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INVITATION TO UNBOUNDED WEIGHTED COMPOSITION OPERATORS IN L^2-SPACES

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1. Introduction

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, $w\colon X \to \mathbb{C}$ be an $\mathcal{A}$-measurable function and $\phi\colon X \to X$ be an $\mathcal{A}$-measurable transformation of X. The operator $C_{\phi,w}$ in $L^2(\mu)$ given by

$$\mathcal{D}(C_{\phi,w}) = \{f \in L^2(\mu)\colon w \cdot (f \circ \phi) \in L^2(\mu)\},$$
$$C_{\phi,w} f = w \cdot (f \circ \phi), \quad f \in \mathcal{D}(C_{\phi,w}),$$

is called $C_{\phi,w}$ a *weighted composition operator.*

The class of weighted composition operators contains important subclasses of operators, e.g., multiplication operators in L^2-spaces, composition operators in L^2-spaces and weighted shifts. As such, it can be found in many areas of mathematics and it has been studied quite extensively (see, e.g., monographs [20, 31, 29, 33, 30] and the literature therein). Until very recently, however, it has not been investigated as a whole in full generality (even in a bounded case). Studies over some of the subclasses have been initiated quite recently as well; we could mention here unbounded composition operators in L^2-spaces (see, e.g., [2, 3, 4, 8, 9, 10, 13, 14, 32]) and weighted shifts on directed trees (see, e.g., [2, 5, 6, 7, 11, 16, 17, 25, 26, 35]). Moreover, many authors have made essential restrictions when considering weighted composition operators in a bounded case (see a discussion of this in [12]).

In this note we survey some recent results concerning unbounded weighted composition operators in L^2-spaces. For extensive information on this topic we refer to a very recent monograph [12]

2. Preliminaries and basic properties

In all what follows $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ stand for the sets of positive integers, real numbers, and complex numbers, respectively, while $\mathbb{Z}_+$, $\mathbb{R}_+$, and $\overline{\mathbb{R}}_+$ denote the sets of nonnegative integers, nonnegative real numbers, and $\mathbb{R}_+ \cup \{\infty\}$, respectively. If X is a topological space, then $\mathfrak{B}(X)$ stands for the family of Borel subsets of X.

Let $\mathcal{H}$ be a (complex) Hilbert space and A be an (linear) operator in $\mathcal{H}$. By $\mathcal{D}(A)$, $\overline{A}$, and A^* we denote the domain, the closure, and the adjoint of A, respectively (if they exists). Given another operator B in $\mathcal{H}$, we write $A \subseteq B$ whenever $\mathcal{D}(A) \subseteq$

$\mathcal{D}(B)$ and $Af = Bf$ for all $f \in \mathcal{D}(A)$. Suppose A i densely defined. A is *normal* if A is closed and $A^*A = AA^*$. If A is closed and $A|A|^2 = |A|^2A$, then A is said to be *quasinormal* (see [28]). We say that A is *subnormal* if there exist a complex Hilbert space $\mathcal{K}$ and a normal operator B in $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding), $A \subseteq B$. A is called *hyponormal* if $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $\|A^*f\| \leqslant \|Af\|$ for all $f \in \mathcal{D}(A)$.

Throughout the paper we assume that $(X, \mathcal{A}, \mu)$ is a σ-finite measure space that $w\colon X \to \mathbb{C}$ and $\phi\colon X \to X$ are $\mathcal{A}$-measurable.

Recall that the weighted composition operator $C_{\phi,w}$ in $L^2(\mu)$ is given by

$$\mathcal{D}(C_{\phi,w}) = \{f \in L^2(\mu)\colon w \cdot (f \circ \phi) \in L^2(\mu)\},$$
$$C_{\phi,w}f = w \cdot (f \circ \phi), \quad f \in \mathcal{D}(C_{\phi,w}).$$

Let μ_w and $\mu_w \circ \phi^{-1}$ be measures on $\mathcal{A}$ defined by

$$\mu_w(\varDelta) = \int_\varDelta |w|^2 \mathrm{d}\mu, \quad \mu_w \circ \phi^{-1}(\varDelta) = \mu_w(\phi^{-1}(\varDelta)), \quad \varDelta \in \mathcal{A}$$

The following theorems addresses the very basic question of when $C_{\phi,w}$ is a well-defined operator.

Theorem 2.1 ([12, Proposition 7]). *The weighted composition operator $C_{\phi,w}$ is well-defined linear operator in $L^2(\mu)$ if and only if the measure $\mu_w \circ \phi^{-1}$ is absolutely continuous with respect to the measure μ.*

As already mentioned in the introduction the class of weighted composition operator contains important subclasses of operators. Below we single out some of the most significant ones.

Example 2.2. See also [12, Section 2.2.3].

(1) Let $\phi = \mathrm{id}_X$, the identity mapping on X. Then the operator $M_w := C_{\mathrm{id}_X,w}$ is well-defined; it is called the *operator of multiplication* by w in $L^2(\mu)$. For more on multiplication operators, the reader is referred to [1, 18, 34, 36].

(2) Let $w = \chi_X$, the characteristic function of the set X. Then, assuming that $\mu \circ \phi^{-1}$ is absolutely continuous with respect to μ, the operator $C_\phi := C_{\phi,\chi_X}$ is well-defined; it is called the *composition operator* in $L^2(\mu)$ with *symbol* ϕ. For more on composition operators we refer the reader to [20, 31, 29, 12].

(3) Let $\{\lambda_n\}_{n=0}^\infty$ be a sequence of complex numbers, $X = \mathbb{Z}_+$, $\mathscr{A} = 2^X$, and let μ be the counting measure on X. Define ϕ and w by

$$\phi(n) = \begin{cases} n-1 & \text{for } n \in \mathbb{N}, \\ 0 & \text{for } n = 0, \end{cases} \quad \text{and} \quad w(n) = \begin{cases} \lambda_{n-1} & \text{for } n \in \mathbb{N}, \\ 0 & \text{for } n = 0. \end{cases}$$

Then the weighted composition operator $C_{\phi,w}$ in $\ell^2(\mathbb{Z}_+)$ is well-defined. Observe that $\{\chi_{\{n\}}\}_{n=0}^\infty \subseteq \mathcal{D}(C_{\phi,w})$ and $C_{\phi,w}\chi_{\{n\}} = \lambda_n \chi_{\{n+1\}}$ for $n \in$

$\mathbb{Z}_+$. The operator $C_{\phi,w}$ is called the *unilateral weighted shift* with *weights* $\{\lambda_n\}_{n=0}^{\infty}$. For more on these operators see [21, 22, 33, 30].

(4) Let $\{\lambda_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{C}$, $X = \mathbb{Z}$, $\mathscr{A} = 2^X$, and let μ be the counting measure on X. Define ϕ and w by $\phi(n) = n - 1$ and $w(n) = \lambda_{n-1}$ for $n \in \mathbb{Z}$. Then the weighted composition operator $C_{\phi,w}$ in $\ell^2(\mathbb{Z})$ is well defined; we call it the *bilateral weighted shift* with *weights* $\{\lambda_n\}_{n\in\mathbb{Z}}$. For more information on bilateral weighted shifts see [24, 33, 19].

(5) Let $\mathscr{T} = (V, E)$ is a directed tree (V denotes the set of vertices and E denotes the set of edges). Given $u \in V$, the unique vertex $v \in V$ such that $(v, u) \in E$ is called a *parent* of u and denoted by $\mathsf{par}(u)$; a vertex with no parent is called a *root* of $\mathscr{T}$ and denoted (provided it exists) by root. Set $V^\circ = V \setminus \{\mathsf{root}\}$ if $\mathscr{T}$ has a root and $V^\circ = V$ otherwise. By a *weighted shift on* $\mathscr{T}$ with *weights* $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ} \subseteq \mathbb{C}$ we understand the operator $S_{\boldsymbol{\lambda}}$ in $\ell^2(V)$ given by

$$\mathcal{D}(S_{\boldsymbol{\lambda}}) = \{f \in \ell^2(V) \colon \varLambda_{\mathscr{T}} f \in \ell^2(V)\},$$
$$S_{\boldsymbol{\lambda}} f = \varLambda_{\mathscr{T}} f, \quad f \in \mathcal{D}(S_{\boldsymbol{\lambda}}),$$

where $\varLambda_{\mathscr{T}}$ is mapping on $\mathbb{C}^V$ given by

$$(\varLambda_{\mathscr{T}} f)(v) = \begin{cases} \lambda_v \cdot f\big(\mathsf{par}(v)\big) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \mathsf{root}, \end{cases} \qquad f \in \mathbb{C}^V.$$

If $\mathrm{card}(V) \leqslant \aleph_0$, then a weighted shift on a directed tree can be regarded as a weighted partial composition operator in $L^2(V, 2^V, \mu)$, where μ is the counting measure (see (d)). The reader is referred to [25] for the foundations of the theory of weighted shifts on directed trees.

The above example show how rich and complex the class of weighted composition operators is. This offers a good reason to study these operators and, in fact, their various subclasses have investigated intensively by many researchers. However, none of the studies have been carried in full generality. In particular, it is worth mentioning that in the past some unnecessary assumptions concerning have been made when studying weighted composition operators. The following elementary example regards that.

Example 2.3 ([12, Example 102]). In majority of studies over weighted composition operators it has been assumed that the corresponding (non-weighted) composition operators were well-defined. This is not always the case. Namely, consider $X = \mathbb{Z}_+$, $\mathscr{A} = 2^X$, and $\mu\colon 2^X \to \overline{\mathbb{R}}_+$ such that $\mu(\{n\}) = 1$ for every $n \geqslant 1$ and $\mu(\{0\}) = 0$. Define ϕ by

$$\phi(n) = \begin{cases} n - 1 & \text{if } n \geqslant 1, \\ 0 & \text{if } n = 0, \end{cases} \qquad n \in X,$$

and w by

$$w(n) = \begin{cases} 0 & \text{if } n \in \{0,1\}, \\ 1 & \text{if } n \geqslant 2, \end{cases} \quad n \in X.$$

Then $C_{\phi,w}$ is well-defined. In fact, it is unitarily equivalent to the unilateral shift of multiplicity 1. On the other hand, the corresponding composition operator C_ϕ is not even well-defined.

Assuming $\mu_w \circ \phi^{-1}$ is absolutely continuous with respect to μ, by the Radon-Nikodym theorem, there exists a unique (up to a set of μ-measure zero) $\mathscr{A}$-measurable $\mathsf{h}_{\phi,w}\colon X \to \overline{\mathbb{R}}_+$ such that

$$\mu_w \circ \phi^{-1}(\Delta) = \int_\Delta \mathsf{h}_{\phi,w} \mathrm{d}\mu, \quad \Delta \in \mathscr{A}.$$

In the compostion operator case, i.e., when $w = \chi_X$, we write simply h_ϕ for h_{ϕ,χ_X}.

Example 2.3 above shows that the operator $C_{\phi,w}$ and the product $M_w C_\phi$ in general do not coincide, a fact that seems to be overlooked by many investigators. The relation between these two is more subtle.

Theorem 2.4 ([12, Proposition 109 & Theorem 110]). *The the following assertions hold:*

(i) *if the composition operator C_ϕ is well-defined, then $C_{\phi,w}$ is well-defined and $M_w C_\phi \subseteq C_{\phi,w}$,*
(ii) *if C_ϕ is well-defined, then $C_{\phi,w} = M_w C_\phi$ if and only if there exists $c \in \mathbb{R}_+$ such that $\mathsf{h}_\phi \leqslant c(1 + \mathsf{h}_{\phi,w})$ a.e. $[\mu]$,*
(iii) *if C_ϕ is a well-defined bounded operator on $L^2(\mu)$, then $C_{\phi,w} = M_w C_\phi$,*
(iv) *if $w \neq 0$ a.e. $[\mu]$ and $C_{\phi,w}$ is well-defined, then C_ϕ is well-defined.*

As we see, the problem of equality between $C_{\phi,w}$ and $M_w C_\phi$ can be solved with help of Radon-Nikodym derivatives h_ϕ and $\mathsf{h}_{\phi,w}$. This is also the case for other problems.

Going back to the basic properties of weighted composition operators we may recall the following. Here, and later on, we will assume that whenever the weighted composition operator is mentioned it is well-defined.

Theorem 2.5 ([12, Proposition 8]). *Then the following assertions are valid:*

(i) $\mathcal{D}(C_{\phi,w}) = L^2((1 + \mathsf{h}_{\phi,w})\mathrm{d}\mu)$,
(ii) *$C_{\phi,w}$ is densely defined if and only if $\mathsf{h}_{\phi,w} < \infty$ a.e. $[\mu]$,*
(iii) *$C_{\phi,w}$ is closed,*
(iv) *$C_{\phi,w} \in \boldsymbol{B}(L^2(\mu))$ if and only if $\mathsf{h}_{\phi,w} \in L^\infty(\mu)$; if this is the case, then $\|C_{\phi,w}\|^2 = \|\mathsf{h}_{\phi,w}\|_{L^\infty(\mu)}$.*

3. More on weighted composition operators

For more advanced considerations one needs the notion of the conditional expectation. Assume that $C_{\phi,w}$ is densely defined. If $f\colon X \to \overline{\mathbb{R}}_+$ or $f\colon X \to \mathbb{C}$ belongs to $L^p(\mu_w)$, $p \in [1,\infty]$, then there exists a (unique) $\phi^{-1}(\mathscr{A})$-measurable function $\mathsf{E}_{\phi,w}(f)$ such that

$$\int_{\phi^{-1}(\Delta)} f \mathrm{d}\mu_w = \int_{\phi^{-1}(\Delta)} \mathsf{E}_{\phi,w}(f) \mathrm{d}\mu_w, \quad \Delta \in \mathscr{A}.$$

Using a well-known description of $\phi^{-1}(\mathscr{A})$-measurable functions one can also show that there exist a (unique up to sets of μ-measure zero) function $\mathsf{E}_{\phi,w}(f) \circ \phi^{-1}$ such that

$$(\mathsf{E}_{\phi,w}(f) \circ \phi^{-1}) \circ \phi = \mathsf{E}_{\phi,w}(f) \quad \text{a.e. } [\mu_w|_{\phi^{-1}(\mathscr{A})}].$$

and $\mathsf{E}_{\phi,w}(f) \circ \phi^{-1} = \chi_{\{\mathsf{h}_{\phi,w}>0\}} \cdot \mathsf{E}_{\phi,w}(f) \circ \phi^{-1}$ a.e. $[\mu]$. Recall that the conditional expectation $\mathsf{E}_{\phi,w}(\cdot)$ can be regarded as a linear contraction on $L^p(\mu_w)$ which leaves invariant the convex cone of $\overline{\mathbb{R}}_+$-valued functions.

The full description of the adjoint of a densely defined weighted composition operators is given below. A partial result on this was given in [15]

Theorem 3.1 ([12, Proposition 17]). *Suppose $C_{\phi,w}$ is densely defined. Then the adjoint of $C_{\phi,w}$ is given by:*

$$\mathcal{D}(C_{\phi,w}{}^*) = \{f \in L^2(\mu) \colon \mathsf{h}_{\phi,w} \cdot \mathsf{E}_{\phi,w}(f_w) \circ \phi^{-1} \in L^2(\mu)\},$$
$$C_{\phi,w}{}^* f = \mathsf{h}_{\phi,w} \cdot \mathsf{E}_{\phi,w}(f_w) \circ \phi^{-1}, \quad f \in \mathcal{D}(C_{\phi,w}{}^*),$$

where $f_w = \chi_{w\neq 0}\frac{f}{w}$.

In turn, the following provides the polar decomposition.

Theorem 3.2 ([12, Theorem 18]). *Suppose $C_{\phi,w}$ is densely defined. Let $C_{\phi,w} = U|C_{\phi,w}|$ be its polar decomposition. Then $|C_{\phi,w}| = M_{\mathsf{h}_{\phi,w}^{1/2}}$ and $U = C_{\phi,\widetilde{w}}$, where $\widetilde{w} = \frac{w}{(\mathsf{h}_{\phi,w}\circ\phi)^{1/2}}$ a.e. $[\mu]$.*

Knowing the descriptions of the adjoint and the polar decomposition allows characterizing many important properties of weighted composition operators. One cans ask about their normality, quasinormality, hyponormality. These can be characterized as follows.

Theorem 3.3 ([12, Theorem 20]). *If $C_{\phi,w}$ is densely defined, then $C_{\phi,w}$ is quasinormal if and only if $\mathsf{h}_{\phi,w} \circ \phi = \mathsf{h}_{\phi,w}$ a.e. $[\mu_w]$.*

Theorem 3.4 ([12, Theorem 53]). *If $C_{\phi,w}$ is densely defined, then $C_{\phi,w}$ is hyponormal if and only if $\mathsf{h}_{\phi,w} > 0$ a.e. $[\mu_w]$ and $\mathsf{E}_{\phi,w}\Big(\frac{\mathsf{h}_{\phi,w}\circ\phi}{\mathsf{h}_{\phi,w}}\Big) \leqslant 1$ a.e. $[\mu_w]$.*

Theorem 3.5 ([12, Theorem 63]). *If $C_{\phi,w}$ is densely defined, then $C_{\phi,w}$ is normal if and only if the following three conditions are satisfied:*

(ii-a) $\mathsf{h}_{\phi,w} = 0$ *on* $\{w = 0\}$ *a.e.* $[\mu]$,
(ii-b) $\mathsf{E}_{\phi,w}(L^2(\mu_w)) = L^2(\mu_w)$,
(ii-c) $\mathsf{h}_{\phi,w} = \mathsf{h}_{\phi,w} \circ \phi$ *a.e.* $[\mu_w]$.

One can ask also about subnormality of weighted composition operators. The known general sufficient conditions are essentially more involved than those describing the above mentioned properties.

Theorem 3.6 ([12, Theorem 29]). *Assume that $C_{\phi,w}$ is densely defined and $\mathsf{h}_{\phi,w} > 0$ a.e. $[\mu_w]$. Suppose there exists an $\mathscr{A}$-measurable family of probability measures $P\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ that satisfies*

$$\mathsf{E}_{\phi,w}(P(\cdot,\sigma))(x) = \frac{\int_\sigma tP(\phi(x),\mathrm{d}t)}{\mathsf{h}_{\phi,w}(\phi(x))} \quad \textit{for } \mu_w\textit{-a.e. } x \in X, \quad \sigma \in \mathfrak{B}(\mathbb{R}_+).$$

Then $C_{\phi,w}$ is subnormal.

Surprisingly, the above sufficient conditions are applicable and lead to quite interesting results. Let us mention here some pathological examples, in particular, the most striking of a subnormal weighted composition operator with trivial square.

Theorem 3.7 ([11, Theorem 3.1]). *There exist a subnormal weighted composition operator $C_{\phi,w}$ such that $\mathcal{D}(C^2_{\phi,w}) = \{0\}$.*

The above result proves that the well-known criteria for the subnormality of a bounded $C_{\phi,w}$ written in terms of Stieltjes moment sequences cannot be generalized to the case of unbounded weighted composition operators.

The studies of subnormality of weighted composition operators led also to another significant example.

Theorem 3.8 ([13, Theorem 5.5.2]). *There exist a non-hyponormal weighted composition operator $C_{\phi,w}$ such that $\bigcap_{n=1}^\infty \mathcal{D}(C^n_{\phi,w})$ is dense in $L^2(\mu)$ and for every $f \in \bigcap_{n=1}^\infty \mathcal{D}(C^n_{\phi,w})$, $\|C^n_{\phi,w} f\|^2 = \int_{\mathbb{R}_+} t^n \mathrm{d}\nu_f(t)$ with some positive Borel measure ν_f on $\mathbb{R}_+$.*

There are many more interesting results and examples concerning both bounded and unbounded weighted composition operators. What is important, there are also interesting unsolved problems waiting for keen researchers.

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References

[1] M. Sh. Birman, M. Z. Solomjak, Spectral theory of selfadjoint operators in Hilbert space, D. Reidel Publishing Co., Dordrecht, 1987.

[2] P. Budzyński, P. Dymek, Z. J. Jabłoński, J. Stochel, Subnormal weighted shifts on directed trees and composition operators in L^2-spaces with non-densely defined powers, Abstr. Appl. Anal. 2014 (2014), Article ID 791817, 6 pp.

[3] P. Budzyński, P. Dymek, A. Płaneta, Unbounded composition operators via inductive limits: cosubnormal operators with matrix symbols, Filomat 31 (2017), 1665-1670.

[4] P. Budzyński, P. Dymek, A. Płaneta, Unbounded composition operators via inductive limits: cosubnormal operators with matrix symbols. II, Banach J. Math. Anal. 11 (2017), 164-187.

[5] P. Budzyński, P. Dymek, M. Ptak, Analytic structure of weighted shifts on directed trees, Math. Nachr. 290 (2017), 1612-1629.

[6] P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel, Unbounded subnormal weighted shifts on directed trees, Math. Anal. Appl. 394 (2012), 819-834.

[7] P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel, Unbounded subnormal weighted shifts on directed trees. II, J. Math. Anal. Appl. 398 (2013), 600-608.

[8] P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel, On unbounded composition operators in L^2-spaces, Ann. Mat. Pura Appl. 193 (2014) 663–688.

[9] P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel, A multiplicative property characterizes quasinormal composition operators in L^2-spaces. J. Math. Anal. Appl. 409 (2014), 576–581.

[10] P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel, Unbounded subnormal composition operators in L^2-spaces, J. Funct. Anal. 269 (2015), 2110-2164.

[11] P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel, Subnormal weighted shifts on directed trees whose nth powers have trivial domain, J. Math. Anal. Appl. 435 (2016), 302-314.

[12] P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel, Unbounded weighted composition operators in L^2-spaces, Lectures Notes in Mathematics (to be published).

[13] P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel, Subnormality of composition operators over directed graphs over one circuit directed graphs: Exotic examples, Adv. Math. 310 (2017), 484-556.

[14] P. Budzyński, A. Płaneta, Dense definiteness and boundedness of composition operators in L^2-spaces via inductive limits, Oper. Matrices, 9 (2015), 853-876.

[15] J. T. Campbell, W. E. Hornor, Seminormal composition operators, J. Operator Theory 29 (1993) 323–343.

[16] S. Chavan, S. Trivedi, An analytic model for left-invertible weighted shifts on directed trees, J. London Math. Soc. **94** (2016), 253-279.

[17] S. Chavan, D. K. Pradhan, S. Trivedi, Multishifts on directed Cartesian products of rooted directed trees, arXiv: https://arxiv.org/abs/1607.03860.

[18] J. B. Conway, A course in functional analysis, Graduate Texts in Mathematics 96, Springer-Verlag, New York, 1990.

[19] R. E. Curto, L. A. Fialkow, Recursively generated weighted shifts and the subnormal completion problem, II, Integr. Equat. Oper. Th. 18 (1994), 369-426.

[20] N. Dunford, J. T. Schwartz, Linear operators, Part I, Interscience, New York-London 1958.

[21] R. Gellar, L. J. Wallen, Subnormal weighted shifts and the Halmos-Bram criterion, Proc. Japan Acad. 46 (1970), 375-378.

[22] P. R. Halmos, Ten problems in Hilbert space, Bull. Amer. Math. Soc. 76 (1970), 887-933.

[23] Z. Jabłoński, Hyperexpansive composition operators, Math. Proc. Camb. Phil. Soc. 135 (2003), 513-526.

[24] D. Herrero, Subnormal bilateral weighted shifts, Notas mimeografiadas, 1971.

[25] Z. J. Jabłoński, I. B. Jung, J. Stochel, Weighted shifts on directed trees, Mem. Amer. Math. Soc. 216 (2012), no. 1017, viii+107pp.

[26] Z. J. Jabłoński, I. B. Jung, J. Stochel, A non-hyponormal operator generating Stieltjes moment sequences, J. Funct. Anal. 262 (2012), 3946-3980.

[27] Z. J. Jabłoński, I. B. Jung, J. Stochel, Normal extensions escape from the class of weighted shifts on directed trees, Complex Anal. Operator Theory, 7 (2013), 409-419.

[28] Z. J. Jabłoński, I. B. Jung, J. Stochel, Quasinormal operators revisited, Integral Equations Operator Theory 79 (2014), 135-149.

[29] R. K. Singh, J. S. Manhas, Composition Operators on Function Spaces, North-Holland, 1993.

[30] N. K. Nikolskii, Treatise on the Shift Operator, Springer-Verlag, 1986.

[31] E. Nordgren, Composition operators on Hilbert spaces, Lecture Notes in Math. 693, Springer-Verlag, Berlin 1978, 37-63.

[32] P. Pietrzycki, The single equality $A^{*n}A^n = (A^*A)^n$ does not imply the quasinormality of weighted shifts on rootless directed trees, J. Math. Anal. Appl. **435** (2016), 338-348.

[33] A. L. Shields, Weighted shift operators and analytic function theory, Topics in operator theory, pp. 49-128. Math. Surveys, No. 13, Amer. Math. Soc., Providence, R.I., 1974.

[34] K. Schmüdgen, Unbounded self-adjoint operators on Hilbert space, Graduate Texts in Mathematics, 265, Springer, Dordrecht, 2012.

[35] J. Trepkowski, Aluthge transforms of weighted shifts on directed trees, J. Math. Anal. Appl. **425** (2015), 886-899.

[36] J. Weidmann, Linear operators in Hilbert spaces, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

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