U_\mu)_{\mathbb{L}^2} \\ &\le \left(\gamma_+ + \kappa^2 + \beta^2 + \frac{1}{2}\right)|U_\mu|^2_{\mathbb{L}^2} + \frac{1}{2}|h|^2_{\mathbb{L}^2} + \frac{1}{2}|F|^2_{\mathbb{L}^2}, \end{aligned} \tag{3.2}$$

where we use the notation $\gamma_+ := \max\{0,\gamma\}$ and the Cauchy-Schwarz inequality. Integrating (3.2) on $(0,S)$ and by Gronwall's inequality, we obtain

$$\begin{aligned} &\frac{1}{2}|U_\mu|^2_{\mathbb{L}^2} + 2\lambda\int_0^t \varphi(U_\mu)d\tau \\ &\le \frac{1}{2}\left(|U_0|^2_{\mathbb{L}^2} + \|h\|^2_{\mathcal{H}^S} + \|F\|^2_{\mathcal{H}^T}\right) \\ &\quad + \left(2(\gamma_+ + \kappa^2 + \beta^2) + 1\right)\int_0^t \left(\frac{1}{2}|U_\mu|^2_{\mathbb{L}^2} + 2\lambda\int_0^\tau \varphi(U_\mu)d\sigma\right)d\tau \\ &\le \frac{1}{2}\left(|U_0|^2_{\mathbb{L}^2} + \|h\|^2_{\mathcal{H}^S} + \|F\|^2_{\mathcal{H}^T}\right)e^{(2(\gamma_+ + \kappa^2 + \beta^2)+1)t} \\ &\le \frac{1}{2}\left(|U_0|^2_{\mathbb{L}^2} + \|h\|^2_{\mathcal{H}^S} + \|F\|^2_{\mathcal{H}^T}\right)e^{(2(\gamma_+ + \kappa^2 + \beta^2)+1)S}, \end{aligned}$$

which implies the desired estimate (3.1). □

Lemma 6 Second Energy Estimates.
Let U_μ be the solution of (AE^h_μ). Then there exists C_2 depending only on $\lambda, \kappa, \beta, \gamma$, $|U_0|_{\mathbb{L}^2}, \varphi(U_0), \|h\|_{\mathcal{H}^S}$ and $\|F\|_{\mathcal{H}^T}$ such that

$$\sup_{t\in[0,S]} \varphi(U_\mu(t)) + \int_0^S |\partial\varphi(U_\mu(t))|^2_{\mathbb{L}^2}dt + \int_0^S \left|\frac{dU_\mu}{dt}(t)\right|^2_{\mathbb{L}^2} dt \le C_2. \tag{3.3}$$

Proof.
Multiplying (AE_μ^h) by $\partial\varphi(U_\mu)$ and using (2.15), we obtain

$$\begin{aligned}&\frac{d}{dt}\varphi(U_\mu) + \lambda|\partial\varphi(U_\mu(t))|^2_{\mathbb{L}^2}\\ &= 2\gamma\varphi(U_\mu) + ((\kappa - \beta I)h + F, \partial\varphi U_\mu)_{\mathbb{L}^2}\\ &\leq 2\gamma_+\varphi(U_\mu) + \frac{1}{\lambda}\left((\kappa^2+\beta^2)|h|^2_{\mathbb{L}^2} + |F|^2_{\mathbb{L}^2}\right) + \frac{3\lambda}{4}|\partial\varphi(U_\mu(t))|^2_{\mathbb{L}^2},\end{aligned}$$

whence follows

$$\frac{d}{dt}\varphi(U_\mu) + \frac{\lambda}{4}|\partial\varphi(U_\mu(t))|^2_{\mathbb{L}^2} \leq 2\gamma_+\varphi(U_\mu) + \frac{1}{\lambda}\left((\kappa^2+\beta^2)|h|^2_{\mathbb{L}^2} + |F|^2_{\mathbb{L}^2}\right). \tag{3.4}$$

Integrating (3.4) on $(0,t)$ for $t \in (0,S]$ and by Lemma 5, we get

$$\begin{aligned}&\varphi(U_\mu) + \frac{\lambda}{4}\int_0^t |\partial\varphi(U_\mu(t))|^2_{\mathbb{L}^2} d\tau\\ &\leq \varphi(U_0) + 2\gamma_+\int_0^t \varphi(U_\mu)d\tau + \frac{1}{\lambda}\left((\kappa^2+\beta^2)\|h\|^2_{\mathcal{H}^S} + \|F\|^2_{\mathcal{H}^T}\right)\\ &\leq \varphi(U_0) + 2\gamma_+ C_1 + \frac{1}{\lambda}\left((\kappa^2+\beta^2)\|h\|^2_{\mathcal{H}^S} + \|F\|^2_{\mathcal{H}^T}\right) \quad \text{for all } t \in (0,S].\end{aligned} \tag{3.5}$$

Thus from (3.5), (2.8) and (AE_μ^h), we derive (3.3). □

Proof of Proposition 4.
Let U_μ be a solution of (AE_μ^h) and fix $T > 0$. First we show $\{U_\mu\}_{\mu>0}$ forms a Cauchy net in $C([0,S]; \mathbb{L}^2(\Omega))$. To this end, we multiply (AE_μ^h)–(AE_ν^h) by $U_\mu - U_\nu$ to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|U_\mu - U_\nu|^2_{\mathbb{L}^2} + 2\lambda\varphi(U_\mu - U_\nu)\\ &= \gamma|U_\mu - U_\nu|^2_{\mathbb{L}^2} + \alpha\left(I\partial\varphi_\mu U_\mu - I\partial\varphi_\nu U_\nu, U_\mu - U_\nu\right)_{\mathbb{L}^2}.\end{aligned}$$

Applying Kōmura's trick, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|U_\mu - U_\nu|^2_{\mathbb{L}^2}\\ &\leq \gamma_+|U_\nu - U_\mu|^2_{\mathbb{L}^2} + |\alpha|\left\{\mu|\partial\varphi_\nu(U_\nu)|_{\mathbb{L}^2}|\partial\varphi_\mu(U_\mu)|_{\mathbb{L}^2} + \nu|\partial\varphi_\mu(U_\mu)|_{\mathbb{L}^2}|\partial\varphi_\nu(U_\nu)|_{\mathbb{L}^2}\right\}\\ &\leq \gamma_+|U_\nu - U_\mu|^2_{\mathbb{L}^2} + \frac{|\alpha|}{2}(\mu+\nu)\left\{|\partial\varphi(U_\mu)|^2_{\mathbb{L}^2} + |\partial\varphi(U_\nu)|^2_{\mathbb{L}^2}\right\}.\end{aligned}$$

Thus Gronwall's inequality yields

$$|U_\mu(t) - U_\nu(t)|^2_{\mathbb{L}^2} \leq |\alpha|(\mu+\nu)e^{2\gamma_+ t}\int_0^t \left\{|\partial\varphi(U_\mu(s))|^2_{\mathbb{L}^2} + |\partial\varphi(U_\nu(s))|^2_{\mathbb{L}^2}\right\} ds,$$

for all $t \in [0,S]$. Then by Lemma 6, we have

$$\sup_{t\in[0,T]} |U_\mu(t) - U_\nu(t)|_{\mathbb{L}^2} \leq e^{\gamma_+ T}\sqrt{2C_2|\alpha|(\mu+\nu)},$$

which assures that $\{U_\mu\}_{\mu>0}$ forms a Cauchy net in $C([0,S];\mathbb{L}^2(\Omega))$. Now let $U_\mu \to U$ in $C([0,S];\mathbb{L}^2(\Omega))$ as $\mu \to 0$. By Lemma 6, $\{\frac{d}{dt}U_\mu\}$ and $\{\partial\varphi(U_\mu)\}$ are bounded in $L^2(0,S;\mathbb{L}^2(\Omega))$. Hence by the demiclosedness of $\frac{d}{dt}$ and $\partial\varphi$ we have

$$\begin{aligned} \frac{dU_{\nu_n}}{dt} &\rightharpoonup \frac{dU}{dt} \quad \text{weakly in } L^2(0,T;\mathbb{L}^2(\Omega)), \\ \partial\varphi(U_{\nu_n}) &\rightharpoonup \partial\varphi(U) \quad \text{weakly in } L^2(0,T;\mathbb{L}^2(\Omega)), \end{aligned}$$

for some sequence $\{\nu_n\}_{n\in\mathbb{N}}$ such that $\nu_n \to 0$ as $n \to \infty$. By the definition of Yosida approximation,

$$\begin{aligned} \|U_{\nu_n} - J^\varphi_{\nu_n}U_{\nu_n}\|^2_{\mathcal{H}^T} &= \int_0^T |U_{\nu_n}(s) - J^\varphi_{\nu_n}U_{\nu_n}(s)|^2_{\mathbb{L}^2}ds \\ &= \nu_n^2 \int_0^T |\partial\varphi_{\nu_n}(U_{\nu_n}(s))|^2_{\mathbb{L}^2}ds \le C_2\nu_n^2 \to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{3.6}$$

whence follows that $J^\varphi_{\nu_n}U_{\nu_n} \to U$ strongly in $\mathcal{H}^T = L^2(0,T;\mathbb{L}^2(\Omega))$. Then since $\partial\varphi_\nu(U_\nu) = \partial\varphi(J^\varphi_\nu U_\nu)$, by the demiclosedness of $\partial\varphi$ we find that U satisfies

$$\frac{dU}{dt} + (\lambda + \alpha I)\partial\varphi(U) - (\kappa + \beta I)h(t) - \gamma U = F \quad \text{in } L^2(0,T;\mathbb{L}^2(\Omega)),$$

i.e., U is the desired solution of (AE^h). □

4 Proof of Theorem 1 (Existence)

Before proving Theorem 1, we deduce some a priori estimates for the unique solutions U_h of auxiliary equations (AE^h), which are given in Proposition 4. First fix a constant R as

$$R := \max\left\{\frac{1}{2}|U_0|^2_{\mathbb{L}^2} + \varphi(U_0) + \frac{1}{\lambda}\|F\|^2_{\mathcal{H}^T}, 1\right\}. \tag{4.1}$$

We assume

$$\|h\|^2_{\mathcal{H}^S} = \int_0^S |h(t)|^2_{\mathbb{L}^2}dt \le R. \tag{4.2}$$

Lemma 7 First Energy Estimate.
Let U_h be the unique solution of (AE^h). Then there exists C_1 depending only on λ, κ, β and γ such that

$$\sup_{t\in[0,S]} |U_h(t)|^2_{\mathbb{L}^2} + \int_0^S \varphi(U_h(t))dt \le C_1R. \tag{4.3}$$

Proof.
We multiply (AE^h) by U_h to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|U_h|^2_{\mathbb{L}^2} + 2\lambda\varphi(U_h) \\ &= \gamma|U_h|^2_{\mathbb{L}^2} + ((\kappa - \beta I)h + F, U_h)_{\mathbb{L}^2} \\ &\le \frac{4\gamma_+ + \kappa^2 + \beta^2 + \lambda}{4}|U_h|^2_{\mathbb{L}^2} + |h|^2_{\mathbb{L}^2} + \frac{1}{\lambda}|F|^2_{\mathbb{L}^2}, \end{aligned} \tag{4.4}$$

where we use the notation $\gamma_+ := \max\{0,\gamma\}$ and the Cauchy-Schwarz inequality. Integrating (4.4) on $(0,S)$ and by (4.1), (4.2) we obtain

$$\begin{aligned}
&\frac{1}{2}|U_h|^2_{\mathbb{L}^2} + 2\lambda\int_0^t \varphi(U_h)d\tau \\
&\le \frac{1}{2}|U_0|^2_{\mathbb{L}^2} + \frac{1}{\lambda}\|F\|^2_{\mathcal{H}^T} + \|h\|^2_{\mathcal{H}^S} \\
&\quad + \frac{4\gamma_+ + \kappa^2 + \beta^2 + \lambda}{2}\int_0^t \left(\frac{1}{2}|U_h|^2_{\mathbb{L}^2} + 2\lambda\int_0^\tau \varphi(U_h)d\sigma\right)d\tau \\
&\le 2R + \frac{4\gamma_+ + \kappa^2 + \beta^2 + \lambda}{2}\int_0^t \left(\frac{1}{2}|U_h|^2_{\mathbb{L}^2} + 2\lambda\int_0^\tau \varphi(U_h)d\sigma\right)d\tau.
\end{aligned} \tag{4.5}$$

We apply Gronwall's inequality to (4.5) to get

$$\begin{aligned}
\frac{1}{2}|U_h|^2_{\mathbb{L}^2} + 2\lambda\int_0^t \varphi(U_h)d\tau &\le 2Re^{\frac{4\gamma_+ + \kappa^2 + \beta^2 + \lambda}{2}t} \\
&\le 2e^{\frac{4\gamma_+ + \kappa^2 + \beta^2 + \lambda}{2}S}R \quad \text{for all } t \in [0,S],
\end{aligned}$$

which implies the desired estimate (4.3). □

Lemma 8 Second Energy Estimates.
Let U_h be the solution of (AE^h). Then there exists C_2 depending only on λ, κ, β and γ such that

$$\sup_{t\in[0,S]} \varphi(U_h(t)) + \int_0^S |\partial\varphi(U_h(t))|^2_{\mathbb{L}^2}dt + \int_0^S \left|\frac{dU_h}{dt}(t)\right|^2_{\mathbb{L}^2} dt \le C_2 R. \tag{4.6}$$

Proof.
Multiplying (AE^h) by $\partial\varphi(U_h)$ and using (2.15), we obtain

$$\begin{aligned}
&\frac{d}{dt}\varphi(U_h) + \lambda|\partial\varphi(U_h(t))|^2_{\mathbb{L}^2} \\
&= 2\gamma\varphi(U_h) + ((\kappa - \beta I)h + F, \partial\varphi(U_h))_{\mathbb{L}^2} \\
&\le 2\gamma_+\varphi(U_h) + \frac{\kappa^2 + \beta^2}{\lambda}|h|^2_{\mathbb{L}^2} + \frac{1}{\lambda}|F|^2_{\mathbb{L}^2} + \frac{3\lambda}{4}|\partial\varphi(U_h(t))|^2_{\mathbb{L}^2},
\end{aligned}$$

whence follows

$$\frac{d}{dt}\varphi(U_h) + \frac{\lambda}{4}|\partial\varphi(U_h(t))|^2_{\mathbb{L}^2} \le 2\gamma_+\varphi(U_h) + \frac{\kappa^2 + \beta^2}{\lambda}|h|^2_{\mathbb{L}^2} + \frac{1}{\lambda}|F|^2_{\mathbb{L}^2}. \tag{4.7}$$

Integrating (4.7) on $(0,t)$ for $t \in (0,S]$ and by Lemma 7, we get

$$\begin{aligned}
&\varphi(U_h) + \frac{\lambda}{4}\int_0^t |\partial\varphi(U_h(t))|^2_{\mathbb{L}^2}d\tau \\
&\le \varphi(U_0) + \frac{1}{\lambda}\|F\|^2_{\mathcal{H}^T} + \frac{\kappa^2 + \beta^2}{\lambda}\|h\|^2_{\mathcal{H}^S} + 2\gamma_+\int_0^t \varphi(U_h)d\tau \\
&\le \left(1 + \frac{\kappa^2 + \beta^2}{\lambda} + 2\gamma_+ C_1\right)R \quad \text{for all } t \in (0,S].
\end{aligned} \tag{4.8}$$

Thus from (4.8) and (AE^h), we derive (4.6). □

Now we are ready to prove the existence part of Theorem 1.

Proof of Theorem 1 (Existence).
We prepare a closed ball in $\mathcal{H}^S$ with radius R:

$$\mathcal{H}^S \supset K_R^S := \left\{ h(t) \in L^2(0,S;\mathbb{L}(\Omega)) \mid \|h\|_{\mathcal{H}^S}^2 = \int_0^S |h(t)|_{\mathbb{L}^2}^2 dt \leq R \right\},$$

and a mapping

$$\mathcal{F} : \mathcal{H}^S \ni h(t) \mapsto \mathcal{F}(h(t)) = \partial\psi_q(U_h) \in \mathcal{H}^S, \tag{4.9}$$

where U_h is the unique solution of (AE^h).

First we show that $\mathcal{F}$ maps K_R^S into itself. By the Gagliardo-Nirenberg-Sobolev inequality, we obtain

$$|\partial\psi_q(U_h)|_{\mathbb{L}^2}^2 = |U_h|_{\mathbb{L}^{2(q-1)}}^{2(q-1)} \leq C_{\mathrm{GN}} |U_h|_{\mathbb{H}^2}^{2(1-\xi)(q-1)} |U_h|_{\mathbb{L}^{2^*}}^{2\xi(q-1)} \tag{4.10}$$

where parameter ξ satisfies

$$\frac{1}{2(q-1)} = \left(\frac{1}{2} - \frac{2}{N}\right)(1-\xi) + \left(\frac{1}{2} - \frac{1}{N}\right)\xi.$$

We apply the elliptic estimate to (4.10) to obtain

$$|U_h|_{\mathbb{H}^2}^{2(1-\xi)(q-1)} |U_h|_{\mathbb{L}^{2^*}}^{2\xi(q-1)} \leq C \left\{ |\partial\varphi(U_h)|_{\mathbb{L}^2}^2 + |U_h|_{\mathbb{L}^2}^2 \right\}^{(1-\xi)(q-1)} \varphi(U_h)^{\xi(q-1)}, \tag{4.11}$$

where C denotes a general constant. Our assumption on q being Sobolev subcritical assures $(1-\xi)(q-1) < 1$. Thus by Young's inequality, for arbitrary $\varepsilon > 0$ and appropriate $\chi > 1$ it holds that

$$\left\{ |\partial\varphi(U_h)|_{\mathbb{L}^2}^2 + |U_h|_{\mathbb{L}^2}^2 \right\}^{(1-\xi)(q-1)} \varphi(U_h)^{\xi(q-1)} \leq \varepsilon \left(|\partial\varphi(U_h)|_{\mathbb{L}^2}^2 + |U_h|_{\mathbb{L}^2}^2 \right) + C_\varepsilon \varphi(U_h)^\chi, \tag{4.12}$$

where the constant C_ε depends on ε. Combining (4.10), (4.11) and (4.12), we get

$$|\partial\psi_q(U_h)|_{\mathbb{L}^2}^2 \leq \varepsilon \left(|\partial\varphi(U_h)|_{\mathbb{L}^2}^2 + |U_h|_{\mathbb{L}^2}^2 \right) + C_\varepsilon \varphi(U_h)^\chi. \tag{4.13}$$

Integrating (4.13) on $[0,S]$ with (4.3) and (4.6) gives

$$\begin{aligned} \int_0^S |\partial\psi_q(U_h)|_{\mathbb{L}^2}^2 dt &\leq \varepsilon \int_0^S \left(|\partial\varphi(U_h)|_{\mathbb{L}^2}^2 + |U_h|_{\mathbb{L}^2}^2 \right) dt + C_\varepsilon \int_0^S \varphi(U_h)^{\chi_2} dt \\ &\leq \varepsilon C_2 R + M_\varepsilon(R) S, \end{aligned}$$

where $M_\varepsilon(\cdot)$ denotes a non-decreasing function depending on ε.

First fix $\varepsilon := \frac{1}{2C_1}$ and then define S by

$$S := \min\left\{ T, \frac{R}{2M_\varepsilon(R)} \right\}. \tag{4.14}$$

Then $\int_0^S |\partial\psi_q(U_h)|_{\mathbb{L}^2}^2 dt = \int_0^S |\mathcal{F}(h)|_{\mathbb{L}^2}^2 dt \leq R$, that is $\mathcal{F}$ maps K_R^S into itself.

Next we prove the weak continuity of $\mathcal{F}$. Since the continuity is a local property, we could focus on compact neighbourhoods, which are metrizable because $L^2(0,S;\mathbb{L}^2(\Omega))$ is a separable Hilbert space. Let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}^S$ such that

$$h_n \rightharpoonup h \text{ weakly in } L^2(0,S;\mathbb{L}^2(\Omega)),$$

and U_{h_n}, U_h be unique solutions of (AE^{h_n}) and (AE^{h}) respectively. Lemma 8 assures the equi-continuity of $\{U_{h_n}(t)\}_{n\in\mathbb{N}}$, indeed:

$$\begin{aligned}|U_{h_n}(t) - U_{h_n}(s)| = \left|\int_s^t \frac{dU_{h_n}}{d\tau}(\tau)d\tau\right| &\leq \int_t^s \left|\frac{dU_{h_n}}{d\tau}(\tau)\right|_{\mathbb{L}^2} d\tau \\ &\leq \left(\int_s^t \left|\frac{dU_{h_n}}{d\tau}(\tau)\right|_{\mathbb{L}^2}^2 d\tau\right)^{\frac{1}{2}} \left(\int_s^t 1 d\tau\right)^{\frac{1}{2}} \\ &\leq \sqrt{C_2 R}\sqrt{t-s}.\end{aligned}$$

Lemmas 7 and 8 and Rellich-Kondrachov theorem read that $\{U_{h_n}(t)\}_{n\in\mathbb{N}}$ is relatively compact in $\mathbb{L}^2(\Omega)$ for all $t \in [0, S]$.

By Ascoli's Theorem and Lemmas 7, 8, there exists a subsequence $\{h_{n'}\}_{n'\in\mathbb{N}} \subset \{h_n\}_{n\in\mathbb{N}}$ and $U \in C([0, S]; \mathbb{L}^2(\Omega))$ such that

$$U_{h_{n'}} \to U \quad \text{strongly in } \mathrm{C}(0,T;\mathbb{L}^2(\Omega)) \text{ and weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)), \tag{4.15}$$

$$\frac{dU_{h_{n'}}}{dt} \rightharpoonup \frac{dU}{dt} \quad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)), \tag{4.16}$$

$$\partial\varphi(U_{h_{n'}}) \rightharpoonup \partial\varphi(U) \quad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)), \tag{4.17}$$

$$\partial\psi_q(U_{h_{n'}}) \rightharpoonup \partial\psi_q(U) \quad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)), \tag{4.18}$$

here we used the weak closedness of $\frac{d}{dt}$ and $\partial\varphi$ in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ in (4.16) and (4.17).

Since U satisfies the following equation:

$$\frac{dU}{dt} + (\lambda + \alpha I)\partial\varphi(U) - (\kappa + \beta I)h - \gamma U = F,$$

U coincides with its unique solution U_h.

We can show that these convergences do not depend on choices of subsequences by contradiction, by the uniqueness of the solution of (AE^h). More precisely, if $\partial\psi_q(U_{h_n}) \not\rightharpoonup \partial\psi_q(U)$ then there exists another subsequence $\{h_{n''}\}_{n''\in\mathbb{N}} \subset \{h_n\}_{n\in\mathbb{N}} \setminus \{h_{n'}\}_{n'\in\mathbb{N}}$ such that $\{\partial\psi_q(U_{h_n''})\}_{n''\in\mathbb{N}}$ does not accumulate to $\partial\psi_q(U)$. However repeating the above argument, we can choose subsequence $\{U_{n'''}\}_{n'''\in\mathbb{N}} \subset \{U_{n''}\}_{n''\in\mathbb{N}}$ such that $\partial\psi_q(U_{h_{n'''}}) \rightharpoonup \partial\psi_q(U)$, which leads to a contradiction.

Then the following convergence holds:

$$\mathcal{F}(h_n) = \partial\psi_q U_{h_n} \rightharpoonup \partial\psi_q U = \partial\psi_q U_h = \mathcal{F}(h),$$

whence follows the weak continuity of $\mathcal{F}$.

Thus, we can apply Schauder-Tychonoff's fixed point theorem on $\mathcal{F}$ and K_R^S to obtain a fixed point h, i.e., h satisfies

$$h = \mathcal{F}(h) = \partial\psi_q(U_h). \tag{4.19}$$

By (4.19) the corresponding solution U_h satisfies:

$$\begin{aligned}&\frac{dU_h}{dt} + (\lambda + \alpha I)\partial\varphi(U_h) - (\kappa + \beta I)h - \gamma U_h \\ &= \frac{dU_h}{dt} + (\lambda + \alpha I)\partial\varphi(U_h) - (\kappa + \beta I)\partial\psi_q(U_h) - \gamma U_h = F.\end{aligned} \tag{4.20}$$

This means U_h is a desired solution of (ACGL_-). □

5 Proof of Theorem 2

Before showing the uniqueness of the solution for (ACGL$_-$), we prove Theorem 2.

Let T_0 be the maximal existence time of a solution of (ACGL$_-$),

$$T_0 := \sup\{S > 0 \mid \exists \text{ a solution of (ACGL) on } [0, S]\}.$$

Proof of Theorem 2.
To ensure the alternative, we rely on proof by contradiction. Assume $T_0 < T$ and the assertion $\lim_{t \uparrow T_0} \varphi(U(t)) = +\infty$ does not hold. Then there exists monotonically increasing sequence $t_n \uparrow T_0$ such that $\varphi(U(t_n)) \leq C$ holds for all $n \in \mathbb{N}$ We repeat the same argument as before with $U(0)$ replaced by $U(t_n)$ to assure the existence of $\sigma > 0$ independent of n such that a solution of (ACGL$_-$) exists on $[t_n, t_n + \sigma]$. Recalling the definition of R (4.1), we define

$$\rho := \max\left\{C(\lambda_1^{-1} + 1) + \frac{1}{\lambda}\|F\|_{\mathcal{H}}^2, 1\right\} \geq 1.$$

Then by Poincaré's inequality, it holds for all $n \in \mathbb{N}$

$$\rho \geq \frac{1}{\lambda}\|F\|_{\mathcal{H}}^2 + \tfrac{1}{2}|U(t_n)|_{\mathbb{L}^2}^2 + \varphi(U(t_n)).$$

Additionally we define σ by

$$\sigma := \min\left\{T - T_0, \frac{1}{2M_\varepsilon(\rho)}\right\},$$

which is independent of n. We can deduce $\mathcal{F}$ maps K_ρ^σ into itself in the same way as before. Thus we can construct solution on $[t_n, t_n + \sigma]$ applying Schauder-Tychonoff's fixed point theorem again.

Since $\{t_n\}_{n \in \mathbb{N}}$ converges upto T_0, there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, it holds that $T_0 < t_n + \frac{\sigma}{2}$. This means the local solution can be extended up to $[0, T_0 + \frac{\sigma}{2}]$, whence follows the contradiction with the definition of T_0. □

6 Proof of Theorem 1 (Uniqueness)

The uniqueness of the solution of (ACGL$_-$) relies on the corollary of the following lemma. Set d_r be

$$d_r = \max\left\{1, \frac{r-1}{2}\right\}. \tag{6.1}$$

Lemma 9.
For all $U = (u_1, u_2), V = (v_1, v_2) \in \mathbb{R}^2$, $i, j = 1, 2$ and $r > 2$, the following inequality holds:

$$\left|\left(|U|^{r-2}u_i - |V|^{r-2}v_i\right)(u_j - v_j)\right| \leq d_r\left(|U|^{r-2} + |V|^{r-2}\right)|U - V|^2. \tag{6.2}$$

Proof.
When we assume $|U| \geq |V|$,

$$\begin{aligned}
&\left(|U|^{r-2}u_i - |V|^{r-2}v_i\right)(u_j - v_j)\\
&= \left\{|U|^{r-2}(u_i - v_i) + \left(|U|^{r-2} - |V|^{r-2}\right)v_i\right\}(u_j - v_j)\\
&\leq |U|^{r-2}|U - V|^2 + \left(|U|^{r-2} - |V|^{r-2}\right)|V||U - V|.
\end{aligned} \tag{6.3}$$

Similarly when $|V| \geq |U|$, it holds:

$$\begin{aligned}
&\left(|U|^{r-2}u_i - |V|^{r-2}v_i\right)(u_j - v_j)\\
&= \left(|V|^{r-2}v_i - |U|^{r-2}u_i\right)(v_j - u_j)\\
&= \left\{|V|^{r-2}(v_i - u_i) + \left(|V|^{r-2} - |U|^{r-2}\right)u_i\right\}(v_j - u_j)\\
&\leq |V|^{r-2}|V-U|^2 + \left(|V|^{r-2} - |U|^{r-2}\right)|U||V-U|.
\end{aligned} \tag{6.4}$$

Let $\tilde{d}_r$ be

$$\tilde{d}_r = \max\left\{\frac{1}{2}, \frac{r-2}{2}\right\}. \tag{6.5}$$

Here we claim for all $U, V \in \mathbb{R}^2$ and $r > 2$, the following inequality holds:

$$\left||U|^{r-2} - |V|^{r-2}\right| \leq \tilde{d}_r \left(|U|^{r-3} + |V|^{r-3}\right)|U - V|. \tag{6.6}$$

Since the above inequality (6.6) holds clearly when $|U||V| = 0$, we assume $|U||V| \neq 0$. When $3 \geq r > 2$, we assume without loss of generality $|U| \geq |V|$, then $\left(\frac{|V|}{|U|}\right)^{r-3} \geq 1$. Factoring out $|U|^{r-3}$ from the left hand side of (6.6), we obtain

$$\begin{aligned}
\left||U|^{r-2} - |V|^{r-2}\right| &= |U|^{r-2} - |V|^{r-2}\\
&= |U|^{r-3}\left\{|U| - \left(\frac{|V|}{|U|}\right)^{r-3}|V|\right\}\\
&= |U|^{r-3}\left(|U| - |V|\right) \leq |U|^{r-3}|U-V| \leq \frac{1}{2}\left(|U|^{r-3} + |V|^{r-3}\right)|U-V|.
\end{aligned}$$

When $r > 3$, we must use concavity of the function $|\cdot|^{r-3}$. First we deform the left hand side of (6.6) to obtain

$$\begin{aligned}
|U|^{r-2} - |V|^{r-2} &= \left[\{|V| + t(|U| - |V|)\}^{r-2}\right]_{t=0}^{t=1}\\
&= \int_0^1 \frac{d}{dt}\{|V| + t(|U| - |V|)\}^{q-2}\,dt\\
&= (r-2)\int_0^1 \{|V| + t(|U| - |V|)\}^{r-3}(|U| - |V|)dt\\
&\leq (r-2)\int_0^1 \{|V| + t(|U| - |V|)\}^{r-3}|U - V|dt\\
&= (r-2)\int_0^1 \{t|U| + (1-t)|V|\}^{r-3}|U - V|dt.
\end{aligned}$$

By the concavity of the function $|\cdot|^{r-3}$ with $r > 3$, it holds that

$$\{t|U| + (1-t)|V|\}^{r-3} \leq t|U|^{r-3} + (1-t)|V|^{r-3},$$

whence follows

$$|U|^{r-2} - |V|^{r-2} \leq \frac{r-2}{2}\left(|U|^{r-3} + |V|^{r-3}\right)|U - V|, \tag{6.7}$$

which leads to the desired inequality (6.6), because of the symmetricity of the right hand side of (6.7) with respect to $|U|$ and $|V|$. When $|U| \geq |V|$, we combine (6.6) with (6.3) to deduce

$$\left|\left(|U|^{r-2}u_i - |V|^{r-2}v_i\right)(u_j - v_j)\right| \leq \left\{|U|^{r-2} + \tilde{d}_r\left(|U|^{r-3}|V| + |V|^{r-2}\right)\right\}|U - V|^2. \tag{6.8}$$

First we assume $2 < r \leq 3$, then $|U|^{r-3} \leq |V|^{r-3}$ holds so that

$$\left|\left(|U|^{r-2}u_i - |V|^{r-2}v_i\right)(u_j - v_j)\right| \leq \left(|U|^{r-2} + |V|^{r-2}\right)|U - V|^2. \tag{6.9}$$

When $r > 3$, applying Yong's inequality to $|U|^{r-3}|V|$, we obtain

$$|U|^{r-3}|V| \leq \frac{r-3}{r-2}|U|^{r-2} + \frac{1}{r-2}|V|^{r-2}, \tag{6.10}$$

whence we have the desired inequality (6.2). For $|V| \geq |U|$, we combine (6.6) with (6.4) and repeat the same argument as above to obtain (6.2). □

As for the corollary of Lemma 9, we obtain the following by Hölder's inequality.

Corollary 10.
For all $U, V \in \mathbb{L}^r(\Omega)$ the following estimates holds with some constant C.

$$|(\partial\psi_r(U) - \partial\psi_r(V), U - V)_{\mathbb{L}^2}| \leq C\left(\psi_r(U)^{r-2} + \psi_r(V)^{r-2}\right)|U - V|^2_{\mathbb{L}^r}, \tag{6.11}$$

$$|(\partial\psi_r(U) - \partial\psi_r(V), I(U - V))_{\mathbb{L}^2}| \leq C\left(\psi_r(U)^{r-2} + \psi_r(V)^{r-2}\right)|U - V|^2_{\mathbb{L}^r}. \tag{6.12}$$

We proceed to the proof of the uniqueness.

Proof of Theorem 1 (Uniqueness).
Let U, V be two solutions of (ACGL$_-$) with $U(0) = U_0$ and $V(0) = V_0$ on $[0, S]$ for any $S \in (0, T_0)$. Multiplying the difference of two equations by $U - V$, using the linearity of $\partial\varphi$, (2.14) and Corollary 10, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|U - V|^2_{\mathbb{L}^2} + 2\lambda\varphi(U - V) \\ &\leq \gamma_+|U - V|^2_{\mathbb{L}^2} + ((\kappa + I\beta)(\partial\psi_q(U) - \partial\psi_q(V)), U - V)_{\mathbb{L}^2} \\ &\leq \gamma_+|U - V|^2_{\mathbb{L}^2} + C\left(\psi_q(U)^{q-2} + \psi_q(V)^{q-2}\right)|U - V|^2_{\mathbb{L}^q}, \end{aligned} \tag{6.13}$$

where the constant C depends only on q, κ, β.

By our assumption on q being Sobolev subcritical, using the parameter η defined by

$$\frac{1}{q} = \left(\frac{1}{2} - \frac{1}{N}\right)(1 - \eta) + \frac{\eta}{2}, \tag{6.14}$$

we obtain

$$|W|_{\mathbb{L}^q} \leq (2\varphi(W))^{\frac{1-\eta}{2}}|W|^{\eta}_{\mathbb{L}^2}. \tag{6.15}$$

Thus by(6.13), (6.15) with $W = U - V$ and Young's inequality,

$$\frac{1}{2}\frac{d}{dt}|U - V|^2_{\mathbb{L}^2} + \lambda\varphi(U - V) \leq C\left(\psi_q(U)^{q-2} + \psi_q(V)^{q-2}\right)^{\frac{1}{\eta}}|U - V|^2_{\mathbb{L}^2}, \tag{6.16}$$

where the constant C depends only on $\lambda, \kappa, \beta, \gamma, \eta$.

Since $S < T_0$, we can derive the uniform boundedness of $\varphi(U)$ and $\varphi(V)$ on $[0, S]$, consequently the boundedness of $|U|_{\mathbb{L}^2}$ and $|V|_{\mathbb{L}^2}$ by Poincaré's inequality. By virtue of (6.15) with $W = U$or V, we get the uniform boundedness on $[0, S]$ for $\psi_q(U)$ and $\psi_q(V)$ as well.

Thus we see that $\psi_q(U)$ and $\psi_q(V)$ are uniformly bounded above by a positive constant M on $[0, S]$. Then the coefficient of $|U - V|^2_{\mathbb{L}^2}$ in the right hand side of (6.16) independent of t. Applying Gronwall's inequality to (6.16), we obtain

$$|U(t) - V(t)|_{\mathbb{L}^2} \leq |U_0 - V_0|_{\mathbb{L}^2}e^{2CM^{\frac{q-2}{\eta}}}, \tag{6.17}$$

whence follows the uniqueness. □

7 Proof of Theorem 3

First we prepare some lemmas.

Lemma 11.
Let all the assumptions in Theorem 3 be satisfied. There exists $\varepsilon_0 > 0$ and $\delta > 0$ such that for all $U \in D(\varphi) = \mathbb{H}_0^1(\Omega)$ satisfying $\varphi(U) < \varepsilon_0$, it holds that

$$(\lambda \partial\varphi U - \kappa \partial\psi_q U - \gamma U, U)_{\mathbb{L}^2} \geq \delta\varphi(U) \geq 2\delta\lambda_1 |U|_{\mathbb{L}^2}^2. \tag{7.1}$$

Proof.
We recall Gagliardo-Nirenberg's inequality with parameter η given in (6.14)

$$\psi_q(W) \leq C_b \varphi(W)^{\frac{q(1-\eta)}{2}} \psi_2(W)^{\frac{\eta}{2}}, \tag{7.2}$$

where C_b denotes the best constant. Combining (7.2) with Poincaré's inequality (2.18), we get

$$\psi_q(W) \leq C_b \sigma_2 \varphi(W)^{\frac{q}{2}}, \tag{7.3}$$

where σ_2 is given by

$$\sigma_2 = \lambda_1^{-\frac{2q-Nq+2N}{4}}. \tag{7.4}$$

We multiply $\lambda\partial\varphi(U) - \kappa\partial\psi_q(U) - \gamma U$ by U and use (7.2) and (2.18) to get

$$\begin{aligned}(\lambda\partial\varphi(U) - \kappa\partial\psi_q(U) - \gamma U, U)_{\mathbb{L}^2} &= 2\lambda\varphi(U) - q\kappa\psi_q(U) - \gamma|U|_{\mathbb{L}^2}^2 \\ &\geq \left(2\lambda - 2\gamma_+\lambda_1^{-1} - q\kappa C_b\sigma_2\varphi(U)^{\frac{q}{2}-1}\right)\varphi(U),\end{aligned} \tag{7.5}$$

where we use the notations $\gamma_+ := \max\{0, \gamma\}$.

By the assumption $\gamma < \lambda\lambda_1$, we can take $\varphi(U) < \left(\frac{2(\lambda - \gamma_+\lambda_1^{-1})}{q\kappa C_b \sigma_2}\right)^{\frac{2}{q-2}} =: \varepsilon_0$ to obtain some $\delta = 2\lambda - 2\gamma_+\lambda_1^{-1} - q\kappa C_b\sigma_2\varphi(U)^{\frac{q}{2}-1} > 0$ in (7.5), which means the first inequality of (7.1). The second inequality of (7.1) follows directly form (2.18). □

Next Lemma is essential for proving Theorem 3.

Lemma 12.
Let all the assumptions in Theorem 3 be satisfied. There exists $k \in (0,1)$ independent of T such that for all $U_0 \in D(\varphi)$ and $T > 0, F \in L^2(0,T;\mathbb{L}^2(\Omega))$, if $\varphi(U_0) \leq (k\varepsilon_0)^2$ and $|||F||| \leq k\varepsilon_0$, then the corresponding solution $U(t)$ on $[0,S]$, $0 < S \leq T$ satisfies

$$\varphi(U(t)) < \varepsilon_0 \quad \forall t \in [0,S]. \tag{7.6}$$

Before proving Lemma 12, we prepare two more Lemmas.

Lemma 13.
Let $T > 0$, $|||F||| \leq r$ and $\delta > 0$. Then the following estimate holds:

$$\int_0^t |F(\tau)|_{\mathbb{L}^2} e^{-\delta(t-\tau)} d\tau \leq r \frac{1 - e^{-\delta} + e^{\delta}}{1 - e^{-\delta}} \quad \text{for all } t \in (0,T). \tag{7.7}$$

Proof.
Fix the floor function $\lfloor t \rfloor$ be the lowest integer less than or equal to t. Then we split the left hand side of (7.7) as

$$\int_0^t |F(\tau)|_{\mathbb{L}^2} e^{-\delta(t-\tau)} d\tau = \int_0^{\lfloor t \rfloor} |F(\tau)|_{\mathbb{L}^2} e^{-\delta(t-\tau)} d\tau + \int_{\lfloor t \rfloor}^t |F(\tau)|_{\mathbb{L}^2} e^{-\delta(t-\tau)} d\tau. \tag{7.8}$$

By the definition of $|||\cdot|||$ and Young's inequality, the second term in the right hand side of (7.8) is bounded above by r. If $0 \le t < 1$, then the first term in the right hand side of (7.8) is zero. When $t \ge 1$ applying Young's inequality to the first term in the right hand side of (7.8), we derive

$$\begin{aligned} \int_0^{\lfloor t \rfloor} |F(\tau)|_{\mathbb{L}^2} e^{-\delta(t-\tau)} d\tau &= e^{-\delta t} \int_0^{\lfloor t \rfloor} |F(\tau)|_{\mathbb{L}^2} e^{-\delta(t-\tau)} d\tau \\ &\le e^{-\delta t} \sum_{s=0}^{\lfloor t \rfloor -1} \int_s^{s+1} |F(\tau)|_{\mathbb{L}^2} e^{\delta\tau} d\tau \\ &\le e^{-\delta t} \sum_{s=0}^{\lfloor t \rfloor -1} \left(\int_s^{s+1} |F(\tau)|_{\mathbb{L}^2}^2 d\tau \right)^{\frac{1}{2}} \left(\int_s^{s+1} e^{2\delta\tau} d\tau \right)^{\frac{1}{2}}. \end{aligned} \tag{7.9}$$

By the definition of $|||\cdot|||$, we obtain

$$\begin{aligned} \int_0^t |F(\tau)|_{\mathbb{L}^2} e^{-\delta(t-\tau)} d\tau &\le r e^{-\delta t} \sum_{s=0}^{\lfloor t \rfloor -1} \left(\int_s^{s+1} e^{2\delta\tau} d\tau \right)^{\frac{1}{2}} \\ &\le r e^{-\delta t} \sum_{s=0}^{\lfloor t \rfloor -1} e^{\delta(s+1)} \\ &= r \sum_{s=0}^{\lfloor t \rfloor -1} e^{\delta(s+1-t)} \\ &= r e^{\delta(\lfloor t \rfloor - t)} \sum_{s=0}^{\lfloor t \rfloor -1} e^{-\delta s} \\ &= r e^{\delta(\lfloor t \rfloor - t)} \frac{1 - e^{-\delta \lfloor t \rfloor}}{1 - e^{-\delta}} = r \frac{e^{\delta(\lfloor t \rfloor - t)} - e^{-\delta t}}{1 - e^{-\delta}}, \end{aligned} \tag{7.10}$$

whence follows (7.7). □

Lemma 14.
Fix $\delta \in \mathbb{R}$. Let $f(t) \in L^1(0,T)$ and $u(t)$ be an absolutely continuous function on $[0,T]$ such that

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \delta |u(t)|^2 \le |f(t)||u(t)| \tag{7.11}$$

Then it holds that

$$|u(t)| \le |u(0)| e^{-\delta t} + \int_0^t |f(\tau)| e^{-\delta(t-\tau)} d\tau \quad \forall t \in [0,T]. \tag{7.12}$$

Proof.
Multiplying $e^{2\delta t} > 0$ by (7.11), we obtain

$$\frac{1}{2} e^{2\delta t} \frac{d}{dt} |u(t)|^2 + \delta e^{2\delta t} |u(t)|^2 = \frac{1}{2}\frac{d}{dt} \left\{ e^{\delta t} |u(t)| \right\}^2 \le e^{\delta t} |f(t)| e^{\delta t} |u(t)|. \tag{7.13}$$

Integrating (7.13) on $[0,t]$ with $t \leq T$, we derive

$$\frac{1}{2}\left\{e^{\delta t}|u(t)|\right\}^2 \leq \frac{1}{2}|u(0)|^2 + \int_0^t e^{\delta\tau}|f(\tau)|e^{\delta\tau}|u(\tau)|d\tau. \tag{7.14}$$

We apply the following Gronwall type inequality to (7.14) to get

$$e^{\delta t}|u(t)| \leq |u(0)| + \int_0^t e^{\delta\tau}|f(\tau)|d\tau,$$

whence follows (7.12).

Lemma (Brézis [1], p. 157.).
Let $m \in L^1(0,T;\mathbb{R})$ such that $m \geq 0$ for a.e. on $(0,T)$ and let a an non-negative constant. Let ϕ be a continuous funtion on $[0,T]$ into $\mathbb{R}$ satisfying

$$\frac{1}{2}\phi^2(t) \leq \frac{1}{2}a^2 + \int_0^t m(s)\phi(s)ds \quad \text{for all } t \in [0,T].$$

Then the following estimate holds:

$$|\phi(t)| \leq a + \int_0^t m(s)ds \quad \text{for all } t \in [0,T].$$

□

Proof of Lemma 12.
By the following Lemma and Theorem 1, it is ensured that $\varphi(U(t))$ is absolutely continuous.

Lemma (Brézis [1], p. 73.).
Let $u \in \mathrm{W}^{1,2}(0,T;H)$ such that $u(t) \in D(\partial\phi)$ for a.e. $(0,T)$. Suppose that there exists $g \in \mathrm{L}^2(0,T;H)$ such that $g(t) \in \partial\phi(u(t))$ for a.e. $(0,T)$.
Then the function $t \mapsto \varphi(u(t))$ is absolutely continuous on $[0,T]$.

We shall prove Lemma 12 by contradiction. If there exists $t_0 \in (0,S]$ such that $\varphi(U(t_0)) \geq \varepsilon_0$, by the continuity of $\varphi(U(t))$ and $\varphi(U(0)) = \varphi(U_0) = k\varepsilon_0 < \varepsilon_0$, there exists $t_1 \in (0,t_0)$ such that $\varphi(U(t))$ attains ε_0 for the first time, i.e. $\varphi(U(t)) < \varepsilon_0$ for all $t \in [0,t_1)$ and $\varphi(U(t_1)) = \varepsilon_0$.
Multiplying (ACGL$_-$) by its solution U and by (7.1) we obtain for a.e. $t \in [0,t_1]$,

$$\frac{1}{2}\frac{d}{dt}|U(t)|^2_{\mathbb{L}^2} + 2\delta\lambda_1|U(t)|^2_{\mathbb{L}^2} \leq |F(t)|_{\mathbb{L}^2}|U(t)|_{\mathbb{L}^2}, \tag{7.15}$$

Then we can apply to (7.15) Lemmas 7.12 and 7.7 to get

$$\begin{aligned}|U(t)|_{\mathbb{L}^2} &\leq |U_0|_{\mathbb{L}^2}e^{-2\delta\lambda_1 t} + \int_{t-}^t |F(t)|_{\mathbb{L}^2}e^{-2\delta\lambda_1(t-\tau)}d\tau \\ &\leq |U_0|_{\mathbb{L}^2} + \int_0^t |F(t)|_{\mathbb{L}^2}e^{-2\delta\lambda_1(t-\tau)}d\tau \\ &\leq \left(2\lambda_1^{-1} + \frac{1-e^{-2\delta\lambda_1}+e^{2\delta\lambda_1}}{1-e^{-2\delta\lambda_1}}\right)k\varepsilon_0 = C_\delta k\varepsilon_0.\end{aligned} \tag{7.16}$$

Multiplying (ACGL$_-$) by $\partial\varphi(U)$, and repeating the same arguements as for (4.7), we get

$$\frac{d}{dt}\varphi(U) + \frac{\lambda}{4}|\partial\varphi(U)|^2_{\mathbb{L}^2} \leq 2\gamma_+\varphi(U) + \frac{\kappa^2+\beta^2}{\lambda}|\partial\psi_q(U)|^2_{\mathbb{L}^2} + \frac{1}{\lambda}|F|^2_{\mathbb{L}^2}. \tag{7.17}$$

From (4.10), (4.12) and (7.17), we derive

$$\frac{d}{dt}\varphi(U)+\frac{\lambda}{8}|\partial\varphi(U)|_{\mathbb{L}^2}^2\le 2\gamma_+\varphi(U)+D_1|U|_{\mathbb{L}^2}^2+D_2\varphi(U)^\chi+\frac{1}{\lambda}|F|_{\mathbb{L}^2}^2, \tag{7.18}$$

where the constants D_1 and D_2 depends only on λ, κ, β and q. Hence by (7.16), we have

$$\frac{d}{dt}\varphi(U(t))\le 2\gamma_+\varepsilon_0+D_1(C_\delta k\varepsilon_0)^2+D_2\varepsilon_0^\chi+\frac{1}{\lambda}|F(t)|_{\mathbb{L}^2}^2. \tag{7.19}$$

Fix a constant $k\in(0,1)$ given by

$$\begin{aligned} k:=\min\Bigg\{&\left(\varepsilon_0+2\gamma_++D_1C_\delta^2\varepsilon_0+D_2\varepsilon_0^{\chi-1}+\frac{1}{\lambda}\varepsilon_0+1\right)^{-1},\\ &\left(\frac{\delta^{-1}}{2}C_\delta\varepsilon_0+\delta^{-1}C_\delta\varepsilon_0+\gamma_++\frac{D_1}{2}C_\delta^2\varepsilon_0+\frac{D_2}{2}\varepsilon_0^{\chi-1}+\frac{1}{2\lambda}\varepsilon_0+1\right)^{-1}\Bigg\}, \end{aligned} \tag{7.20}$$

Integrating (7.19) on $[s,t_1]$ with $0\le s<t_1$, we obtain

$$\varphi(U(t_1))\le\varphi(U(s))+2\gamma_+\varepsilon_0(t_1-s)+D_1(C_\delta k\varepsilon_0)^2(t_1-s)+D_2\varepsilon_0^\chi(t_1-s)+\frac{1}{\lambda}(k\varepsilon_0)^2(t_1-s). \tag{7.21}$$

If $t_1\le k$ then we take $s=0$ to deduce

$$\begin{aligned} \varphi(U(t_1))&\le\varphi(U_0)+2\gamma_+k\varepsilon_0t_1+D_1(C_\delta k\varepsilon_0)^2t_1+D_2k\varepsilon_0^\chi t_1+\frac{1}{\lambda}(k\varepsilon_0)^2t_1\\ &\le\left(k\varepsilon_0+2\gamma_+k+D_1C_\delta^2k^2\varepsilon_0+D_2\varepsilon_0^{\chi-1}+\frac{1}{\lambda}k^2\varepsilon_0\right)k\varepsilon_0\\ &<\left(\varepsilon_0+2\gamma_++D_1C_\delta^2\varepsilon_0+D_2\varepsilon_0^{\chi-1}+\frac{1}{\lambda}\varepsilon_0\right)k\varepsilon_0. \end{aligned} \tag{7.22}$$

By the definition (7.20) of k, we find $\varphi(U(t_1))<\varepsilon_0$, which contradicts the definition of t_1.

We consider next the case $t_1-k\ge 0$. Again (7.1) and multiplying (ACGL$_-$) by U we obtain for a.e. $t\in[0,t_1]$,

$$\frac{1}{2}\frac{d}{dt}|U(t)|_{\mathbb{L}^2}^2+\delta C_P\varphi(U(t))\le|F(t)|_{\mathbb{L}^2}|U(t)|_{\mathbb{L}^2}. \tag{7.23}$$

We integrate (7.23) on $[t_1-k,t_1]$, then we obtain by (7.16)

$$\begin{aligned} \frac{1}{2}|U(t_1)|_{\mathbb{L}^2}^2+\delta\int_{t_1-k}^{t_1}\varphi(U(t))dt&\le\frac{1}{2}|U(t_1-k)|_{\mathbb{L}^2}^2+\int_{t_1-k}^{t_1}|F(t)|_{\mathbb{L}^2}|U(t)|_{\mathbb{L}^2}dt\\ &\le\frac{1}{2}(C_\delta k\varepsilon_0)^2+(C_\delta k\varepsilon_0)k^2\varepsilon_0, \end{aligned} \tag{7.24}$$

where we used

$$\begin{aligned} \int_{t_1-k}^{t_1}|F(t)|_{\mathbb{L}^2}dt&\le\left(\int_{t_1-k}^{t_1}|F(t)|_{\mathbb{L}^2}^2dt\right)^{\frac{1}{2}}\left(\int_{t_1-k}^{t_1}dt\right)^{\frac{1}{2}}\\ &\le\left\{k(k\varepsilon_0)^2\right\}^{\frac{1}{2}}k^{\frac{1}{2}}=k^2\varepsilon_0. \end{aligned}$$

On the other hand, integration of (7.21) with respect to s on $[t_1-k,t_1]$ yields

$$\varphi(U(t_1))k\le\int_{t_1-k}^{t_1}\varphi(U(s))ds+\gamma_+k^2\varepsilon_0+\frac{D_1}{2}(C_\delta k\varepsilon_0)^2k^2+\frac{D_2}{2}\varepsilon_0^\chi k^2+\frac{1}{2\lambda}(k\varepsilon_0)^2k^2. \tag{7.25}$$

Combining (7.25) with (7.24), we deduce

$$\begin{aligned}\varphi(U(t_1))k &\leq \left(\frac{\delta^{-1}}{2}C_\delta\varepsilon_0 + \delta^{-1}C_\delta k\varepsilon_0 + \gamma_+ + \frac{D_1}{2}(C_\delta k)^2\varepsilon_0 + \frac{D_2}{2}\varepsilon_0^{\chi-1} + \frac{1}{2\lambda}k^2\varepsilon_0\right)k^2\varepsilon_0 \\ &< \left(\frac{\delta^{-1}}{2}C_\delta\varepsilon_0 + \delta^{-1}C_\delta\varepsilon_0 + \gamma_+ + \frac{D_1}{2}C_\delta^2\varepsilon_0 + \frac{D_2}{2}\varepsilon_0^{\chi-1} + \frac{1}{2\lambda}\varepsilon_0\right)k^2\varepsilon_0.\end{aligned} \tag{7.26}$$

By the definition (7.20) of k, we obtain again that $\varphi(U(t_1)) < \varepsilon_0$, which leads to the contradiction with the definition of t_1. Therefore $\varphi(U(t)) < \varepsilon_0$ for all $t \in [0, S]$. □

Proof of Theorem 3.
Theorem 3 is a direct consequence of the uniform boundedness of $\varphi(U)$ based on Lemma 12 and Theorem 2 with $r = k\varepsilon_0$. □

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