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Remark on global regularity for the rotating Navier-Stokes equations in a periodic domain

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1. Introduction

In the study of the initial value problems for nonlinear dispersive equations under the periodic boundary condition, since the pioneering work of Bourgain [3] about nonlinear Schrödinger equations and the KdV equation, tools from combinatorics or number theory such as the divisor bound (Lemma 3.2 below) have been exploited to estimate the strength of specific nonlinear interactions by counting the number of sets of frequency modes satisfying a specific condition. The aim of this article is to develop this idea in the case of more involved dispersion relation arising in equations of fluids.

We consider the dispersive effect of the rotating incompressible Navier-Stokes equations in a periodic domain $\mathbb{T}^3 := (\mathbb{R}/2\pi\mathbb{Z})^3$:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \Omega J u - \nu\Delta u = -\nabla p, & t>0, \quad x\in\mathbb{T}^3, \\ \nabla\cdot u = 0 \quad \text{and} \quad u\big|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where[1] $u = (u^1(t,x), u^2(t,x), u^3(t,x))$ and $p = p(t,x)$ are respectively the unknown velocity vector field and scalar pressure at the time $t \geq 0$ and the point $x = (x_1, x_2, x_3)$ in space, while $u_0 = (u_0^1(x), u_0^2(x), u_0^3(x))$ is the given initial velocity field satisfying $\nabla\cdot u_0 = 0$. Here, the Coriolis term $\Omega J u$ with the skew-symmetric matrix

$$J := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

represents the effect of rotation around the vertical x_3 axis. $\Omega \in \mathbb{R}$ is the Coriolis parameter, which is twice the angular velocity of the rotation, and $\nu > 0$ is the kinematic viscosity coefficient.

The Coriolis force appears in almost all of the models of meteorology and geophysics dealing with large-scale phenomena. In 1868 Kelvin observed that a sphere moving along the axis of uniformly rotating water takes with it a column of liquid as if this were a rigid mass (see [7] for references). After that, Taylor [17] and Proudman [16] did important contributions. Mathematically, linear wave dynamics for rotating fluids was investigated by Poincaré [15].

It is known that the dispersive effect of the rotation ensures the existence of global smooth solutions to (1.1). A typical statement is the following:

In this note, we reorganize and summarize the paper [11] by the author and Tsuyoshi Yoneda (University of Tokyo). More information and detailed proofs can be found in [11].

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[1] Vectors in $\mathbb{R}^3$ should be considered as *column vectors*, but we will write them as *row vectors* throughout the article for notational convenience.

> "For any $\nu > 0$ and u_0 in a suitable space, there exists $\Omega_0 > 0$ depending on ν and (the size of) u_0 such that the initial value problem (1.1) with $|\Omega| \geq \Omega_0$ has a unique global-in-time smooth solution."

In the case of $\mathbb{R}^3$, this was proved by Chemin, Desjardins, Gallagher and Grenier [5] and Koh, Lee and Takada [13] by means of the Strichartz-type estimates. In the $\mathbb{T}^3$ case, where the global-in-time Strichartz estimates are no longer true, Babin, Mahalov and Nicolaenko [1, 2] proved the above statement through the analysis of the *resonant equation*, which can be regarded as the formal limit of (1.1) as $|\Omega| \to \infty$.

We notice that Babin et al. proved the result for three-dimensional tori with *any* aspect ratios. They also pointed out that the estimates on the obtained global solutions depend discontinuously on the aspect ratio of the torus. For instance, the global-in-time bound of Sobolev norm obtained in their works is independent of the viscosity coefficient ν for generic periods ([1]), whereas exponential-in-ν^{-1} dependence may occur in the "worst case" ([2]).

In this article, we shall make a refined analysis on the resonant equation. We will focus on the specific torus $\mathbb{T}^3$ with the common period in each direction; this is the situation where the combinatorial techniques work most effectively. Our refined estimate on nonlinear interactions in the resonant equation (Theorem 3.1 below) will enable us to answer the question of existence of global smooth solutions in the case of the rotating Navier-Stokes equations (1.1) but with a slightly less viscosity $(-\Delta)^\alpha$ $(1 > \alpha > \frac{3}{4})$ instead of the usual Laplacian[2], as well as to give a polynomial-in-ν^{-1} estimate on the global solutions.

Fractional Laplacian operators have been employed in many theoretical and numerical works instead of the usual viscosity; see, for example, [4] and [18]. Here, we regard the study of fractional Navier-Stokes equations as the first step towards the *inviscid* case. In the spatially decaying setting, Koh et al. [14] showed long time existence of solutions to the Euler equations under fast rotation assumption by combining the Strichartz estimates with Beale-Kato-Majda's blow-up criterion. In [1], Babin et al. considered long time solvability of the rotating Euler equations in the periodic setting, but only for specific periodic domains (specific aspect ratios) for which the "non-trivial resonant part"[3] is excluded in the nonlinear interactions. On the other hand, in a cylinder case, Golse, Mahalov and Nicolaenko [8] considered bursting dynamics of the inviscid resonant equation. Thus, we may not expect existence of inviscid smooth global flow in general periodic cases where "non-trivial resonances" do exist. Nevertheless, we can progress a less viscosity effect case (fractional Laplacian case) in the periodic domain $\mathbb{T}^3 = [0, 2\pi)^3$.

In the following analysis, we essentially use the spatial Fourier transform denoted

[2] This can be easily proved using estimates given in [1, 2] unless we consider a torus with an aspect ratio that is of the "worst case" of [2]. It can be shown that the regular torus $\mathbb{T}^3 = [0, 2\pi)^3$ is in fact among the "worst case"; see the discussion in [11, Section 4.4].

[3] This part is essentially related to the three-wave resonances of the Rossby waves in physics (see [12, 19] for example).

by $\mathcal{F}$ or $\widehat{\cdot}$:

$$u(x) = \sum_{n\in\mathbb{Z}^3} \widehat{u}(n)e^{in\cdot x} \quad \text{with} \quad (\mathcal{F}u)(n) = \widehat{u}(n) := \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3} u(x)e^{-in\cdot x}\,dx.$$

We will assume that all the vector fields are *mean-zero*. This assumption is valid from the following observation: Let

$$f(t) := \left(\widehat{u}_0^1(0)\cos\Omega t + \widehat{u}_0^2(0)\sin\Omega t,\ \ -\widehat{u}_0^1(0)\sin\Omega t + \widehat{u}_0^2(0)\cos\Omega t,\ \ \widehat{u}_0^3(0)\right).$$

Note that $f(t)$, which is the solution to the following ODE:

$$f'(t) + \Omega J f(t) = 0, \qquad f(0) = \widehat{u}_0(0),$$

is the average over $\mathbb{T}^3$ of the velocity component of the solution to (1.1) at t. Then the following invertible transforms

$$u(t,x) \mapsto u\Big(t, x + \int_0^t f(s)ds\Big) - f(t) \quad \text{and} \quad p(t,x) \mapsto p\Big(t, x + \int_0^t f(s)ds\Big)$$

preserve the equation (1.1), and the new velocity field has zero mean for all time.

We therefore define Sobolev spaces $H^s(\mathbb{T}^3)$ without distinguishing homogeneous and inhomogeneous ones, as follows:

$$H^s(\mathbb{T}^3) := \left\{ u = \sum_{n\in\mathbb{Z}^3\setminus\{0\}} \widehat{u}(n)e^{in\cdot x} \;\middle|\; \|u\|_{H^s} := \Big(\sum_{n\in\mathbb{Z}^3\setminus\{0\}} |n|^{2s}|\widehat{u}(n)|^2\Big)^{1/2} < \infty \right\}.$$

Note that $s_1 < s_2$ implies $\|u\|_{H^{s_1}} \le \|u\|_{H^{s_2}}$ and $H^{s_2} \subset H^{s_1}$.

2. Review of previous results

Before discussing our results, we briefly recall the strategy of [1, 2] (see also [6]).

2.1. Poincaré propagator

Let $\mathbb{P}$ be the Helmholtz-Leray projection onto divergence-free fields, which acts as multiplication by the matrix $\widehat{\mathbb{P}}(n)$ in the Fourier space:

$$\widehat{\mathbb{P}}(n) = \mathrm{Id} - \Big(\frac{n_i n_j}{|n|^2}\Big)_{1\le i,j\le 3} = \frac{1}{|n|^2}\begin{pmatrix} n_2^2+n_3^2 & -n_1n_2 & -n_1n_3 \\ -n_2n_1 & n_1^2+n_3^2 & -n_2n_3 \\ -n_3n_1 & -n_3n_2 & n_1^2+n_2^2 \end{pmatrix}, \quad n\in\mathbb{Z}^3\setminus\{0\}.$$

As usual, we apply $\mathbb{P}$ to (1.1) and consider the equation for the velocity field only:

$$\partial_t u + \mathbb{P}(u\cdot\nabla)u + \Omega\mathbb{P}J\mathbb{P}u + \nu A u = 0, \qquad u|_{t=0} = u_0 \quad \text{with } \nabla\cdot u_0 = 0, \tag{2.1}$$

where $A := -\mathbb{P}\Delta\mathbb{P}$ is the Stokes operator.

The *Poincaré propagator* $\mathcal{L}(\Omega t) = e^{-\Omega t\mathbb{P}J\mathbb{P}}$ is defined as the unitary group associated with the linear problem

$$\partial_t\Phi + \Omega\mathbb{P}J\mathbb{P}\Phi = 0, \qquad \Phi\big|_{t=0} = \Phi_0 \quad \text{with } \nabla\cdot\Phi_0 = 0.$$

We observe that the operator $\mathbb{P}J\mathbb{P}$ can be written in Fourier space as multiplication by a matrix

$$\widehat{\mathbb{P}}(n)J\widehat{\mathbb{P}}(n) = \frac{n_3}{|n|^2}\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix},$$

which has eigenvalues $\pm i\frac{n_3}{|n|}, 0$.

Moreover, for each $n \in \mathbb{Z}^3 \setminus \{0\}$, the vectors $e^{\pm}(n) \in \mathbb{C}^3$ defined by

$$e^{\pm}(n) = \begin{cases} \dfrac{1}{\sqrt{2}|n||n^h|}\big(n_1n_3 \pm in_2|n|,\ n_2n_3 \mp in_1|n|,\ -|n^h|^2\big) & \text{if } n^h := (n_1,n_2) \neq 0, \\ \dfrac{1}{\sqrt{2}}\big(1,\ \mp i\,\mathrm{sgn}(n_3),\ 0\big) & \text{if } n^h = 0 \end{cases}$$

are eigenvectors corresponding to $\pm i\frac{n_3}{|n|}$ and form an orthonormal basis of

$$\{\, \widehat{a} \in \mathbb{C}^3 \,|\, n \cdot \widehat{a} = 0 \,\} = \mathrm{Ran}\,\widehat{\mathbb{P}}(n).$$

We define the orthogonal projections $\Pi_n^{\pm}\widehat{a} := \langle \widehat{a}, e^{\pm}(n)\rangle_{\mathbb{C}^3} e^{\pm}(n)$, so that the Poincaré propagator $\mathcal{L}(\Omega t)$ acts on a divergence-free and mean-free vector field as

$$a(x) = \sum_{n\neq 0}\sum_{\sigma\in\{\pm\}} \Pi_n^{\sigma}\widehat{a}(n)e^{in\cdot x} \quad\Longrightarrow\quad \big[\mathcal{L}(\Omega t)a\big](x) = \sum_{n\neq 0}\sum_{\sigma\in\{\pm\}} e^{-\sigma i\Omega t\frac{n_3}{|n|}}\Pi_n^{\sigma}\widehat{a}(n)e^{in\cdot x}.$$

2.2. Van der Pol transformation, Resonant equation

Next, we introduce *van der Pol transformation* $v(t) = \mathcal{L}(-\Omega t)u(t)$. Since $\mathcal{L}(\Omega t)$ commutes with A, the equation (2.1) becomes

$$\partial_t v + \nu A v + B(\Omega t; v(t), v(t)) = 0, \qquad v|_{t=0} = u_0 \quad \text{with } \nabla\cdot u_0 = 0, \tag{2.2}$$

where

$$B(\Omega t; a, b) := \mathcal{L}(-\Omega t)\mathbb{P}\big(\mathcal{L}(\Omega t)a\cdot\nabla\big)\mathcal{L}(\Omega t)b$$

for divergence-free mean-zero vector fields a, b, so that

$$\big[\mathcal{F}B(\Omega t; a,b)\big](n) = \sum_{\sigma=(\sigma_1,\sigma_2,\sigma_3)\in\{\pm\}^3}\ \sum_{n=k+m} e^{-i\Omega t\omega^{\sigma}_{nkm}}\big(\Pi_k^{\sigma_1}\widehat{a}(k)\cdot im\big)\Pi_n^{\sigma_3}\big[\Pi_m^{\sigma_2}\widehat{b}(m)\big],$$

$$\omega^{\sigma}_{nkm} := \sigma_1\frac{k_3}{|k|} + \sigma_2\frac{m_3}{|m|} - \sigma_3\frac{n_3}{|n|}.$$

Now we decompose $B(\Omega t; a, b)$ into the *resonant* and the *non-resonant* parts as

$$B(\Omega t; a, b) = B_R(a,b) + B_{NR}(\Omega t; a, b),$$

where

$$\big[\mathcal{F}B_R(a,b)\big](n) := \sum_{\sigma\in\{\pm\}^3}\ \sum_{\substack{n=k+m\\ \omega^{\sigma}_{nkm}=0}} \big(\Pi_k^{\sigma_1}\widehat{a}(k)\cdot im\big)\Pi_n^{\sigma_3}\big[\Pi_m^{\sigma_2}\widehat{b}(m)\big],$$

$$[\mathcal{F}B_{NR}(\Omega t;a,b)](n) := \sum_{\sigma\in\{\pm\}^3} \sum_{\substack{n=k+m\\ \omega^\sigma_{nkm}\neq 0}} e^{-i\Omega t\omega^\sigma_{nkm}} \big(\Pi^{\sigma_1}_k \widehat{a}(k)\cdot im\big)\Pi^{\sigma_3}_n \big[\Pi^{\sigma_2}_m \widehat{b}(m)\big].$$

It is expected that the contribution from the non-resonant part becomes smaller as $|\Omega|$ gets larger due to the fast oscillation $e^{-i\Omega t\omega^\sigma_{nkm}}$. Therefore, we are led to consider the following limit equation, which we call the *resonant equation*:

$$\partial_t U + \nu AU + B_R(U(t),U(t)) = 0, \qquad U\big|_{t=0} = u_0 \quad \text{with } \nabla\cdot u_0 = 0. \tag{2.3}$$

In fact, this intuition can be verified by an integration-by-parts argument in t, and existence of the global smooth solution of (2.3) will imply that of (2.2) for $|\Omega|$ large enough; see [11, Section 6] for a proof of this fact and Appendix below for an outline of it.

Let us take initial data from H^1 and focus on how to obtain a global-in-time a priori estimate on the H^1 norm of the smooth solutions to the resonant equation (2.3).[4]

2.3. Reduction to estimating non-trivial resonances

The set of resonant frequency triplets $\{(n,k,m)\}$ is divided into two classes according to $n_3k_3m_3 = 0$ and $n_3k_3m_3 \neq 0$. We call the former *trivial resonances* (this corresponds to two-wave resonances considered in [1]) and the latter *non-trivial resonances* (this corresponds to strict three-wave resonances [2]). The matter is then reduced to estimating the contribution from the non-trivial resonances, as follows.

For a 3D-3C (three-dimensional three-component) vector field $a = (a_1,a_2,a_3) : \mathbb{T}^3 \to \mathbb{R}^3$, we define

- 2D-3C vector field $\overline{a}$ by $\overline{a}(x^h) := \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} a(x)\,dx_3$, or $\overline{a}(x^h) = \displaystyle\sum_{n_3=0} \widehat{a}(n)e^{in\cdot x}$,
- 3D-3C vector field a_{osc} by $a_{\mathrm{osc}}(x) := a(x) - \overline{a}(x^h)$, or $a_{\mathrm{osc}}(x) = \displaystyle\sum_{n_3\neq 0} \widehat{a}(n)e^{in\cdot x}$,
- 3D-2C vector field a^h by $a^h(x) := \big(a_1(x),\, a_2(x)\big)$.

It is easily verified that for any divergence-free and mean-zero vector fields a,b,

$$\overline{B_R(\overline{a},b_{\mathrm{osc}})} = \overline{B_R(a_{\mathrm{osc}},\overline{b})} = B_R(\overline{a},\overline{b})_{\mathrm{osc}} = 0,$$
$$\overline{B_R(\overline{a},\overline{b})} = B_R(\overline{a},\overline{b}) = \big(\mathbb{P}_h(\overline{a}^h\cdot\nabla^h)\overline{b}^h,\ (\overline{a}^h\cdot\nabla^h)\overline{b}_3\big),$$

where $\mathbb{P}_h$ is the 2D Helmholtz-Leray projection and $\nabla^h = (\partial_{x_1},\partial_{x_2})$. Note that $\nabla^h\cdot\overline{u_0}^h = 0$ if $\nabla\cdot u_0 = 0$. Moreover, it is known ([1, Theorem 3.1], [6, Proposition 6.2(1)]; see also [11, Lemma 3.1]) that

$$\overline{B_R(a_{\mathrm{osc}},a_{\mathrm{osc}})} = 0.$$

[4] The existence of local-in-time smooth solutions can be shown by a standard fixed-point argument. See [11, Section 2] for details. Also, the same argument applies to the case of H^s data for $s > \frac{1}{2}$.

Therefore, we have $\overline{B_R(U,U)} = B_R(\overline{U},\overline{U})$, and the equation (2.3) is decoupled into the following three equations:

$$\begin{cases} \partial_t \overline{U}^h + \nu A_h \overline{U}^h + \mathbb{P}_h(\overline{U}^h \cdot \nabla^h)\overline{U}^h = 0, & t>0, \quad x \in \mathbb{T}^2, \\ \overline{U}^h\big|_{t=0} = \overline{u_0}^h \quad \text{with } \nabla^h \cdot \overline{u_0}^h = 0, \end{cases} \tag{2.4}$$

$$\begin{cases} \partial_t \overline{U}_3 + \nu A_h \overline{U}_3 + (\overline{U}^h \cdot \nabla^h)\overline{U}_3 = 0, & t>0, \quad x \in \mathbb{T}^2, \\ \overline{U}_3\big|_{t=0} = \overline{u_{0,3}}, \end{cases} \tag{2.5}$$

$$\begin{cases} \partial_t U_{\text{osc}} + \nu A U_{\text{osc}} + B_R(\overline{U}, U_{\text{osc}}) + B_R(U_{\text{osc}}, \overline{U}) + B_R(U_{\text{osc}}, U_{\text{osc}}) = 0, & t>0, \quad x \in \mathbb{T}^3, \\ U_{\text{osc}}\big|_{t=0} = u_{0,\text{osc}} \quad \text{with } \nabla \cdot u_{0,\text{osc}} = 0, \end{cases} \tag{2.6}$$

where $A_h = -\mathbb{P}_h \Delta_h \mathbb{P}_h$.

Using (2.4) and (2.5), a global-in-time a priori estimate for the 2D part $\overline{U}(t)$ can be obtained straightforwardly (see Section 4.2 below):

$$\|\overline{U}(t)\|_{H^1}^2 + \nu \int_0^t \|\overline{U}(\tau)\|_{H^2}^2 d\tau \leq C(\nu, \|\overline{u_0}\|_{H^1}) < \infty, \tag{2.7}$$

where the constant C depends polynomially in ν^{-1} and $\|u_0\|_{H^1}$. For the remaining part U_{osc}, we use the fact that

$$\langle B_R(\overline{U}, U_{\text{osc}}) \,,\, U_{\text{osc}} \rangle_{H^1} = \langle B_R(U_{\text{osc}}, \overline{U}) \,,\, U_{\text{osc}} \rangle_{H^1} = 0.$$

([1, Theorem 5.3], [6, Proposition 6.2(2)]; see also [11, Lemma 3.2].) Then, we only need to control the term $\langle B_R(U_{\text{osc}}, U_{\text{osc}}) \,,\, U_{\text{osc}} \rangle_{H^1}$ corresponding to the non-trivial resonances in the H^1 energy estimate.

2.4. Previous estimate on the size of non-trivial resonances

For $n \in \mathbb{Z}^3_* := \mathbb{Z}^3 \cap \{n_3 \neq 0\}$, let $\Lambda(n)$ be the set of all $k \in \mathbb{Z}^3$ such that $(n, k, n-k)$ is non-trivially resonant, and for $K > 0$ let $\Lambda_K(n) := \Lambda(n) \cap \{|k| \leq K\}$. The key observation in [2] is the following:

Lemma 2.1 (cf. [2], Proof of Theorem 3.1). *There exists $C > 0$ such that*

$$\sup_{n \in \mathbb{Z}^3_*} \#\Lambda_K(n) \leq CK^2, \qquad \forall K \geq 1.$$

Proof. Fix an arbitrary $n \in \mathbb{Z}^3_*$. For $k = (k^h, k_3) \in \Lambda_K(n)$, there are at most $O(K^2)$ choices for k^h. Now, we observe that $k \in \Lambda(n)$ implies

$$0 = \prod_{\sigma_1, \sigma_2 \in \{\pm\}^2} \omega_{nk(n-k)}^{(\sigma_1, \sigma_2, +)} = \left(\frac{k_3^2}{|k|^2} + \frac{(n_3 - k_3)^2}{|n-k|^2} - \frac{n_3^2}{|n|^2} \right)^2 - \frac{4k_3^2(n_3-k_3)^2}{|k|^2|n-k|^2} = \frac{P(n,k)}{|n|^4|k|^4|n-k|^4},$$

where $P(n,k)$ is a non-degenerate polynomial of degree 8 in k_3. Hence, if we also fix k^h, then there are at most 8 choices for k_3. This implies the desired estimate. $\square$

Roughly speaking, Lemma 2.1 says that the number of non-trivial resonant frequencies is at most "2D like", though the interactions are genuinely 3D. This upper bound, together with an argument using the Littlewood-Paley decomposition, allows us to derive a 2D-like estimate:

$$\left|\langle B_R(U_{\mathrm{osc}}, U_{\mathrm{osc}}) \,,\, U_{\mathrm{osc}}\rangle_{H^1}\right| \le C\|U_{\mathrm{osc}}\|_{H^1}^2\|U_{\mathrm{osc}}\|_{H^2} \le \nu\|U_{\mathrm{osc}}\|_{H^2}^2 + \frac{C}{\nu}\|U_{\mathrm{osc}}\|_{H^1}^4.$$

Combining this estimate with the H^1 energy argument on (2.6), Gronwall's inequality and the straightforward L^2 energy equality

$$\|U_{\mathrm{osc}}(t)\|_{L^2}^2 + 2\nu\int_0^t \|U_{\mathrm{osc}}(\tau)\|_{H^1}^2\,d\tau = \|u_{0,\mathrm{osc}}\|_{L^2}^2, \tag{2.8}$$

we have

$$\begin{aligned}\|U_{\mathrm{osc}}(t)\|_{H^1}^2 + \nu\int_0^t \|U_{\mathrm{osc}}(\tau)\|_{H^2}^2\,d\tau &\le \|u_{0,\mathrm{osc}}\|_{H^1}^2 \exp\Big(\frac{C}{\nu}\int_0^t \|U_{\mathrm{osc}}(\tau)\|_{H^1}^2\,d\tau\Big)\\ &\le \|u_{0,\mathrm{osc}}\|_{H^1}^2 \exp\Big(\frac{C}{\nu^2}\|u_{0,\mathrm{osc}}\|_{L^2}^2\Big), \qquad t > 0.\end{aligned} \tag{2.9}$$

This and (2.7) will imply the H^1 a priori estimate on $U(t)$, as desired.

We remark that the above argument yields only an exponential-in-ν^{-1} bound on the solution. Moreover, it seems difficult to obtain any global-in-time bound from the above estimates in the case of fractional viscosity $(-\Delta)^\alpha$ with $\alpha < 1$.[5]

The above argument can be clearly applied to the torus with *arbitrary* aspect ratios;[6]

$$\mathbb{T}^3_{\theta_1,\theta_2} := (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\theta_1\mathbb{Z}) \times (\mathbb{R}/2\pi\theta_2\mathbb{Z}), \qquad \forall\theta_1,\theta_2 > 0.$$

The only difference is to consider frequencies $n \in \mathbb{Z}^3_{\theta_1,\theta_2} := \mathbb{Z}\times(\theta_1^{-1}\mathbb{Z})\times(\theta_2^{-1}\mathbb{Z})$ instead of $n \in \mathbb{Z}^3$. In fact, it is easily shown ([1]) that for *almost all* (θ_1, θ_2) it holds $\cup_n\Lambda(n) = \emptyset$, hence $\langle B_R(U_{\mathrm{osc}}, U_{\mathrm{osc}}) \,,\, U_{\mathrm{osc}}\rangle_{H^1} \equiv 0$, which implies much better results. However, concerning the *regular* torus $\mathbb{T}^3$, no estimate better than Lemma 2.1 has been obtained.

3. Main result and its proof

Proof of the "2D-like" estimate in Lemma 2.1 is quite simple and applies to the torus with arbitrary aspect ratios. It is actually almost "trivial" in the sense that the constraint $\omega^\sigma_{nkm} = 0$ should reduce possibility for k by at least one dimension.

Now, it is natural to expect that the non-trivial resonance is in fact much rarer event, since the resonance relation determines a surface of nonzero curvature in the frequency space. For instance, the number of 3D integer points on a sphere of radius N, which are determined by one constraint $|n| = N$ and therefore initially expected to be at most $O(N^2)$, is in fact known to be $O(N^{1+\varepsilon})$ for any $\varepsilon > 0$.

[5] Although the non-trivial resonant interactions are quantitatively "2D-like", they are actually 3D interactions, and thus we cannot exploit the vorticity framework for these interactions as in 2D.

[6] We may always assume the period in the x_1 direction to be equal to 2π by rescaling the torus.

For nonlinear dispersive equations such as nonlinear Schrödinger equations and the KdV equation, tools from elementary number theory are used to derive better bounds. However, such analysis seems less developed for equations of rotating fluids due to complicated dispersion relations.

Our main result is a justification of this intuition for the regular torus $\mathbb{T}^3$:

Theorem 3.1 ([11], Lemma 5.1). *For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\sup_{n\in\mathbb{Z}^3_*} \#\Lambda_K(n) \leq C_\varepsilon K^{1+\varepsilon}, \qquad \forall K \geq 1.$$

Namely, the number of non-trivial resonant frequencies is actually at most "(1+ε)D-like". Note that Theorem 3.1 also holds in the case of *rational* torus, i.e., torus with rational aspect ratios.

Our proof is based on a combinatorial argument with the following *divisor bound*:

Lemma 3.2 (cf. Theorems 278 and 315 in [9]). *For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that the following estimates hold for any positive integer N.*

(i) $\#\{\text{divisors of } N\} \leq C_\varepsilon N^\varepsilon$.

(ii) $\#\{(x,y) \in \mathbb{Z}^2 \,|\, x^2+y^2 = N\} \leq C_\varepsilon N^\varepsilon$.

Proof of Theorem 3.1. For given $n, k, m \in \mathbb{Z}^3_*$, positive integers $\nu, \kappa, \mu, d_n, d_k, d_m$ are uniquely determined so that

$$|n| = \nu\sqrt{d_n}, \quad |k| = \kappa\sqrt{d_k}, \quad |m| = \mu\sqrt{d_m}, \quad d_n, d_k, d_m : \text{square-free}.$$

We first see that $d_n = d_k = d_m$ if $\omega^\sigma_{nkm} = 0$. In fact, we have

$$\frac{n_3^2}{|n|^2} - 2\sigma_2\sigma_3\frac{n_3k_3}{|n||k|} + \frac{k_3^2}{|k|^2} = \frac{m_3^2}{|m|^2},$$

hence $|n||k| = \nu\kappa\sqrt{d_nd_k}$ must be a rational number, which means $d_n = d_k$ since both d_n and d_k are square-free. Similarly we have $d_n = d_m$. Therefore, we may write uniquely as

$$|n| = \nu\sqrt{d}, \quad |k| = \kappa\sqrt{d}, \quad |m| = \mu\sqrt{d}, \quad d : \text{square-free}.$$

Given an arbitrary $n \in \mathbb{Z}^3_*$, we need to count the number of $k \in \mathbb{Z}^3_*$ such that $n_3 \neq k_3$, $\omega^\sigma_{nk(n-k)} = 0$ and $|k| \leq K$. We focus on the case $\sigma = (+,+,+)$; a similar proof applies for other cases. Note that ν, d are determined once n is fixed.

Since $|k| \leq K$ and $k_3 \neq 0$, there are at most $2K$ choices for k_3. We fix k_3, so that $n_3 - k_3$ is also fixed. We shall prove that there are at most $O(K^{\varepsilon/2})$ choices for κ. Before proving it, we note that there are at most $O(K^{\varepsilon/2})$ choices for (k_1, k_2) after fixing k_3 and κ, because $k_1^2 + k_2^2 = |k|^2 - k_3^2 = \kappa^2 d - k_3^2 =: N$ is now a fixed positive integer and we can apply Lemma 3.2 (ii), noticing $N \leq |k|^2 \leq K^2$. These estimates imply the desired bound on the number of k's.

Now we estimate the total number of possible κ's for fixed n and k_3, considering the following three cases separately.

(I) $|n| \lesssim K^6$: By the argument at the beginning of the proof (with $m = n - k$), we see that

$$\omega^{\sigma}_{nk(n-k)} = 0 \iff \frac{k_3}{\kappa} + \frac{n_3 - k_3}{\mu} = \frac{n_3}{\nu}$$
$$\iff (n_3\kappa - k_3\nu)\big(n_3\mu - (n_3 - k_3)\nu\big) = k_3(n_3 - k_3)\nu^2.$$

Therefore, $n_3\kappa - k_3\nu \in \mathbb{Z}$ divides the fixed integer $k_3(n_3 - k_3)\nu^2$ of size $O(K^{1+6+6\cdot 2})$. By Lemma 3.2 (i), there are at most $O(K^{\varepsilon/2})$ choices for $n_3\kappa - k_3\nu \in \mathbb{Z}$. This implies that there are at most $O(K^{\varepsilon/2})$ possibilities for κ, because n_3, k_3, ν are all already determined.

(II) $|n| \gg K^6$, $|n_3| \lesssim |n|^{1/2}$: We see that this case does not occur. In fact, it would hold that $|n - k| \sim |n|$ and $|k| \leq K \ll |n|^{1/2}$ in this case. Then, we would have

$$\frac{1}{K} \leq \frac{1}{|k|} \leq \left|\frac{k_3}{|k|}\right| \leq \left|\frac{n_3}{|n|}\right| + \left|\frac{n_3 - k_3}{|n - k|}\right| \lesssim \frac{|n|^{1/2}}{|n|} = \frac{1}{|n|^{1/2}},$$

which is not consistent with $|n| \gg K^6$.

(III) $|n| \gg K^6$, $|n_3| \gg |n|^{1/2}$: In this case we use the classical geometric argument of Jarník [10] to show that there are at most four choices for κ's. Suppose for contradiction that there are five possibilities for κ. Since $(\kappa, \mu) \in \mathbb{Z}^2$ must be on the fixed hyperbola

$$\left\{(x, y) \in \mathbb{R}^2 \,\middle|\, \left(x - \frac{k_3\nu}{n_3}\right)\left(y - \frac{(n_3 - k_3)\nu}{n_3}\right) = \frac{k_3(n_3 - k_3)\nu^2}{n_3^2}\right\},$$

at least three different (non-collinear) points $P_j := (\kappa_j, \mu_j) \in \mathbb{Z}^2$ $(j = 1, 2, 3)$ are on the same component of this curve in this order. Now, under the assumptions $|n| \gg K^6$ and $|n_3| \gg |n|^{1/2}$, we can show that the (non-zero) curvature of this curve is so small that the area of the region surrounded by the curve and the segment $\overline{P_1P_3}$ is less than $\frac{1}{2}$. This is a contradiction, however, because the area of a non-degenerate lattice triangle is bounded from below by $\frac{1}{2}$. Therefore, we finish the case (III).

This completes the proof of Theorem 3.1. $\square$

At the level of the H^1 energy estimate, Theorem 3.1 yields the following:

Corollary 3.3 ([11], Lemma 4.1). *For any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for any real-valued, divergence-free and mean-zero smooth vector field a on $\mathbb{T}^3$, we have*

$$\left|\langle B_R(a_{\rm osc}, a_{\rm osc})\,,\, a_{\rm osc}\rangle_{H^1}\right| \leq C_\varepsilon \|a_{\rm osc}\|_{H^1}^2 \|a_{\rm osc}\|_{H^{\frac{3+\varepsilon}{2}}}.$$

Proof is based on the Littlewood-Paley decomposition technique and essentially the same as the corresponding result in [2] (cf. [2, Lemma 3.1], [6, Lemma 6.2]).

At the moment, it is not clear whether our estimate with $O(K^{1+\varepsilon})$ in Theorem 3.1 is optimal or not. To conclude this section, let us see that $\#\Lambda(n)$ is at least *not uniformly bounded in n*; in other words, $\lim_{K\to\infty} \sup_n \#\Lambda_K(n) = \infty$.[7]

[7] The example used in the proof of Proposition 3.4 also ensures that the case of regular torus $\mathbb{T}^3$ is included in the "worst case" in [2], where no estimate better than Lemma 2.1 was obtained. On the other hand, this example does not exclude the possibility that the estimate with $O(K^{1+\varepsilon})$ may be improved to $O(K^\varepsilon)$.

Proposition 3.4 (cf. [11], Lemma 4.2). *We have* $\sup_{n\in\mathbb{Z}^3_*} \#\Lambda(n) = \infty$.

Proof. Let us look for non-trivially resonant frequency triplets (n, k, m) of the form

$$n = (x+y, 0, x+y), \quad k = (x, 1, y), \quad m = (y, -1, x) \tag{3.1}$$

for some $x, y \in \mathbb{Z}$ with $xy(x+y) \neq 0$. Since any (non-zero) scalar multiple of a non-trivially resonant frequency triplet is again non-trivially resonant, it suffices to show that there are infinitely many distinct triplets of the form (3.1).

For frequencies (3.1) to be non-trivially resonant, we impose the condition

$$\omega_{nkm}^{(+,+,+)} = 0 \quad \text{or} \quad \omega_{nkm}^{(+,+,-)} = 0.$$

This implies that

$$\Big(\frac{x+y}{\sqrt{x^2+y^2+1}}\Big)^2 = \Big(\frac{1}{\sqrt{2}}\Big)^2 \quad \Longleftrightarrow \quad x^2+4xy+y^2 = 1.$$

Hence, it suffices to find infinitely many $(x, y) \in \mathbb{Z}^2$ satisfying $x^2 + 4xy + y^2 = 1$ and $xy(x+y) \neq 0$.

This can be shown by the theory of *Pell's equations*. In fact, we notice that $x^2 + 4xy + y^2 = (x+2y)^2 - 3y^2$ and $X^2 - 3Y^2 = 1$ is one of Pell's equations with the minimal solution $(X_1, Y_1) = (2, 1)$. Then, all of the (positive) integer solutions of $X^2 - 3Y^2 = 1$ are given by (X_j, Y_j) with $X_j + Y_j\sqrt{3} = (2+\sqrt{3})^j$, $j = 1, 2, \ldots$, or the recurrence formulae $X_{j+1} = 2X_j + 3Y_j$, $Y_{j+1} = X_j + 2Y_j$. Therefore, going back to the equation $x^2 + 4xy + y^2 = 1$, we obtain a family of solution $\{(x_j, y_j)\}_{j\geq 1}$ defined by $(x_1, y_1) = (0, 1)$, $x_{j+1} = -y_j$ and $y_{j+1} = x_j + 4y_j$. It is not hard to see that $\{(x_j, y_j)\}_{j\geq 2}$ gives infinitely many non-trivial resonances through (3.1), as required. □

4. Applications

4.1. Polynomial bound on the global solutions

Concerning the rotating Navier-Stokes equations (1.1), our result can be applied to improve the exponential-in-ν^{-1} estimate (2.9) to a polynomial one.

To this end, we use Corollary 3.3 and an interpolation argument:

$$\begin{aligned}\big|\langle B_R(U_{\rm osc}, U_{\rm osc}),\, U_{\rm osc}\rangle_{H^1}\big| &\leq C_\varepsilon \|U_{\rm osc}\|_{H^1}^2 \|U_{\rm osc}\|_{H^{\frac{3+\varepsilon}{2}}} \leq C_\varepsilon \|U_{\rm osc}\|_{H^2}^{\frac{3+\varepsilon}{2}} \|U_{\rm osc}\|_{H^1}^{\frac{1-\varepsilon}{2}} \|U_{\rm osc}\|_{L^2} \\ &\leq \nu \|U_{\rm osc}\|_{H^2}^2 + C_\varepsilon \nu^{-\frac{3+\varepsilon}{1-\varepsilon}} \|U_{\rm osc}\|_{H^1}^2 \|U_{\rm osc}\|_{L^2}^{\frac{4}{1-\varepsilon}}\end{aligned}$$

for $0 < \varepsilon < 1$, and hence by the H^1 energy estimate on (2.6)

$$\frac{d}{dt}\|U_{\rm osc}(t)\|_{H^1}^2 + \nu\|U_{\rm osc}(t)\|_{H^2}^2 \leq C_\varepsilon \nu^{-\frac{3+\varepsilon}{1-\varepsilon}} \|U_{\rm osc}(t)\|_{H^1}^2 \|U_{\rm osc}(t)\|_{L^2}^{\frac{4}{1-\varepsilon}}, \qquad t > 0.$$

Integrating on $(0, t)$ and applying (2.8), we obtain

$$\|U_{\rm osc}(t)\|_{H^1}^2 + \nu \int_0^t \|U_{\rm osc}(\tau)\|_{H^2}^2\, d\tau \leq \|u_{0,\rm osc}\|_{H^1}^2 + C_\varepsilon \nu^{-\frac{4}{1-\varepsilon}} \|u_{0,\rm osc}\|_{L^2}^{\frac{4}{1-\varepsilon}+2}, \qquad t > 0.$$

This and (2.7) yield a global-in-time a priori estimate on $\|U(t)\|_{H^1}$ depending polynomially in ν^{-1} and $\|u_0\|_{H^1}$.

4.2. Fractional Navier-Stokes equations

As another simple application of Theorem 3.1, we consider the existence of global-in-time smooth solutions to the rotating Navier-Stokes equations with *fractional Laplacian*:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \Omega J u + \nu(-\Delta)^\alpha u = -\nabla p, & t>0, \quad x\in\mathbb{T}^3, \\ \nabla\cdot u = 0 \quad \text{and} \quad u\big|_{t=0} = u_0. \end{cases} \tag{4.1}$$

Let us consider the case of less dissipation: $1\geq\alpha>0$. Applying the Helmholtz-Leray projection $\mathbb{P}$, we investigate the following Cauchy problem instead of (4.1):

$$\partial_t u + \mathbb{P}(u\cdot\nabla)u + \Omega\mathbb{P}J\mathbb{P}u + \nu A^\alpha u = 0, \qquad u|_{t=0} = u_0 \quad \text{with } \nabla\cdot u_0 = 0. \tag{4.2}$$

As before, we concentrate here on deriving a global-in-time H^1 a priori estimate on smooth solutions $U(t)$ to the resonant equation:

$$\partial_t U + \nu A^\alpha U + B_R(U(t),U(t)) = 0, \qquad U\big|_{t=0} = u_0 \quad \text{with } \nabla\cdot u_0 = 0. \tag{4.3}$$

We decompose (4.3) into equations for $\overline{U}^h$, $\overline{U}_3$ and U_{osc} similarly to (2.4)–(2.6).

For the 2D horizontal part $\overline{U}^h$, we consider equation for the vorticity $\omega = \nabla_h^\perp\cdot\overline{U}^h := -\partial_{x_2}\overline{U}_1 + \partial_{x_1}\overline{U}_2$. Note that $\overline{U}^h$ can be recovered from ω by the Biot-Savart law $\overline{U}^h = -(-\Delta_h)^{-1}\nabla_h^\perp\omega$ and $\|\omega\|_{H^s}\sim\|\overline{U}^h\|_{H^{s+1}}$ for $s\in\mathbb{R}$, whenever ω is mean-zero. From the L^2 energy estimate on ω, we obtain the following inequality for $\overline{U}^h$:

$$\|\overline{U}^h(t)\|_{H^1}^2 + \nu\int_0^t \|\overline{U}^h(\tau)\|_{H^{1+\alpha}}^2\,d\tau \leq C\|\overline{u_0}^h\|_{H^1}^2, \qquad t>0. \tag{4.4}$$

For the 2D vertical part $\overline{U}_3$, we begin with the easy L^2 energy equality:

$$\|\overline{U}_3(t)\|_{L^2}^2 + 2\nu\int_0^t \|\overline{U}_3(\tau)\|_{H^\alpha}^2\,d\tau = \|\overline{u_{0,3}}\|_{L^2}^2, \qquad t>0. \tag{4.5}$$

For the H^1 energy estimate, we see that the 2D Sobolev inequality and an interpolation argument yield that

$$\begin{aligned}\big|\langle(\overline{U}^h\cdot\nabla_h)\overline{U}_3\,,\,\overline{U}_3\rangle_{H^1}\big| &= \big|\langle\nabla_h\overline{U}^h\,,\,\nabla_h\overline{U}_3\otimes\nabla_h\overline{U}_3\rangle_{L^2}\big| \leq C\|\overline{U}^h\|_{H^1}\|\overline{U}_3\|_{H^{3/2}}^2 \\ &\leq C\|\overline{U}^h\|_{H^1}\|\overline{U}_3\|_{H^{1+\alpha}}^{\frac{3-2\alpha}{\alpha}}\|\overline{U}_3\|_{H^\alpha}^{\frac{2\alpha-1}{\alpha}} \leq \nu\|\overline{U}_3\|_{H^{1+\alpha}}^2 + C_\alpha\nu^{-\frac{3-2\alpha}{2\alpha-1}}\|\overline{U}^h\|_{H^1}^{\frac{2}{2\alpha-1}}\|\overline{U}_3\|_{H^\alpha}^2.\end{aligned}$$

Note that this estimate is available as long as $\frac{3}{2}\geq\alpha>\frac{1}{2}$. From this we have

$$\frac{d}{dt}\|\overline{U}_3(t)\|_{H^1}^2 + \nu\|\overline{U}_3(t)\|_{H^{1+\alpha}}^2 \leq C_\alpha\nu^{-\frac{3-2\alpha}{2\alpha-1}}\|\overline{U}^h(t)\|_{H^1}^{\frac{2}{2\alpha-1}}\|\overline{U}_3(t)\|_{H^\alpha}^2, \qquad t>0.$$

Integrating both sides in t and applying (4.4), (4.5), we obtain that

$$\|\overline{U}_3(t)\|_{H^1}^2 + \nu\int_0^t\|\overline{U}_3(\tau)\|_{H^{1+\alpha}}^2\,d\tau \leq \|\overline{u_{0,3}}\|_{H^1}^2 + C_\alpha\nu^{-\frac{2}{2\alpha-1}}\|\overline{u_0}^h\|_{H^1}^{\frac{2}{2\alpha-1}}\|\overline{u_{0,3}}\|_{L^2}^2. \tag{4.6}$$

For the non-trivial resonance part $U_{\rm osc}$, we have the following L^2 energy equality:

$$\|U_{\rm osc}(t)\|_{L^2}^2 + 2\nu \int_0^t \|U_{\rm osc}(\tau)\|_{H^\alpha}^2 \, d\tau = \|u_{0,{\rm osc}}\|_{L^2}^2, \qquad t > 0. \tag{4.7}$$

We proceed the H^1 energy estimate as in Section 4.1. If $\alpha, \varepsilon > 0$ satisfy $4\alpha > 3 + \varepsilon$ (this requires $\alpha > \frac{3}{4}$), then Corollary 3.3 and interpolation imply that

$$\begin{aligned}
\left| \langle B_R(U_{\rm osc}, U_{\rm osc}) \, , \, U_{\rm osc} \rangle_{H^1} \right| &\le C_\varepsilon \|U_{\rm osc}\|_{H^1}^2 \|U_{\rm osc}\|_{H^{\frac{3+\varepsilon}{2}}} \\
&\le C_\varepsilon \|U_{\rm osc}\|_{H^{1+\alpha}}^{\frac{7-4\alpha+\varepsilon}{2}} \|U_{\rm osc}\|_{H^\alpha}^{\frac{4\alpha-3-\varepsilon}{2}} \|U_{\rm osc}\|_{L^2} \\
&\le \nu \|U_{\rm osc}\|_{H^{1+\alpha}}^2 + C_{\alpha,\varepsilon} \nu^{-\frac{7-4\alpha+\varepsilon}{4\alpha-3-\varepsilon}} \|U_{\rm osc}\|_{H^\alpha}^2 \|U_{\rm osc}\|_{L^2}^{\frac{4}{4\alpha-3-\varepsilon}},
\end{aligned}$$

and hence,

$$\frac{d}{dt} \|U_{\rm osc}(t)\|_{H^1}^2 + \nu \|U_{\rm osc}(t)\|_{H^{1+\alpha}}^2 \le C_{\alpha,\varepsilon} \nu^{-\frac{7-4\alpha+\varepsilon}{4\alpha-3-\varepsilon}} \|U_{\rm osc}(t)\|_{H^\alpha}^2 \|U_{\rm osc}(t)\|_{L^2}^{\frac{4}{4\alpha-3-\varepsilon}}.$$

Integrating both sides in t and applying (4.7), we obtain that

$$\|U_{\rm osc}(t)\|_{H^1}^2 + \nu \int_0^t \|U_{\rm osc}(\tau)\|_{H^{1+\alpha}}^2 \, d\tau \le \|u_{0,{\rm osc}}\|_{H^1}^2 + C_{\alpha,\varepsilon} \nu^{-\frac{4}{4\alpha-3-\varepsilon}} \|u_{0,{\rm osc}}\|_{L^2}^{\frac{4}{4\alpha-3-\varepsilon}+2}. \tag{4.8}$$

Combining (4.4), (4.6) and (4.8), we obtain desired H^1 a priori bound on $U(t)$:

$$\|U(t)\|_{H^1}^2 + \nu \int_0^t \|U(\tau)\|_{H^{1+\alpha}}^2 \, d\tau \le C_{\alpha,\varepsilon} \|u_0\|_{H^1}^2 \big(1 + \nu^{-1} \|u_0\|_{H^1}\big)^{\frac{4}{4\alpha-3-\varepsilon}}, \quad t > 0. \tag{4.9}$$

Some additional arguments (see Appendix below) then yield the following conclusion:

Proposition 4.1 ([11], Theorem 1.3). *Let $1 \ge \alpha > 3/4$, $\nu > 0$. Then, for any $E > 0$ there exists $\Omega_0 = \Omega_0(\alpha, \nu, E) > 0$ such that for any real-valued and divergence-free $u_0 \in H^1(\mathbb{T}^3)$ with $\|u_0\|_{H^1} \le E$ and any $\Omega \in \mathbb{R}$ with $|\Omega| \ge \Omega_0$, a global-in-time smooth solution (u, p) to (4.1) exists and obeys a polynomial bound:*

$$\|u(t)\|_{H^1}^2 + \nu \int_0^t \|u(\tau)\|_{H^{1+\alpha}}^2 \, d\tau \le C E^2 \big(1 + \nu^{-1} E\big)^C, \qquad t > 0. \tag{4.10}$$

Here, $C > 0$ is a constant depending only on α. Moreover, Ω_0 can be taken as

$$\Omega_0 = E \exp\big[C(\nu^{-1}E)^C\big].$$

5. Future works

The analysis on the resonant interactions for the fluid equations is still in progress, and many problems are left open. It is likely that the estimate in Theorem 3.1 is not optimal and holds not only for the regular (or a rational) torus. We are also interested in the case of other equations (i.e., other dispersion relations); see [12] for an observation on the β-plane model.

We note that the Navier-Stokes system with the fractional Laplacian (4.1) is not regarded as a physical model, though the fractional Laplacian operator itself appears in physically important equations, such as the quasi-geostrophic equation. Therefore, applications of the estimate in Theorem 3.1 to more physical models should also be investigated. For instance, application to the equations for non-Newtonian fluids could be a good problem to try.

For the inviscid case, global or long-time existence of smooth solutions to the rotating Euler equatinos or the corresponding resonant equation is completely open in the periodic setting. We hope that a deep study on the resonant interactions will enable us to attack these problems in future.

A. Appendix: Proof of Proposition 4.1

Here, we present an outline of the proof of Proposition 4.1. Our proof based on the framework of mild solutions is in a sense different from the previous argument [2, 6] using the framework of weak solutions.

Let $1 \geq \alpha > 3/4$, $\nu > 0$, $E > 0$,[8] and let $u_0 \in H^1(\mathbb{T}^3)$ be a real-valued, divergence-free vector field satisfying $\|u_0\|_{H^1} \leq E$. Applying van der Pol transformation, we consider the following Cauchy problem instead of (4.2):

$$\partial_t v + \nu A^\alpha v + B(\Omega t; v(t), v(t)) = 0, \qquad v|_{t=0} = u_0. \tag{A.1}$$

In fact, by the unitarity of the Poincaré propagator, Proposition 4.1 is reduced to the same problem for the equation (A.1).

We divide the proof into three steps.

Step 1: Local existence of mild solutions.

We first prepare local-in-time results. Using the semigroup $\{e^{-\nu t A^\alpha}\}_{t\geq 0}$, (A.1) is transformed into the integral equation

$$v(t) = e^{-\nu t A^\alpha} u_0 - \int_0^t e^{-\nu(t-\tau)A^\alpha} B(\Omega\tau; v(\tau), v(\tau))\, d\tau.$$

By a fixed point argument with some appropriate norm, for instance,

$$\|v\|_{X_T} := \sup_{0<t\leq T} \Big(\|v(t)\|_{H^1} + (\nu t)^{\frac{1}{2}} \|v(t)\|_{H^{1+\alpha}} \Big),$$

we can show existence of a unique local-in-time solution to (A.1) on $[0, T_l]$ with $T_l = T_l(\|u_0\|_{H^1}) > 0$, which belongs to $C([0,T_l]; H^1) \cap C((0,T_l]; H^\infty)$. Furthermore, by the H^1 energy estimate, we can show that the solution is also in $L^2((0,T_l); H^{1+\alpha})$.

[8] In the following argument, any constants may depend on α, ν and E. However, we do not track the precise dependence on these parameters for brevity.

Step 2: Global existence for the resonant equation.
We next solve the resonant equation (4.3). Clearly, the local result in Step 1 also holds for (4.3). Then, by the a priori estimate (4.9) established in Section 4.2, we have a global solution $U(t)$ with $U(0) = v(0) = u_0$ satisfying

$$\|U(t)\|_{H^1}^2 + \nu \int_0^t \|U(\tau)\|_{H^{1+\alpha}}^2 \, d\tau \le \widetilde{E}^2 < \infty, \qquad t \ge 0, \tag{A.2}$$

where $\widetilde{E}$ is a constant depending on α, ν and E.

Step 3: Error estimate.
To prove global existence for (A.1), it suffices to ensure that, under the large Coriolis parameter assumption, the solutions $v(t)$, $U(t)$ stay close to each other until an arbitrarily given time $t = T$. More precisely, we claim the following: There exists $\Omega_0 > 0$ such that if $|\Omega| \ge \Omega_0$, then for any $T > 0$, $v(t)$ extends to $[0, T]$ and

$$\|w(t)\|_{H^1}^2 + \nu \int_0^t \|w(\tau)\|_{H^{1+\alpha}}^2 \, d\tau \le \widetilde{E}^2, \qquad t \in [0, T], \tag{A.3}$$

where $w(t) := v(t) - U(t)$ is a solution to

$$\partial_t w + \nu A^\alpha w + B_R(w, v) + B_R(U, w) + B_{NR}(\Omega t; v, v) = 0, \quad w|_{t=0} = 0. \tag{A.4}$$

We show this by induction. (A.3) is obviously true for $T = 0$, so we assume that this is true for some $T \ge 0$. Then, from (A.2), the same estimate but with a bound $(2\widetilde{E})^2$ is true for v on $[0, T]$. Let $\widetilde{T}_l = T_l(2\widetilde{E})$ be the local existence time for data of size $2\widetilde{E}$. By the local theory in Step 1, v extends to $t = T + \widetilde{T}_l$ and we have

$$\|v(t)\|_{H^1}^2 + \nu \int_0^t \|v(\tau)\|_{H^{1+\alpha}}^2 \, d\tau \le L^2, \qquad t \in [0, T + \widetilde{T}_l], \tag{A.5}$$

where $L = L(\widetilde{E}) > 0$ is independent of T.

We now proceed the H^1 energy estimate on w. By the Sobolev inequality and an interpolation argument, together with (A.2), we can show that

$$\begin{aligned}
&\frac{d}{dt}\|w(t)\|_{H^1}^2 + 2\nu \|w(t)\|_{H^{1+\alpha}}^2 \\
&\le C\Big(\|v(t)\|_{H^{1+\alpha}} \|w(t)\|_{H^1} \|w(t)\|_{H^{1+\alpha}} + \|U(t)\|_{H^1} \|w(t)\|_{H^1}^{\frac{1}{2}} \|w(t)\|_{H^{1+\alpha}}^{\frac{3}{2}} \Big) \\
&\quad - 2\langle B_{NR}(\Omega t; v(t), v(t)) \, , \, w(t) \rangle_{H^1} \\
&\le \nu \|w(t)\|_{H^{1+\alpha}}^2 + C\Big(\|v(t)\|_{H^{1+\alpha}}^2 + \widetilde{E}^2 \|U(t)\|_{H^{1+\alpha}}^2 \Big) \|w(t)\|_{H^1}^2 \\
&\quad - 2\langle B_{NR}(\Omega t; v(t), v(t)) \, , \, w(t) \rangle_{H^1},
\end{aligned}$$

and hence,

$$\begin{aligned}
&\|w(t)\|_{H^1}^2 + \nu \int_0^t \|w(\tau)\|_{H^{1+\alpha}}^2 \, d\tau \\
&\le C \int_0^t \Big(\|v(\tau)\|_{H^{1+\alpha}}^2 + \widetilde{E}^2 \|U(\tau)\|_{H^{1+\alpha}}^2 \Big) \|w(\tau)\|_{H^1}^2 \, d\tau \\
&\quad - 2 \int_0^t \langle B_{NR}(\Omega \tau; v(\tau), v(\tau)) \, , \, w(\tau) \rangle_{H^1} \, d\tau, \qquad t \in [0, T + \widetilde{T}_l].
\end{aligned} \tag{A.6}$$

In order to keep w small, we need to make the last term in (A.6) small by taking $|\Omega|$ large, exploiting the non-resonant property of it. Roughly speaking, an integration by parts in τ yields the factor $|\Omega\omega^\sigma_{nkm}|^{-1}$ from the nonlinear interaction between (n,k,m) (at the cost of appearance of $\partial_t v$ and $\partial_t w$).

The problem is that $|\omega^\sigma_{nkm}|$, though it never vanishes thanks to the non-resonant property, does not have a positive lower bound. To deal with this problem, we divide v into the high- and low-frequency parts; $v = v_{>N} + v_{\le N}$, $v_{>N} := \sum_{|n|>N} \widehat{v}(n) e^{in\cdot x}$. The terms with at least one $v_{>N}$ are estimated with some negative power of N at the cost of regularity.[9] For the low-frequency contribution $\langle B_{NR}(\Omega t; v_{\le N}, v_{\le N}), w\rangle_{H^1}$, we can show a positive lower bound

$$\inf\left\{ |\omega^\sigma_{nkm}| \;\middle|\; \begin{array}{c} \sigma\in\{\pm\}^3,\ n,k,m\in\mathbb{Z}^3\setminus\{0\} \text{ s.t.} \\ n=k+m,\ \omega^\sigma_{nkm}\neq 0,\ |k|\le N,\ |m|\le N \end{array}\right\} \gtrsim N^{-12},$$

thus we have a factor $N^{12}/|\Omega|$ by an integration by parts.[10] Therefore, for given $\delta>0$, we first choose $N = N(\delta, \widetilde{E}, L)$ large, and then take $\Omega_0 = \Omega_0(N,\delta,\widetilde{E},L)$ large to obtain at the end

$$\left| 2\int_0^t \langle B_{NR}(\Omega\tau; v(\tau), v(\tau)), w(\tau)\rangle_{H^1}\, d\tau \right| \le \delta + \frac{1}{2}\Big(\|w(t)\|_{H^1}^2 + \nu \int_0^t \|w(\tau)\|_{H^{1+\alpha}}^2\, d\tau \Big)$$

for $t\in[0, T+\widetilde{T}_l]$, provided that $|\Omega|\ge\Omega_0$. Inserting this into (A.6), we have

$$\begin{aligned} &\|w(t)\|_{H^1}^2 + \nu\int_0^t \|w(\tau)\|_{H^{1+\alpha}}^2\, d\tau \\ &\le 2\delta + C\int_0^t \Big(\|v(\tau)\|_{H^{1+\alpha}}^2 + \widetilde{E}^2 \|U(\tau)\|_{H^{1+\alpha}}^2 \Big) \|w(\tau)\|_{H^1}^2\, d\tau \end{aligned}$$

for $t\in[0,T+\widetilde{T}_l]$. By the Gronwall inequality and (A.2), (A.5) again, we obtain

$$\|w(t)\|_{H^1}^2 + \nu\int_0^t \|w(\tau)\|_{H^{1+\alpha}}^2\, d\tau \le 2\delta e^{C(L^2+\widetilde{E}^4)}, \qquad t\in[0,T+\widetilde{T}_l].$$

Choosing $\delta = \delta(L,\widetilde{E})$ sufficiently small, we finally show (A.3) for $t\in[0,T+\widetilde{T}_l]$. Since all the constants in the above argument do not depend on T, we conclude the proof by an induction argument.

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[9] Since the problem (4.1) is H^1-subcritical if $\alpha > 3/4$, there is a room to accept such regularity loss.
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