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Spectrum of the artificial compressible system near bifurcation point

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1 Introduction

This article gives a summary of [5]. We consider the artificial compressible system

$$\begin{aligned} \epsilon^2\partial_t p + \operatorname{div}\boldsymbol{v} &= 0, && (1.1)\\ \partial_t\boldsymbol{v} - \nu\Delta\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p &= \boldsymbol{g}. && (1.2)\end{aligned}$$

on a bounded domain Ω of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Here $\boldsymbol{v} = {}^\top(v^1(x,t), v^2(x,t), v^3(x,t))$ and $p = p(x,t)$ denote the unknown velocity field and pressure, respectively, at time $t > 0$ and position $x \in \Omega$; $\boldsymbol{g} = \boldsymbol{g}(x)$ is a given external force; and $\epsilon > 0$ is a small parameter, called the artificial Mach number.

We consider the system (1.1)–(1.2) under the boundary condition

$$\boldsymbol{v}|_{\partial\Omega} = \boldsymbol{v}_*. \qquad (1.3)$$

Here $\boldsymbol{v}_* = \boldsymbol{v}_*(x)$ is a given velocity field satisfying $\int_{\partial\Omega}\boldsymbol{v}_*\cdot\boldsymbol{n}\,dS = 0$, where $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega$.

A. Chorin proposed the system (1.1)–(1.2) in numerical computation to find a stationary solution of the incompressible Navier-Stokes equations:

$$\begin{aligned} \operatorname{div}\boldsymbol{v} &= 0, && (1.4)\\ \partial_t\boldsymbol{v} - \nu\Delta\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p &= \boldsymbol{g} && (1.5)\end{aligned}$$

with the boundary condition (1.3). The idea of the method proposed by Chorin is stated as follows. Obviously, the sets of stationary solutions of (1.1)–(1.2) and (1.4)–(1.5) are the same ones. If solutions of the artificial compressible system (1.1)–(1.2) converge to a function $u_s = {}^\top(p_s, \boldsymbol{v}_s)$ as $t \to \infty$, then the limit u_s is a stationary solution of (1.1)–(1.2), and thus, u_s is a stationary solution of (1.4)–(1.5). By using this method, Chorin numerically obtained stationary cellular convection patterns of the Bénard convection problem described by the Oberbeck-Boussinesq equation.

A mathematical basis for Chorin's method was given by Kagei and Nishida ([3, 4]). The limit function u_s in Chorin's method is a large time limit of solutions of (1.1)–(1.2), and so, u_s is stable as a solution of (1.1)–(1.2). In [3], it was shown that if u_s is stable as a solution of (1.1)–(1.2), then it is also stable as a solution of (1.4)–(1.5). This means that stationary solutions obtained by Chorin's method represents observable flows in the real world.

It was also shown in [3] that, conversely, if stable stationary solutions of (1.4)–(1.5) are also stable as a solution of (1.1)–(1.2) when $0 < \epsilon \ll 1$, then one can conclude that (1.1)–(1.2) give a good approximation of (1.4)–(1.5) in the stability view point. Furthermore, a sufficient condition for a stable stationary solution of (1.4)–(1.5) to be stable as a solution of (1.1)–(1.2) was obtained in [3]. The condition was then improved in in [4].

We briefly explain the result in [4]. Let us introduce the linearized operators around a stationary solution $u_s = {}^\top(p_s, \boldsymbol{v}_s)$ for the systems (1.1)–(1.2) and (1.4)–(1.5) with (1.3). Here and in what follows ${}^\top\cdot$ stands for the transposition. Let $\mathbb{L} : L^2_\sigma(\Omega) \to L^2_\sigma(\Omega)$ be the operator defined by

$$\mathbb{L} = -\nu\mathbb{P}\Delta + \mathbb{P}(\boldsymbol{v}_s \cdot \nabla + {}^\top(\nabla \boldsymbol{v}_s))$$

with domain $D(\mathbb{L}) = [H^2(\Omega) \cap H^1_0(\Omega)]^3 \cap L^2_\sigma(\Omega)$. Here $H^k(\Omega)$ denotes the k th order L^2-Sobolev space on Ω, $H^1_0(\Omega)$ is the set of all functions f satisfying $f|_{\partial\Omega} = 0$, $\mathbb{P}$ is the orthogonal projection, called the Helmholtz projection from $L^2(\Omega)^3$ to $L^2_\sigma(\Omega)$, and $L^2_\sigma(\Omega)$ denotes the set of all L^2-vector fields $\boldsymbol{w}$ on Ω satisfying $\operatorname{div} \boldsymbol{w} = 0$ and $\boldsymbol{w} \cdot \boldsymbol{n}|_{\partial\Omega} = 0$. We define the operator $L_\epsilon : H^1_*(\Omega) \times L^2(\Omega)^3 \to H^1_*(\Omega) \times L^2(\Omega)^3$, acting on $u = {}^\top(p, \boldsymbol{w})$, by

$$L_\epsilon = \begin{pmatrix} 0 & \frac{1}{\epsilon^2}\operatorname{div} \\ \nabla & -\nu\Delta + \boldsymbol{v}_s \cdot \nabla + {}^\top(\nabla \boldsymbol{v}_s) \end{pmatrix}$$

with domain $D(L_\epsilon) = H^1_*(\Omega) \times [H^2(\Omega) \cap H^1_0(\Omega)]^3$. Here $H^1_*(\Omega)$ denotes the set of all H^1 functions on Ω that have zero mean value over Ω.

The result of [4] is stated as follows: if $\rho(-\mathbb{L}) \supset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -b_0\}$ for some positive constant b_0, then there exist positive constants ϵ_0, κ_0 and b_1 such that $\rho(-L_\epsilon) \supset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -b_1\}$ for $0 < \epsilon \leq \epsilon_0$, provided that

$$\inf_{\boldsymbol{w}\in H_0^1(\Omega)^3, \boldsymbol{w}\neq 0} \frac{\mathrm{Re}\,(\mathbb{Q}\boldsymbol{w}\cdot\nabla \boldsymbol{v}_s, \mathbb{Q}\boldsymbol{w})_{L^2}}{\|\nabla \mathbb{Q}\boldsymbol{w}\|_{L^2}^2} \geq -\kappa_0. \tag{1.6}$$

Here $\mathbb{Q} = I - \mathbb{P}$ is the orthogonal projection from $L^2(\Omega)^3$ to the space $G^2(\Omega) = \{\nabla p; p \in H_*^1(\Omega)\}$ which is the orthogonal complement of $L_\sigma^2(\Omega)$. In general, ϵ_0 depends on b_0, and so it may occur $\epsilon_0 \to 0$ as $b_0 \to 0$. This implies that if b_0 approaches to zero, we have to take the range of ϵ smaller and smaller. This situation can happen when a stationary bifurcation occurs. Therefore, when one considers the stability of a bifurcating stationary solution near the bifurcation point, the range of ϵ shrinks when the bifurcation parameter approaches its critical value.

In this article we will investigate the spectrum of $-L_\epsilon$ near the origin when a stationary bifurcation occurs, following [5]. We will show that the range of ϵ in the result of [4] can be taken uniformly near the bifurcation point in the case of the stability of a bifurcating solution from a simple eigenvalue. Our result is applicable to the Taylor and Bénard problems, i.e., a bifurcation of the Taylor vortex from the Couette flow and a bifurcation of spatially periodic convective patterns from the motionless state, respectively.

2 Main Results

In this section we summarize the results in [5]. For $1 \leq p \leq \infty$ we denote by $L^p(\Omega)$ the usual Lebesgue space over Ω and its norm is denoted by $\|\cdot\|_p$. The mth order L^2 Sobolev space over Ω is denoted by $H^m(\Omega)$, and its norm is denoted by $\|\cdot\|_{H^m}$. The inner product of $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$, i.e.,

$$(f, g) = \int_\Omega f(x)\overline{g(x)}dx.$$

Here $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. We also defined the weighted inner product $\langle\langle\cdot,\cdot\rangle\rangle_\epsilon$ by

$$\langle\langle u_1, u_2\rangle\rangle_\epsilon = \epsilon^2(p_1, p_2) + (\boldsymbol{w}_1, \boldsymbol{w}_2)$$

for $u_j = {}^\top(p_j, \boldsymbol{w}_j)$, $j = 1, 2$. The functions spaces $L_\sigma^2(\Omega)$, $H_0^1(\Omega)$, and $H_*^1(\Omega)$ are the ones defined in section 1.

We are interested in the stability of a stationary solution bifurcating from a basic stationary flow. Let $\mathcal{R}$ be the Reynolds number and let $\boldsymbol{v}_{\mathcal{R}}$ be a basic stationary flow. We consider the following situation.

(A0) There exists a positive number $\mathcal{R}_c$ such that if $\mathcal{R}$ is smaller than $\mathcal{R}_c$, then $\boldsymbol{v}_{\mathcal{R}}$ is stable; and if $\mathcal{R}$ is larger than $\mathcal{R}_c$, then $\boldsymbol{v}_{\mathcal{R}}$ is unstable and a stationary bifurcation occurs at $\mathcal{R} = \mathcal{R}_c$.

Let us introduce a bifurcation parameter $\eta = \mathcal{R} - \mathcal{R}_c$ and write $\boldsymbol{v}_{\mathcal{R}}$ as $\boldsymbol{v}_\eta$. The linearized operator $\mathbb{L}_\eta$ around $\boldsymbol{v}_\eta$ then takes the form,

$$\begin{aligned}\mathbb{L}_\eta &= -\mathbb{P}\Delta + (\mathcal{R}_c + \eta)\mathbb{P}(\boldsymbol{v}_\eta \cdot \nabla + ({}^{\mathrm{T}}\nabla \boldsymbol{v}_\eta)) \\ &= \mathbb{A} + (\mathcal{R}_c + \eta)\mathbb{P}\mathbb{M}[\boldsymbol{v}_\eta],\end{aligned}$$

with domain $D(\mathbb{L}_\eta) = D(\mathbb{A}) = [H^2(\Omega) \cap H_0^1(\Omega)]^3 \cap L_\sigma^2(\Omega)$, where

$$\mathbb{A} = -\mathbb{P}\Delta, \ \mathbb{M}[\boldsymbol{v}]\boldsymbol{w} = \boldsymbol{v} \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}.$$

The adjoint operator of $\mathbb{L}_\eta$ is defined by $\mathbb{L}_\eta^*$:

$$\mathbb{L}_\eta^* = \mathbb{A} + (\mathcal{R}_c + \eta)\mathbb{P}\mathbb{M}^*[\boldsymbol{v}_\eta]$$

with domain $D(\mathbb{L}_\eta) = D(\mathbb{A})$, where

$$\mathbb{M}^*[\boldsymbol{v}]\boldsymbol{w} = -\boldsymbol{v} \cdot \nabla \boldsymbol{w} + (\nabla \boldsymbol{v})\boldsymbol{w}.$$

The following assumptions are made in this article.

(A1) $\boldsymbol{v}_\eta$ is a smooth stationary solution.

(A2) $\boldsymbol{v}_\eta$ is analytic in η in $(H^2 \cap H_0^1)(\Omega)^3$.

(A3) 0 is a simple eigenvalue of $-\mathbb{L}_0$ with $\mathrm{Ker}(\mathbb{L}_0) = \mathrm{span}\{\boldsymbol{w}_0\}$. The eigenprojection P_0 for the eigenvalue 0 is

$$P_0 \boldsymbol{w} = \langle \boldsymbol{w} \rangle \boldsymbol{w}_0.$$

Here and in what follows the symbol $\langle \boldsymbol{w} \rangle$ for $\boldsymbol{w} \in L^2(\Omega)^3$ is defined by

$$\langle \boldsymbol{w} \rangle = (\boldsymbol{w}, \boldsymbol{w}_0^*),$$

where $\boldsymbol{w}_0^*$ is the eigenfunction for the eigenvalue 0 of $\mathbb{L}_0^*$ satisfying $\langle \boldsymbol{w}_0 \rangle = 1$.

(A4) $\langle \mathbb{M}[\boldsymbol{v}_0]\boldsymbol{w}_0 + \mathcal{R}_c \mathbb{M}[\boldsymbol{v}_1]\boldsymbol{w}_0 \rangle \neq 0$, where $\boldsymbol{v}_1 = \partial_\eta \boldsymbol{v}_\eta|_{\eta=0}$.

(A5) There exists a positive constant $\widetilde{b_0} > 0$ such that

$$\{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda \geq -\widetilde{b_0}\} \backslash \{0\} \subset \rho(-\mathbb{L}_0)$$

.

Our interest is concerned with a nontrivial solution branch $\{\eta, \boldsymbol{w}_\eta\}$, $\boldsymbol{w}_\eta \neq \mathbf{0}$, of

$$(\mathrm{NS})_\eta \qquad \mathbb{L}_\eta \boldsymbol{w}_\eta + (\mathcal{R}_c + \eta)\mathbb{P}\mathbb{N}(\boldsymbol{w}_\eta, \boldsymbol{w}_\eta) = 0$$

near $\{\eta, \boldsymbol{w}\} = \{0, 0\}$. Here $\mathrm{N}(\boldsymbol{w}_\eta, \boldsymbol{w}_\eta) = \boldsymbol{w}_\eta \cdot \nabla \boldsymbol{w}_\eta$. We note that $\boldsymbol{w}_\eta = \mathbf{0}$ is a solution of $(\mathrm{NS})_\eta$ for all η. Under (A1)–(A4) we have a nontrivial solution branch. In fact, by applying the standard bifurcation theory ([2]), one can prove the following proposition.

Proposition 2.1. *Assume (A1)-(A4). There exist a positive constant δ_0 and a solution branch $\{\eta(\delta),\ \boldsymbol{w}_\eta(\delta)\}$ of $(\mathrm{NS})_\eta$ with $\eta = \eta(\delta)$ of the form*

$$\eta(\delta) = \delta\sigma(\delta),$$

$$\boldsymbol{w}_\eta(\delta) = \delta(\boldsymbol{w}_0 + \delta \boldsymbol{w}_1(\delta)),$$

where $\sigma(\delta)$ is analytic in δ ($|\delta| \leq \delta_0$), and $\boldsymbol{w}_1(\delta)$ is analytic in δ in $H^2(\Omega)$ ($|\delta| \leq \delta_0$).

Our next issue is the stability of $\tilde{\boldsymbol{v}}(\delta) = \boldsymbol{v}_{\eta(\delta)} + \boldsymbol{w}_\eta(\delta)$. The linearized operator around $\tilde{\boldsymbol{v}}(\delta)$ is denoted by

$$\mathbb{L}(\delta) = -\mathbb{P}\Delta + (\mathcal{R}_c + \eta(\delta))\mathbb{M}[\tilde{\boldsymbol{v}}(\delta)].$$

The spectrum of $-\mathbb{L}(\delta)$ has the following properties.

Proposition 2.2. *Assume (A1)-(A5). There exists a positive number δ_0 such that*

$$\rho(-\mathbb{L}(\delta)) \supset \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda \geq -\frac{3}{4}\tilde{b}_0,\ |\lambda| > \frac{\tilde{b}_0}{4}\},$$

$$\sigma(-\mathbb{L}(\delta)) \cap \{\lambda \in \mathbb{C};\ |\lambda| \leq \frac{\widetilde{b_0}}{4}\} = \{\lambda(\delta)\},$$

for all $\delta \in (-\delta_0, \delta_0)$. *Here* $\lambda(\delta)$ *is a simple eigenvalue given by*

$$\lambda(\delta) = -\alpha(\delta)\delta\frac{d\eta}{d\delta}(\delta),$$

where $\alpha(\delta)$ *is an analytic function of* $\delta \in (-\delta_0, \delta_0)$ *satisfying*

$$\alpha(0) = -\langle \mathbb{M}[\boldsymbol{v}_0]\boldsymbol{w}_0 + \mathcal{R}_c\mathbb{M}[\boldsymbol{v}_1]\boldsymbol{w}_0\rangle (\neq 0).$$

Proposition 2.2 was obtained by Crandall-Rabinowitz [2] (See also [1, Theorem 27.2]).

Assuming (A0), we have $\alpha(0) > 0$. Therefore, we have the following proposition.

Proposition 2.3. *Assume (A0)-(A5).*

(i) $\alpha(0) = -\langle \mathbb{M}[\boldsymbol{v}_0]\boldsymbol{w}_0 + \mathcal{R}_c\mathbb{M}[\boldsymbol{v}_1]\boldsymbol{w}_0\rangle > 0.$

(ii) $\lambda(\delta) = \lambda_k\delta^k + \mathcal{O}(\delta^{k+1})$ *if and only if* $\eta(\delta) = \eta_k\delta^k + \mathcal{O}(\delta^{k+1})$. *In this case, it follows that* $\lambda_k = -k\alpha(0)\eta_k$. *Therefore,* $\operatorname{sgn}(\lambda(\delta)) = -\operatorname{sgn}(\eta(\delta))$ *for* $0 < |\delta| \ll 1$.

We next consider relations between $\lambda^{(l)}$ and $\eta^{(l)}$. We can prove the following proposition by induction on k.

Proposition 2.4. *The following (a)-(c) are equivalent:*

(a) $\lambda^{(l)}(0) = 0$ *for* $l = 1, \cdots, k$.

(b) $\eta^{(l)}(0) = 0$ *for* $l = 1, \cdots, k$.

(c) $\sigma^{(l-1)}(0) = 0$ *for* $l = 1, \cdots, k$.

Under the above situation we consider the stability of the bifurcating solution $\tilde{\boldsymbol{v}}(\delta)$ as a solution of the artificial compressible system (1.4)–(1.5). The linearized operator around $\tilde{\boldsymbol{v}}(\delta)$ is defined by $L(\epsilon, \delta)$ which is an operator on $H^1_*(\Omega) \times L^2(\Omega)^3$ given by

$$L(\epsilon, \delta) = \begin{pmatrix} 0 & \frac{1}{\epsilon^2}\operatorname{div} \\ \nabla & -\Delta + (\mathcal{R}_c + \eta(\delta))\mathbb{M}[\tilde{v}(\delta)] \end{pmatrix}$$

with domain $D(L(\epsilon,\delta)) = D := H^1_*(\Omega)\times[H^2(\Omega)\cap H^1_0(\Omega)]^3$. We also introduce $\mathbb{K}(\delta)$ and $K(\delta)$ defined by

$$\mathbb{K}(\delta) = (\mathcal{R}_c + \eta(\delta))\mathbb{M}[\tilde{\boldsymbol{v}}(\delta)] - \mathcal{R}_c\mathbb{M}[\boldsymbol{v}_0],$$

$$K(\delta) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{K}(\delta) \end{pmatrix}.$$

Proposition 2.1 implies that $\mathbb{M}(\delta)$ and $M(\delta)$ can be expanded as

$$\mathbb{K}(\delta) = \sum_{k=1}^{\infty} \delta^k \mathbb{K}_k,$$

$$K(\delta) = \sum_{k=1}^{\infty} \delta^k K_k, \quad K_k = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{K}_k \end{pmatrix}.$$

Here $\mathbb{K}_k$ satisfies the estimate

$$\|\mathbb{K}_k \boldsymbol{w}\|_2 \le c_k \|\boldsymbol{w}\|_{H^1} \tag{2.1}$$

uniformly for $\boldsymbol{w} \in H^1(\Omega)$ with positive constant c_k satisfying $\sum_{k=1}^{\infty} c_k \delta^k < \infty$ for $|\delta| \le \delta_1$.

We now state the result on the spectrum of $-L(\epsilon,\delta)$ near the origin.

Theorem 2.5. ([5]) *Let $\lambda(\delta) = \lambda_k \delta^k + \mathcal{O}(\delta^{k+1})$ with $\lambda_k \neq 0$ for some $k \ge 1$. Then there exist positive constants $\delta_1 = \delta_1(\tilde{b}_0, \boldsymbol{v}_0)$ and $\epsilon_1 = \epsilon_1(\tilde{b}_0, \boldsymbol{v}_0)$ such that*

$$\sigma(-L(\epsilon,\delta)) \cap \{\lambda \in \mathbb{C};\ |\lambda| \ge \frac{\tilde{b}_0}{4}\} = \{\lambda(\epsilon,\delta)\},$$

$$\lambda(\epsilon,\delta) = \delta^k((1 + c_1(\epsilon^2))\lambda_k + \Lambda_k(\epsilon,\delta))$$

with some $\Lambda_k(\epsilon,\delta) = \mathcal{O}(\delta)$ uniformly for $0 < \epsilon \le \epsilon_1$, $0 < |\delta| \le \delta_1$. Here $c_1(\epsilon^2)$ satisfies $|c_1(\epsilon^2)| \le \frac{1}{2}$ for $0 < \epsilon \le \epsilon_1$.

Theorem 2.5, together with the argument of the proof of [4, Theorem 2.1], yields the following result on the stability of the bifurcating solution $\tilde{\boldsymbol{v}}(\delta)$ as a solution of the artificial compressible system (1.4)–(1.5).

Theorem 2.6. ([5]) *Assume that (A0)-(A5). Then there exist positive constants* $\epsilon_1 = \epsilon_1(\widetilde{b_0}, \boldsymbol{v}_0)$ *and* $\delta_1 = \delta_1(\widetilde{b_0}, \boldsymbol{v}_0)$ *such that the following assertions hold true for* $0 < |\delta| \leq \delta_1$.

(i) If $\tilde{\boldsymbol{v}}(\delta)$ *is unstable as a solution of* (1.1)–(1.2) *then so is* $\tilde{\boldsymbol{v}}(\delta)$ *as a solution of* (1.4)–(1.5) *for* $0 < \epsilon \leq \epsilon_1$.

(ii) Let $\tilde{\boldsymbol{v}}(\delta)$ *be stable as a solution of* (1.1)–(1.2). *Then there exist positive constants* $\epsilon_2 = \epsilon_2(\tilde{b}_0, \boldsymbol{v}_0)$ *and* κ *such that if*

$$\inf_{\boldsymbol{w} \in H_0^1(\Omega)^3, \boldsymbol{w} \neq \boldsymbol{0}} \frac{\operatorname{Re}(\mathbb{Q}\boldsymbol{w} \cdot \nabla \tilde{\boldsymbol{v}}(\delta), \mathbb{Q}\boldsymbol{w})}{\|\nabla \mathbb{Q}\boldsymbol{w}\|^2} \geq -\kappa, \tag{2.2}$$

then $\tilde{\boldsymbol{v}}(\delta)$ *is stable as a solution of* (1.4)–(1.5) *for* $0 < \epsilon \leq \epsilon_2$.

Similarly to the proof of Theorems 2.5 and 2.6, one can prove the stability and instability of the basic flow $\boldsymbol{v}_\eta$. In fact, it is possible to show that the spectrum of the linearized operator $\mathbb{L}_\eta$ satisfies

$$\sigma(-\mathbb{L}_\eta) = \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda \geq -\frac{3}{4}\tilde{b}_0\} \cup \{\lambda_\eta\},\ \eta \in [-\eta_0, \eta_0]$$

for some positive constant η_0. Here λ_η is a simple eigenvalue of $-\mathbb{L}_\eta$ and satisfies

$$\lambda_\eta = \alpha(0)\eta + \mathcal{O}(\eta^2).$$

Let $L_{\epsilon,\eta}$ be the linearized operator around $u_\eta = {}^\top(p_\eta, \boldsymbol{v}_\eta)$ of the artificial compressible system. Here p_η is the pressure corresponding to $\boldsymbol{v}_\eta$. We have the following result.

Theorem 2.7. ([5]) *There exist positive constants* $\tilde{\eta}_1 = \tilde{\eta}_1(\tilde{b}_0, \boldsymbol{v}_0)$ *and* $\epsilon_3 = \epsilon_3(\tilde{b}_0, \boldsymbol{v}_0)$ *such that*

$$\sigma(-L_{\epsilon,\eta}) \cap \{\lambda \in \mathbb{C};\ |\lambda| \leq \frac{\tilde{b}_0}{4}\} = \{\lambda_{\epsilon,\eta}\}$$

$$\lambda_{\epsilon,\eta} = \eta(c_1(\epsilon^2)\alpha(0) + \Lambda_{\epsilon,\eta})$$

with some $\Lambda_{\epsilon,\eta} = \mathcal{O}(\eta)$ *uniformly for* $0 < \epsilon \leq \epsilon_3$ *and* $0 < |\eta| \leq \tilde{\eta}_1$.

Theorems 2.5 and 2.7 imply that the same exchange of stability as in the case of (1.1)–(1.2) holds for the case of (1.4)–(1.5) uniformly for small ϵ. For definiteness, we consider the case where k is even and η_k is positive in Proposition 2.3 (ii). In this case one can prove the following result.

Theorem 2.8. ([5]) *Let k be even and η_k be positive in Proposition 2.3 (ii). Then there exist positive constants ϵ_4 and δ_2 such that*

(i) The basic flow $\boldsymbol{v}_{\eta(\delta)}$ is unstable for $0 < |\delta| \leq \delta_2$ and $0 < \epsilon \leq \epsilon_4$.

(ii) There exist positive constants ϵ_5, δ_3, $\tilde{\eta}_2$ and $\tilde{\kappa}$ such that if

$$\inf_{\boldsymbol{w}\in H_0^1(\Omega)^3, \boldsymbol{w}\neq \boldsymbol{0}} \frac{\mathrm{Re}\,(\mathbb{Q}\boldsymbol{w}\cdot \nabla \boldsymbol{v}_0, \mathbb{Q}\boldsymbol{w})}{\|\nabla \mathbb{Q}\boldsymbol{w}\|^2} \geq -\tilde{\kappa},$$

then $\boldsymbol{v}_\eta$ is stable for $-\tilde{\eta}_2 \leq \eta < 0$ and $0 < \epsilon \leq \epsilon_5$ and $\tilde{\boldsymbol{v}}(\delta)$ is stable for $0 < |\delta| \leq \delta_3$ and $0 < \epsilon \leq \epsilon_5$.

The other cases where k is odd or η_k is negative, we have similar results.

Remark 2.9. *Theorem 2.8 is applicable to the Taylor and Bénard problems, i.e., a bifurcation of the Taylor vortex from the Couette flow and a bifurcation of spatially periodic convective patterns from the motionless state, respectively.*

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