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ASYMMETRIC TRUNCATED TOEPLITZ OPERATORS AND ITS CHARACTERIZATIONS BY RANK TWO OPERATORS

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ABSTRACT. When investigating truncated Toeplitz operators, the question of considering two different model spaces naturally appears. The goal of this paper is to present asymmetric truncated Toeplitz operators with L^2 symbols between two different model spaces given by inner functions such that one divides the other. Asymmetric truncated Toeplitz operators can be characterized in terms of operators of rank at most two. Mainly, the results from [6] are presented.

1. INTRODUCTION

Toeplitz operators on the Hardy space H^2, which are compositions of a multiplication operator and the orthogonal projection from L^2 onto H^2, constitute a classical topic in operator theory. In the important paper ([20]) Sarason investigated truncated Toeplitz operators, thus generating huge interest in this class of operators; see, for example [1, 7, 9, 10, 11, 12, 13, 16]. Instead of the classical Hardy space H^2, they act on a model space $K_\theta^2 = H^2 \ominus \theta H^2$ associated with a given nonconstant inner function θ, and a multiplication operator is composed with the orthogonal projection from H^2 onto K_θ^2.

Asymmetric truncated Toeplitz operators involve the composition of a multiplication operator with two projections from H^2 onto a model space, associated with (possibly different) nonconstant inner functions α and θ. They are natural generalizations of rectangular Toeplitz matrices, which appear in various contexts, such as the study of finite-time convolution equations, signal processing, control theory, probability, approximation theory, diffraction problems (see for instance [2, 3, 4, 17, 18, 1, 21]).

Asymmetric truncated Toeplitz operators were introduced (in the context of the Hardy space H^p on the half-plane, with $1 < p < \infty$) and studied in the case of bounded symbols in [8]. The following review paper presents properties of an asymmetric truncated Toeplitz operator on the unit disc and is mainly based on the results from [6]. This work was inspired by the work of Sarason ([20]), where many interesting properties of truncated Toeplitz operators were given.

Here we consider bounded asymmetric truncated Toeplitz operators with L^2 symbols, defined between two model spaces K_θ^2 and K_α^2, where α divides θ

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$(\alpha \leqslant \theta)$. We study various properties of these operators and their relations with the corresponding symbols, and we present a necessary and sufficient condition for a bounded operator between two model spaces to be an asymmetric truncated Toeplitz operator in terms of rank two operators, thus generalizing a corresponding result of Sarason for the case where $\alpha = \theta$. In the asymmetric case, however, a more complex connection between the operators and their symbols is revealed, which is not apparent when the two model spaces involved are the same.

2. Model spaces and decompositions

Let L^2 denote the space $L^2(\mathbb{T}, m)$, where $\mathbb{T}$ is the unit circle and m is the normalized Lebesgue measure on $\mathbb{T}$, and let H^2 be the Hardy space on the unit disc $\mathbb{D}$, identified as usual with a subspace of L^2. Similarly, $L^\infty = L^\infty(\mathbb{T}, m)$ and we denote by H^∞ the space of all analytic and bounded functions on $\mathbb{D}$. Denoting by H_0^2 the subspace consisting of all functions in H^2 which vanish at 0, we have $L^2 \ominus H^2 = \overline{H_0^2}$, and we denote by P and P^- the orthogonal projections from L^2 onto H^2 and $\overline{H_0^2}$, respectively.

With any given inner function θ we associate the so called *model space* K_θ^2, defined by $K_\theta^2 = H^2 \ominus \theta H^2$. We also have $K_\theta^2 = H^2 \cap \theta\, \overline{H_0^2}$, and thus

$$f \in K_\theta^2 \quad \text{if and only if} \quad \bar{\theta} f \in \overline{H_0^2} \quad \text{and } f \in H^2.$$

In particular, if $f \in K_\theta^2$, then $\theta \bar{f} \in H_0^2$. Let P_θ be the orthogonal projection $P_\theta \colon L^2 \to K_\theta^2$.

Model spaces are also equipped with *conjugations* (antilinear isometric involutions), which are important tools in the study of model spaces and truncated Toeplitz operators (see for example [14, 15, 19]). For a given inner function θ, the conjugation C_θ is defined by $C_\theta \colon L^2 \to L^2$,

$$C_\theta f(z) = \theta \overline{z f(z)}.$$

It is worth noting that C_θ preserves the space K_θ^2 and maps θH^2 onto $L^2 \ominus H^2$.

Recall that for $\lambda \in \mathbb{D}$ the *kernel function* in H^2 denoted by k_λ is given by $k_\lambda(z) = \frac{1}{1-\bar{\lambda} z}$. Similarly, for an inner function θ, in K_θ^2 the kernel function k_λ^θ is given by $k_\lambda^\theta = P_\theta k_\lambda$, i.e., $k_\lambda^\theta = k_\lambda(1 - \overline{\theta(\lambda)}\,\theta)$. The set $\{k_\lambda^\theta : \lambda \in \mathbb{D}\}$ is linearly dense in K_θ^2. Since $k_\lambda^\theta \in K_\theta^\infty$, where K_θ^∞ denotes the subspace $H^\infty \cap K_\theta^2$, the space K_θ^∞ is dense in K_θ^2 (see [20]).

Defining $\tilde{k}_\lambda^\theta = C_\theta k_\lambda^\theta$, we have in particular

$$k_0^\theta(z) = 1 - \overline{\theta(0)}\theta(z)\,, \quad \tilde{k}_0^\theta(z) = \bar{z}(\theta(z) - \theta(0)).$$

It is easy to see that, for all $f \in K_\theta^2$,

$$\langle f, k_0^\theta \rangle = f(0)\,, \quad \langle f, \tilde{k}_0^\theta \rangle = \overline{(C_\theta f)(0)}.$$

Now let us consider two nonconstant inner functions α and θ. If $\bar{\alpha}\theta$ is an inner function, we say that α *divides* θ and we write $\alpha \leqslant \theta$.

Proposition 2.1. *Let α, θ be nonconstant inner functions such that $\alpha \leqslant \theta$. The following holds:*

(1) $K_\theta^2 = K_\alpha^2 \oplus \alpha K_{\frac{\theta}{\alpha}}^2$,
(2) $P_\theta = P_\alpha + \alpha P_{\frac{\theta}{\alpha}} \bar{\alpha}$,
(3) $k_0^\theta = k_0^\alpha + \overline{\alpha(0)} \alpha k_0^{\frac{\theta}{\alpha}}$,
(4) $\tilde{k}_0^\theta = \frac{\theta}{\alpha}(0) \tilde{k}_0^\alpha + \alpha \tilde{k}_0^{\frac{\theta}{\alpha}}$,
(5) $P_\alpha k_0^\theta = k_0^\alpha$, $P_\alpha \tilde{k}_0^\theta = \frac{\theta}{\alpha}(0) \tilde{k}_0^\alpha$.

The following proposition describes some relations between decompositions and conjugations. Note that, if $\alpha \leqslant \theta$, any $f \in K_\theta^2$ can be uniquely decomposed as $f = f_1 + \alpha f_2$ for some $f_1 \in K_\alpha^2$ and some $f_2 \in K_{\frac{\theta}{\alpha}}^2$, or as $f = f_2 + \frac{\theta}{\alpha} f_1$, for some $f_1 \in K_\alpha^2$ and some $f_2 \in K_{\frac{\theta}{\alpha}}^2$. Then the conjugation C_θ can be seen as $C_\theta \colon K_\theta^2 = K_\alpha^2 \oplus \alpha K_{\frac{\theta}{\alpha}}^2 \to K_\theta^2 = K_{\frac{\theta}{\alpha}}^2 \oplus \frac{\theta}{\alpha} K_\alpha^2$, or as $C_\theta \colon K_\theta^2 = K_{\frac{\theta}{\alpha}}^2 \oplus \frac{\theta}{\alpha} K_\alpha^2 \to K_\theta^2 = K_\alpha^2 \oplus \alpha K_{\frac{\theta}{\alpha}}^2$. Now we have:

Proposition 2.2. *Let α, θ be nonconstant inner functions such that $\alpha \leqslant \theta$. Then, if $f_1 \in K_\alpha^2$ and $f_2 \in K_{\frac{\theta}{\alpha}}^2$,*
(1) $C_\theta(f_1 + \alpha f_2) = C_{\frac{\theta}{\alpha}} f_2 + \frac{\theta}{\alpha} C_\alpha f_1$,
(2) $C_\theta(f_2 + \frac{\theta}{\alpha} f_1) = C_\alpha f_1 + \alpha C_{\frac{\theta}{\alpha}} f_2$.

Let S be the unilateral shift on the Hardy space H^2 and, for a nonconstant inner function θ, let $S_\theta = P_\theta S_{|K_\theta^2}$ be the compression of S to K_θ^2. The space K_θ^2 is invariant for S^*, thus $(S_\theta)^* = S^*{}_{|K_\theta^2}$. Note that, for any $f \in K_\theta^2$,

$$S_\theta f = zf - \overline{(C_\theta f)(0)}\, \theta = Sf - \langle f, \tilde{k}_0^\theta \rangle \theta, \tag{2.1}$$

$$S_\theta^* f = \bar{z}(f - f(0)). \tag{2.2}$$

In particular,

$$S_\theta^* k_0^\theta = -\overline{\theta(0)}\, \tilde{k}_0^\theta, \quad S_\theta \tilde{k}_0^\theta = -\theta(0) k_0^\theta. \tag{2.3}$$

The function k_0^θ is a cyclic vector for S_θ and $\tilde{k}_0^\theta$ is a cyclic vector for S_θ^* (see [20, Lemma 2.3]). We can define the *defect operators* $I_{K_\theta^2} - S_\theta S_\theta^* = k_0^\theta \otimes k_0^\theta$ and $I_{K_\theta^2} - S_\theta^* S_\theta = \tilde{k}_0^\theta \otimes \tilde{k}_0^\theta$, using the notation $(x \otimes y)z = \langle z, y \rangle x$ for any x, y, z in a Hilbert space H ([20, Lemma 2.4]).

3. Asymmetric truncated Toeplitz operators

Let α, θ be nonconstant inner functions. For $\varphi \in L^2$ we define an operator $A_\varphi^{\theta,\alpha} \colon \mathcal{D} \subset K_\theta^2 \to K_\alpha^2$, as $A_\varphi^{\theta,\alpha} f = P_\alpha(\varphi f)$ having domain $\mathcal{D} = \mathcal{D}(A_\varphi^{\theta,\alpha}) = \{f \in K_\theta^2 \colon \varphi f \in L^2\}$. The operator $A_\varphi^{\theta,\alpha}$ is closed and densely defined in K_θ^2. Note that $K_\theta^\infty \subset \mathcal{D}(A_\varphi^{\theta,\alpha})$. The operator $A_\varphi^{\theta,\alpha}$ will be called *an asymmetric truncated Toeplitz operator.* If this operator is bounded, then it admits a unique bounded extension to K_θ^2, $A_\varphi^{\theta,\alpha} \colon K_\theta^2 \to K_\alpha^2$. By $\mathcal{T}(\theta, \alpha)$ we denote the space of all bounded asymmetric truncated Toeplitz operators. For $\alpha = \theta$ we will write A_φ^θ instead of $A_\varphi^{\theta,\theta}$ (such operators are called *truncated Toeplitz operators* and were studied by Sarason in [20]) and $\mathcal{T}(\theta)$ instead of $\mathcal{T}(\theta, \theta)$.

It is easy to see that the following holds.

Proposition 3.1. *Let α, θ be any inner functions and $\varphi \in L^2$. Then*

$$\langle A_\varphi^{\theta,\alpha} f, g\rangle = \langle f, A_{\bar\varphi}^{\alpha,\theta} g\rangle \quad \textit{for all} \quad f \in \mathcal{D}(A_\varphi^{\theta,\alpha}),\ g \in \mathcal{D}(A_{\bar\varphi}^{\alpha,\theta}).$$

Moreover, $\mathcal{D}(A_{\bar\varphi}^{\alpha,\theta}) = \mathcal{D}\big((A_\varphi^{\theta,\alpha})^*\big)$ *and* $\big(A_\varphi^{\theta,\alpha}\big)^* = A_{\bar\varphi}^{\alpha,\theta}$.

The following shows the first difference between asymmetric truncated and truncated Toeplitz operators

Proposition 3.2. *Let α, θ be nonconstant inner functions such that $\alpha \leqslant \theta$. Let $A_\psi^{\theta,\alpha}$ be an asymmetric truncated Toeplitz operator with $\psi \in H^2$. Then*

$$S_\alpha A_\psi^{\theta,\alpha} f = A_\psi^{\theta,\alpha} S_\theta f \quad \textit{for all} \quad f \in K_\theta^\infty.$$

Remark 3.3. Theorem 3.1.16 [5] implies that for nonconstant inner functions α, θ such that $\alpha \leqslant \theta$, if a bounded operator $A\colon K_\theta^2 \to K_\alpha^2$ intertwines S_α, S_θ, i.e., $S_\alpha A = A S_\theta$, then $A = A_\psi^{\theta,\alpha}$ for some $\psi \in H^\infty$.

Example 3.4. One can ask, whether a similar result as in Proposition 3.2 can be obtained for $A_\psi^{\alpha,\theta}$ with $\alpha \leqslant \theta$ and $\psi \in H^2$, but the answer is negative. For example, let $\alpha = z^2$, $\theta = z^n$, $n > 5$, $\psi = z^3$ and $f = z$. Then $S_\theta A_\psi^{\alpha,\theta} f = z^5$ but $A_\psi^{\alpha,\theta} S_\alpha f = 0$.

The next theorem shows a necessary and sufficient condition for a bounded asymmetric truncated Toeplitz operator to be the zero operator in terms of its symbol.

Theorem 3.5. *Let α, θ be nonconstant inner functions such that $\alpha \leqslant \theta$. Let $A_\varphi^{\theta,\alpha}\colon K_\theta^2 \to K_\alpha^2$ be a bounded asymmetric truncated Toeplitz operator with $\varphi \in L^2$. Then $A_\varphi^{\theta,\alpha} = 0$ if and only if $\varphi \in \alpha H^2 + \overline{\theta H^2}$.*

Corollary 3.6. *Let $\alpha \leqslant \theta$ be nonconstant inner functions and let $A_\varphi^{\theta,\alpha} \in \mathcal{T}(\theta, \alpha)$. For $\varphi \in L^2$ there are functions $\psi \in K_\alpha^2$ and $\chi \in K_\theta^2$ such that $A_\varphi^{\theta,\alpha} = A_{\psi+\bar\chi}^{\theta,\alpha}$. Moreover, $A_{\psi+\bar\chi}^{\theta,\alpha} = A_{\psi_1+\bar\chi_1}^{\theta,\alpha}$ iff $\psi_1 = \psi + ck_0^\alpha$, $\chi_1 = \chi - \bar c k_0^\theta$ for some constant c.*

The following properties can be immediately obtained from the previous results by taking adjoint.

Corollary 3.7. *Let $A_\varphi^{\alpha,\theta}\colon K_\alpha^2 \to K_\theta^2$, $A_\varphi^{\alpha,\theta} \in \mathcal{T}(\alpha,\theta)$, $\alpha \leqslant \theta$, $\varphi \in L^2$. Then $A_\varphi^{\alpha,\theta} = 0$ iff $\varphi \in \theta H^2 + \overline{\alpha H^2}$.*

Corollary 3.8. *Let $A_\varphi^{\alpha,\theta}\colon K_\alpha^2 \to K_\theta^2$, $A_\varphi^{\alpha,\theta} \in \mathcal{T}(\alpha,\theta)$, $\alpha \leqslant \theta$, $\varphi \in L^2$. Then there are functions $\psi \in K_\alpha^2$, $\chi \in K_\theta^2$ such that $A_\varphi^{\alpha,\theta} = A_{\bar\psi+\chi}^{\alpha,\theta}$.*

4. Characterizations in terms of rank–two operators

In [20, Theorem 4.1] a characterization of truncated Toeplitz operators in $\mathcal{T}(\theta)$ was presented using rank two operators defined in terms of the kernel function k_0^θ. Following [6], an analogous result for asymmetric truncated Toeplitz operators $\mathcal{T}(\theta, \alpha)$ using the kernel functions k_0^α and k_0^θ can be presented.

Theorem 4.1 (Theorem 5.1 [6]). *Let α, θ be nonconstant inner functions such that $\alpha \leqslant \theta$ and let $A : K_\theta^2 \to K_\alpha^2$ be a bounded operator. Then $A \in \mathcal{T}(\theta, \alpha)$ if and only if there are $\psi \in K_\alpha^2$, $\chi \in K_\theta^2$ such that*

$$A - S_\alpha A S_\theta^* = \psi \otimes k_0^\theta + k_0^\alpha \otimes \chi. \tag{4.1}$$

It can be obtained a similar characterization for operators from $\mathcal{T}(\alpha, \theta)$ by taking adjoints in (4.1).

Corollary 4.2 (Corollary 5.2 [6]). *Let α, θ be nonconstant inner functions such that $\alpha \leqslant \theta$ and let $A\colon K_\alpha^2 \to K_\theta^2$ be a bounded operator. Then $A \in \mathcal{T}(\alpha, \theta)$ if and only if there are $\psi \in K_\alpha^2$, $\chi \in K_\theta^2$ such that*

$$A - S_\theta A S_\alpha^* = k_0^\theta \otimes \psi + \chi \otimes k_0^\alpha.$$

Sarason obtained also a characterization for truncated Toeplitz operators belonging to $\mathcal{T}(\theta)$ using the function $\tilde{k}_0^\theta = C_\theta k_0^\theta$ instead of k_0^θ, by a simple application of the conjugation C_θ to the result of Theorem 4.1 in the case $\alpha = \theta$. Here, following [6] we will present that an analogous result holds for operators belonging to $\mathcal{T}(\theta, \alpha)$, $\alpha \leqslant \theta$. However, in the case of asymmetric truncated Toeplitz operators the situation is more complex. The relation between a symbol of an asymmetric truncated Toeplitz operator and a rank two operator appearing in (4.2) is more involved.

Theorem 4.3 (Theorem 6.1 [6]). *Let α, θ be nonconstant inner functions such that $\alpha \leqslant \theta$, and let $A\colon K_\theta^2 \to K_\alpha^2$ be a bounded operator. Then $A \in \mathcal{T}(\theta, \alpha)$ if and only if there are $\mu \in K_\alpha^2$ and $\nu \in K_\theta^2$ such that*

$$A - S_\alpha^* A S_\theta = \mu \otimes \tilde{k}_0^\theta + \tilde{k}_0^\alpha \otimes \nu. \tag{4.2}$$

Moreover, if $A = A_{\psi+\bar{\chi}}^{\theta,\alpha}$ with $\psi \in K_\alpha^2$ and $\chi \in K_\theta^2$, then A satisfies (4.2) with

$$\mu = C_\alpha P_\alpha(\tfrac{\bar{\theta}}{\bar{\alpha}}\chi), \quad \nu = C_\alpha \psi + S^*(\alpha P_{\frac{\theta}{\alpha}} \chi). \tag{4.3}$$

By taking adjoints in (4.2) we obtain a similar characterization for operators from $\mathcal{T}(\alpha, \theta)$:

Corollary 4.4 (Corollary 6.2 [6]). *Let α, θ be nonconstant inner functions such that $\alpha \leqslant \theta$, and let $A\colon K_\alpha^2 \to K_\theta^2$ be a bounded operator. Then $A \in \mathcal{T}(\alpha, \theta)$ if and only if there are $\mu \in K_\alpha^2$, $\nu \in K_\theta^2$ such that*

$$A - S_\theta^* A S_\alpha = \tilde{k}_0^\theta \otimes \mu + \nu \otimes \tilde{k}_0^\alpha.$$

It is clear that if an asymmetric truncated Toeplitz operator A satisfies equation (4.2) with some μ, ν, then that equation is also satisfied if μ, ν are replaced by

$$\mu' = \mu + \bar{b}\,\tilde{k}_0^\alpha\,,\ \nu' = \nu - b\,\tilde{k}_0^\theta, \tag{4.4}$$

respectively, for any $b \in \mathbb{C}$. On the other hand, it is also true that the symbol of $A = A_{\psi+\bar{\chi}}^{\theta,\alpha} \in \mathcal{T}(\theta, \alpha)$ is not unique, and by Corollary 3.6 we can replace $\psi \in K_\alpha^2$ and $\chi \in K_\theta^2$ by

$$\psi' = \psi + c\,k_0^\alpha \in K_\alpha^2\,,\ \chi' = \chi - \bar{c}\,k_0^\theta \in K_\theta^2, \tag{4.5}$$

respectively, for any $c \in \mathbb{C}$. Using (4.3), it is easy to see that the following relation between the freedom of choice of μ, ν on the one hand, and ψ, χ on the other, holds.

Corollary 4.5 (Corollary 6.3 [6]). *Let $\mu \in K_\alpha^2$ and $\nu \in K_\theta^2$ be defined by (4.3) for given $\psi \in K_\alpha^2$ and $\chi \in K_\theta^2$, and let $\mu' \in K_\alpha^2$ and $\nu' \in K_\theta^2$ be defined analogously for $\psi' \in K_\alpha^2$ and $\chi' \in K_\theta^2$. If (4.5) holds, then*

$$\mu' = \mu - c\, \tfrac{\theta}{\alpha}(0)\tilde{k}_0^\alpha\ , \ \nu' = \nu + \bar{c}\, \overline{\tfrac{\theta}{\alpha}(0)}\tilde{k}_0^\theta.$$

The examples below illustrate the result of Theorem 4.3 in the case of Toeplitz matrices.

Example 4.6 (Example 6.4 [6]). Let us consider $\alpha = z^2$, $\theta = z^5$ and a Toeplitz operator $A = A_{\psi+\bar{\chi}}^{z^5}$. Assume that $\psi = a_0 + a_1 z$ and $\chi = \bar{b}_0 + \bar{b}_{-1}z + \bar{b}_{-2}z^2 + \bar{b}_{-3}z^3 + \bar{b}_{-4}z^4 = (\bar{b}_0 + \bar{b}_{-1}z + \bar{b}_{-2}z^2) + z^3(\bar{b}_{-3} + \bar{b}_{-4}z)$. Then $C_{z^2}\psi = \bar{a}_1 + \bar{a}_0 z$, $C_{z^2}P_{z^2}\bar{z}^3\chi = b_{-4} + b_{-3}z$ and $S^*(z^2(\bar{b}_0 + \bar{b}_{-1}z + \bar{b}_{-2}z^2)) = \bar{b}_0 z + \bar{b}_{-1}z^2 + \bar{b}_{-2}z^3$. Note that $A - S_{z^2}^* A S_{z^5}$ has a matrix representation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & b_{-4} \\ a_1 & a_0 + b_0 & b_{-1} & b_{-2} & b_{-3} \end{pmatrix},$$

which can be expressed as

$$(b_{-4} + b_{-3}z) \otimes z^4 + z \otimes (\bar{a}_1 + (\bar{a}_0 + \bar{b}_0)z + \bar{b}_{-1}z^2 + \bar{b}_{-2}z^3).$$

On the other hand, let $A - S_{z^2}^* A S_{z^5}$ have a matrix representation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & b_0 \\ a_0 & a_1 & a_2 & a_3 & a_4 + b_1 \end{pmatrix},$$

which can be expressed as

$$\mu \otimes z^4 + z \otimes \nu = (b_0 + b_1 z) \otimes z^4 + z \otimes (\bar{a}_0 + \bar{a}_1 z + \bar{a}_2 z^2 + \bar{a}_3 z^3 + \bar{a}_4 z^4).$$

Note that $\nu = \nu_{z^2} + z^2\nu_{z^3} = (\bar{a}_0 + \bar{a}_1 z) + z^2(\bar{a}_2 + \bar{a}_3 z + \bar{a}_4 z^2)$. Then $\psi = C_{z^2}P_{z^2}\nu = a_1 + a_0 z$ and $\chi = \bar{a}_2 z + \bar{a}_3 z^2 + (\bar{b}_1 + \bar{a}_4)z^3 + \bar{b}_0 z^4$. Hence by Theorem 4.1 we have

$$A - S_{z^2} A S_{z^5}^* = (a_1 + a_0 z) \otimes 1 + 1 \otimes (\bar{a}_2 z + \bar{a}_3 z^2 + (\bar{b}_1 + \bar{a}_4)z^3 + \bar{b}_0 z^4) \tag{4.6}$$

Requiring that ν_{z^3} is orthogonal to z^2 (see Theorem 6.3) determines that $a_4 = 0$.

On the other hand, we have some freedom in defining ψ and χ; namely $\psi_1 = s + a_0 z$ and $\chi_1 = \bar{t} + \bar{a}_2 z + \bar{a}_3 z^2 + (\bar{b}_1 + \bar{a}_4)z^3 + \bar{b}_0 z^4$ also satisfy (4.6) if we assume that $t + s = a_1$.

Example 4.7 (Example 6.5 [6]). Let us now take $\alpha = z^3$, $\theta = z^3((\lambda - z)/(1 - \bar{\lambda}z))^2$, $\lambda \in \mathbb{D}$ and consider the operator $A = A_{\psi+\bar{\chi}}^{\theta,\alpha}$, where $\psi = a_0 + a_1 z + a_2 z^2 \in K_\alpha^2$ and $\chi = (\bar{b}_0 + \bar{b}_1 z + \bar{b}_2 z^2 + \bar{b}_3 z^3 + \bar{b}_4 z^4)(1 - \bar{\lambda}z)^{-2} \in K_\theta^2$ (see [13, Corollary 5.7.3]). Then by Theorem 4.3

$$A - S_\alpha^* A S_\theta = \mu \otimes (\lambda^2 z^2 - 2\lambda z^3 + z^4)(1 - \bar{\lambda}z)^{-2} + z^2 \otimes \nu,$$

where $\mu = b_4 + (b_3 + 2\bar{\lambda}b_4)z + (b_2 + 3\bar{\lambda}^2 b_4 + \bar{\lambda}b_3)z^2$ and $\nu = \big(\bar{a}_2 + (\bar{a}_1 - 2\bar{\lambda}\bar{a}_2)z + (\bar{b}_0 + \bar{a}_0 - 2\bar{\lambda}\bar{a}_1 + \bar{\lambda}\bar{a}_2)z^2 + (\bar{b}_1 + \bar{\lambda}^2\bar{a}_1 - 2\bar{\lambda}\bar{a}_0)z^3 + \bar{\lambda}^2\bar{a}_0 z^4\big)(1 - \bar{\lambda}z)^{-2}$.

5. Characterizations in terms of rank–one operators

Our aim now is to describe the classes of symbols of an operator $A \in \mathcal{T}(\theta,\alpha)$ for which the right hand side of (4.2) is a rank one operator. The corresponding question regarding the equation (4.1) is trivial by Corollary 3.6, since the right side of (4.1) is a rank one operator if and only if $\psi = c \cdot k_0^\alpha$ or $\chi = c \cdot k_0^\theta$ with $c \in \mathbb{C}$. In the case $\alpha = \theta$ the question regarding the equality (4.2) also has an easy answer, since the relation between the symbols in (4.1) and (4.2) is $\psi = C_\theta \nu$ and $\chi = C_\theta \mu$.

Theorem 5.1 (Theorem 7.2 [6]). *Let $\alpha \leqslant \theta$ be nonconstant inner functions and let $A^{\theta,\alpha}_{\psi+\bar\chi} \in \mathcal{T}(\theta,\alpha)$, where $\psi \in K^2_\alpha$ and $\chi \in K^2_\theta$. Then*

(1) $A^{\theta,\alpha}_{\psi+\bar\chi} - S^*_\alpha A^{\theta,\alpha}_{\psi+\bar\chi} S_\theta = \mu \otimes \tilde{k}^\theta_0$ *for $\mu \in K^2_\alpha$ if and only if there is $s \in \mathbb{C}$ such that $\psi = s k^\alpha_0$, $P_{\frac{\theta}{\alpha}} \chi = -\bar{s} k_0^{\frac{\theta}{\alpha}}$,*

(2) $A^{\theta,\alpha}_{\psi+\bar\chi} - S^*_\alpha A^{\theta,\alpha}_{\psi+\bar\chi} S_\theta = \tilde{k}^\alpha_0 \otimes \nu$ *for $\nu \in K^2_\theta$ if and only if $P_\alpha(\chi \frac{\bar\theta}{\bar\alpha}) = const \cdot k^\alpha_0$.*

Remark 5.2 (Remark 7.3 [6]). When the right hand side of the characterization (4.1) reduces to a rank one operator $const \cdot k^\alpha_0 \otimes k^\theta_0$ it is immediate that this operator can be expressed in terms of the symbol $\psi + \bar\chi$ as

$$const \cdot k^\alpha_0 \otimes k^\theta_0 = P_{\mathbb{C}k^\alpha_0}\psi \otimes k^\theta_0 + k^\alpha_0 \otimes P_{\mathbb{C}k^\theta_0}\chi = \left(\psi(0)\|\tilde{k}^\alpha_0\|^{-2} + \overline{\chi(0)}\|\tilde{k}^\theta_0\|^{-2}\right) k^\alpha_0 \otimes k^\theta_0.$$

It might be of independent interest to consider the case when the right hand side in the equation (4.2) reduces to a rank one operator $const \cdot \tilde{k}^\alpha_0 \otimes \tilde{k}^\theta_0$. In fact this operator can be expressed in terms of the symbol $\psi + \bar\chi$ as

$$\begin{aligned} const \cdot \tilde{k}^\alpha_0 \otimes \tilde{k}^\theta_0 &= P_{\mathbb{C}\tilde{k}^\alpha_0}\mu \otimes \tilde{k}^\theta_0 + \tilde{k}^\alpha_0 \otimes P_{\mathbb{C}\tilde{k}^\theta_0}\nu = \\ &\left(\overline{\chi_\alpha(0)}\|\tilde{k}^\alpha_0\|^{-2} + \tfrac{\theta}{\alpha}(0)(\psi(0) - \overline{\chi_{\frac{\theta}{\alpha}}(0)}\|\tilde{k}^\theta_0\|^{-2})\right)\ \tilde{k}^\alpha_0 \otimes \tilde{k}^\theta_0. \end{aligned}$$

A similar question can be asked regarding the case when the right hand side of the equation (4.2) reduces to a rank one operator $const \cdot \tilde{k}^\alpha_0 \otimes \tilde{k}^\alpha_0$. We have

$$\begin{aligned} const \cdot \tilde{k}^\alpha_0 \otimes \tilde{k}^\alpha_0 &= P_{\mathbb{C}\tilde{k}^\alpha_0}\mu \otimes \tilde{k}^\alpha_0 + \tilde{k}^\alpha_0 \otimes P_{\mathbb{C}\tilde{k}^\alpha_0}\nu = \\ &\left(\overline{\chi(0)} - \overline{\chi_{\frac{\theta}{\alpha}}(0)}|\alpha(0)|^2 + \psi(0)\right)\ \|k^\alpha_0\|^{-2}\ \tilde{k}^\alpha_0 \otimes \tilde{k}^\alpha_0. \end{aligned}$$

6. From the operator to the symbol

In the case of a classical Toeplitz operator T_φ on H^2, the (unique) symbol φ can be obtained from the operator by the formula $\lim_{n\to\infty} \bar{z}^n T_\varphi z^n$. In the case of a truncated Toeplitz operator, i.e., of the form $A^{\alpha,\theta}_\varphi$ with $\alpha = \theta$, one can obtain a symbol belonging to $H^2 + \overline{H^2}$ from the action of A^θ_φ on k^θ_0 and $\tilde{k}^\theta_0$ ([4]). A similar result can be obtained for an asymmetric truncated Toeplitz operator $A \in \mathcal{T}(\theta,\alpha)$ by considering the action of the operator A and its adjoint on reproducing kernel functions of the same kind, see [6]. Here we concentrate on the question wether the characterizations of asymmetric truncated Toeplitz operators in terms of operators of rank two at most, presented in the previous sections, allow us also to obtain a symbol for the operator.

Regarding the first characterization, it follows from Theorem 4.1 that, if A is a bounded operator and satisfies the equality (4.1), then $A = A^{\theta,\alpha}_{\psi+\bar{\chi}}$. Remark that by Corollary 3.6 we know that ψ and χ are not unique and we can adjust the value of either ψ or χ at the origin.

For $\alpha = \theta$ the characterization (4.2) of truncated Toeplitz operators in Theorem 4.3 reduces to Sarason's ([20, Remark, p. 501]). In that case the relation between ψ, χ in the symbol of $A^{\theta}_{\psi+\bar{\chi}}$ and μ, ν is given by the conjugation C_θ, namely $\mu = C_\theta\chi$ and $\nu = C_\theta\psi$. Thus one can also immediately associate a symbol of the form $\psi + \bar{\chi}$ to a truncated Toeplitz operator satisfying that equality. In the asymmetric case, however, Theorem 4.3 unveils a more complex connection between the rank-two operator on the right-hand side of (4.2) and the symbols of $A^{\theta,\alpha}_{\psi+\bar{\chi}}$, and finding a symbol in terms of μ and ν for an operator A satisfying equality (4.2) is more difficult.

To solve that problem in the case of asymmetric truncated Toeplitz operators we start with two auxiliary results.

Lemma 6.1 (Lemma 8.3 [6]). *Let $\psi \in K^2_\alpha$, $\chi \in K^2_\theta$. Assume that $\chi = \chi_{\frac{\theta}{\alpha}} + \frac{\theta}{\alpha}\chi_\alpha$ according to the decomposition $K^2_\theta = K^2_{\frac{\theta}{\alpha}} \oplus \frac{\theta}{\alpha}K^2_\alpha$. If*

$$\mu = C_\alpha P_\alpha(\tfrac{\bar{\theta}}{\bar{\alpha}}\,\chi) + \bar{b}\tilde{k}^\alpha_0, \quad \nu = C_\alpha\psi + S^*(\alpha P_{\frac{\theta}{\alpha}}\,\chi) - b\tilde{k}^\theta_0$$

for fixed $b \in \mathbb{C}$, then

$$\psi = C_\alpha\nu_\alpha - \left(\overline{\chi_{\frac{\theta}{\alpha}}(0)} - \bar{b}\,\overline{\tfrac{\theta}{\alpha}(0)}\right)k^\alpha_0,$$

$$\chi_\alpha = C_\alpha\mu - bk^\alpha_0, \quad \chi_{\frac{\theta}{\alpha}} = S_{\frac{\theta}{\alpha}}\nu_{\frac{\theta}{\alpha}} + \left(\chi_{\frac{\theta}{\alpha}}(0) - b\tfrac{\theta}{\alpha}(0)\right)k_0^{\frac{\theta}{\alpha}},$$

where $\nu = \nu_\alpha + \alpha\nu_{\frac{\theta}{\alpha}}$ according to the decomposition $K^2_\theta = K^2_\alpha \oplus \alpha K^2_{\frac{\theta}{\alpha}}$.

Lemma 6.2 (Lemma 8.4 [6]). *Let $A \in \mathcal{T}(\theta, \alpha)$ satisfy the equation*

$$A - S^*_\alpha AS_\theta = \mu \otimes \tilde{k}^\theta_0 + \tilde{k}^\alpha_0 \otimes \nu$$

for $\mu \in K^2_\alpha$, $\nu \in K^2_\theta$. Then μ and ν can be chosen such that $P_{\frac{\theta}{\alpha}}(\bar{\alpha}\nu)$ is orthogonal to $\tilde{k}_0^{\frac{\theta}{\alpha}}$. In this case, μ and ν are uniquely determined.

When investigating symbols of the asymmetric truncated Toeplitz operator, it is worth to have in mind Corollary 3.6 saying that it is enough to find one of them. Following [6] we have

Theorem 6.3 (Theorem 8.5 [6]). *Let α, θ be nonconstant inner functions such that $\alpha \leqslant \theta$ and let A be a bounded operator satisfying*

$$A - S^*_\alpha AS_\theta = \mu \otimes \tilde{k}^\theta_0 + \tilde{k}^\alpha_0 \otimes \nu \tag{6.1}$$

for $\mu \in K^2_\alpha$, $\nu \in K^2_\theta$. *Then* $A = A^{\theta,\alpha}_{\psi+\bar{\chi}}$, *where*

$$\begin{aligned}
\psi =& C_\alpha P_\alpha(\nu - c\tilde{k}^\theta_0) = C_\alpha P_\alpha \nu - \bar{c}\,\overline{\tfrac{\theta}{\alpha}(0)}k^\alpha_0 \in K^2_\alpha \quad \textit{and} \\
\chi =& S_{\frac{\theta}{\alpha}} P_{\frac{\theta}{\alpha}} \bar{\alpha}(\nu - c\,\tilde{k}^\theta_0) + \tfrac{\theta}{\alpha} C_\alpha(\mu + \bar{c}\tilde{k}^\alpha_0) \\
=& (S_{\frac{\theta}{\alpha}} P_{\frac{\theta}{\alpha}} \bar{\alpha}\nu + c\,\tfrac{\theta}{\alpha}(0)k^{\frac{\theta}{\alpha}}_0) + \tfrac{\theta}{\alpha}(C_\alpha \mu + c\,k^\alpha_0) \in K^2_\theta = K^2_{\frac{\theta}{\alpha}} \oplus \tfrac{\theta}{\alpha} K^2_\alpha \quad \textit{with} \\
c =& \langle P_{\frac{\theta}{\alpha}} \bar{\alpha}\nu, \tilde{k}^{\frac{\theta}{\alpha}}_0 \rangle \| \tilde{k}^{\frac{\theta}{\alpha}}_0 \|^{-2}.
\end{aligned}$$

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