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On the structure of Banach spaces with James constant $\sqrt{2}$

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1 introduction

This note is a survey on [8].

In the theory of Banach space geometry, several geometric constants of normed spaces indicate characteristics of normed spaces from various geometric viewpoints, and sometimes play very important roles. In this paper, we deal with two of the most important such geometric constants so-called *von Neumann-Jordan constant* and *James constant.* Let X be a Banach space, and let S_X be its unit sphere. The the von Neumann-Jordan constant $C_{NJ}(X)$ and James constant $J(X)$ of X are defined by

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : (x,y) \neq (0,0)\right\},$$
$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}.$$

The following are basic properties of these constants.

(i) $1 \leq C_{NJ}(X) \leq 2$ for any Banach space X.

(ii) $C_{NJ}(X) = 1$ if and only if X is a Hilbert space.

(iii) $C_{NJ}(X) < 2$ if and only if X is uniformly non-square, that is, there exists a $\delta > 0$ such that $\min\{\|x+y\|, \|x-y\|\} < 2(1-\delta)$ whenever $x, y \in S_X$ ([11]).

(iv) $\sqrt{2} \leq J(X) \leq 2$ for any Banach space X ([3]).

(v) If X is a Hilbert space, then $J(X) = \sqrt{2}$. Moreover, provided that $\dim X \geq 3$, then $J(X) = \sqrt{2}$ if and only if X is a Hilbert space ([5]).

(vi) There are many non-Hilbert two-dimensional normed space X with $J(X) = \sqrt{2}$ ([5, 6, 7]).

There are some relationship between $C_{NJ}(X)$ and $J(X)$. In particular, inequalities comparing these two constants are especially interesting. For example, Wang [13], Takahashi and Kato [12] and Yang and Li [14] independently showed the simple inequality $C_{NJ}(X) \leq J(X)$. Moreover, as a further improvement (which is what Wang actually showed), a bit complicated estimation

$$C_{NJ}(X) \leq 1 + \frac{2(J(X)-1)}{\sqrt{J(X)^2+(2-J(X))^2}+2-J(X)} (\leq J(X))$$

also holds.

The goal of the above topic is to find the function $F : [\sqrt{2}, 2] \to \mathbb{R}$ satisfying the following two conditions:

(I) for each normed space X, the inequality

$$C_{NJ}(X) \leq F(J(X))$$

holds; and

(II) if $G : [\sqrt{2}, 2] \to \mathbb{R}$ also satisfies

$$C_{NJ}(X) \leq G(J(X))$$

for each normed space X, then $F \leq G$.

In fact, the function F can be defined explicitly by

$$F(s) := \sup\{C_{NJ}(X) : X \text{ is a normed space with } J(X) = s\}. \tag{1}$$

for each $s \in [\sqrt{2}, 2]$; but, then, we only know $F(2) = 2$.

The purpose of this note is to provide the first step of the study on the function F defined by (1). We present a partial estimation of $F(\sqrt{2})$.

2 The Banach-Mazur compactum

Let X and Y be normed spaces isomorphic to each other, and let $GL(X, Y)$ be the collection of all isomorphisms from X onto Y. Then the Banach-Mazur distance between X and Y is defined by

$$\delta(X, Y) := \log(\inf\{\|T\| \, \|T^{-1}\| : T \in GL(X, Y)\}).$$

If $\dim X = \dim Y = n \in \mathbb{N}$, it can be shown that $\delta(X, Y) = 0$ if and only if X is isometrically isomorphic to Y. Indeed, if $\delta(X, Y) = 0$ then there exists a sequence (T_m) in $GL(X, Y)$ such that $\lim_m \|T_m\| \, \|T_m^{-1}\| = 1$. Putting $R_m = \|T_m^{-1}\| T_m$ and $S_m = \|T_m\| T_m^{-1}$ yields the bounded sequences (R_m) and (S_m). Since $\dim X = \dim Y = n < \infty$, we may assume that (R_m) and (S_m) converge in the operator norm topology to some R and S, respectively. It follows that $\|R\| = \|S\| = 1$, $SR = I_X$ and $RS = I_Y$, that is, R is an isometric isomorphism from X onto Y. The converse is obvious.

For an n-dimensional normed space X, let $[X]$ be the collection of all n-dimensional normed spaces isometrically isomorphic to X. Then the Banach-Mazur compactum $Q(n)$ is defined as the set of all isometry class $[X]$ of n-dimensional normed spaces equipped with the metric given by

$$\delta([X], [Y]) := \delta(X, Y).$$

It is well-known that $Q(n)$ is a compact metric space.

If $n = 2$, we can find a specific representative elements for each $[X]$ in $Q(2)$. This is based on the result of Alonso [1]. Let Ψ_2 be the set of all convex functions ψ on $[0, 1]$ satisfying $\max\{1 - t, t\} \leq \psi(t) \leq 1$ for each $t \in [0, 1]$. Then, for each $\psi \in \Psi_2$, the formula

$$\|(a, b)\|_\psi = \begin{cases} (|a| + |b|)\psi\left(\dfrac{|b|}{|a| + |b|}\right) & ((a, b) \neq (0, 0)) \\ 0 & ((a, b) = (0, 0)) \end{cases}$$

defines an absolute normalized norm on $\mathbb{R}^2$, that is, $\|(a,b)\|_\psi = \|(|a|,|b|)\|_\psi$ for each (a,b) and $\|(1,0)\|_\psi = \|(0,1)\|_\psi = 1$; see [2, 10]. Furthermore, using this kind of norms, we can introduce more general normed spaces. Namely, for each $\varphi, \psi \in \Psi_2$, let

$$\|(a,b)\|_{\varphi,\psi} = \begin{cases} \|(a,b)\|_\varphi & (ab \geq 0) \\ \|(a,b)\|_\psi & (ab \leq 0) \end{cases}.$$

Then $\|\cdot\|_{\varphi,\psi}$ is a norm on $\mathbb{R}^2$. The space $(\mathbb{R}^2, \|\cdot\|_{\varphi,\psi})$ is called a Day-James space, and denoted by $\ell_{\varphi,\psi}$ ([9]).

Theorem 2.1 (Alonso [1]). *Every two-dimensional normed space is isometrically isomorphic to some Day-James space.*

From this, we can always take a Day-James space as a representative element for each isometry class in $Q(2)$. Moreover, we have a useful estimation of Banach-Mazur distance between two Day-James spaces in terms of corresponding convex functions in Ψ_2. For the convenience, for $\varphi, \psi \in \Psi_2$, put

$$m_{\varphi,\psi} = \min_{t\in[0,1]} \frac{\psi(t)}{\varphi(t)} \quad \text{and} \quad M_{\varphi,\psi} = \max_{t\in[0,1]} \frac{\psi(t)}{\varphi(t)},$$

respectively. It is easy to check that $m_{\psi,\varphi} = 1/M_{\varphi,\psi}$.

The following result is important in this paper. The same idea can be found in [9, Lemma 3.1].

Proposition 2.2. *Let $\varphi_i, \psi_i \in \Psi_2$, $i = 1, 2$. Then*

$$m\|\cdot\|_{\varphi_1,\psi_1} \leq \|\cdot\|_{\varphi_2,\psi_2} \leq M\|\cdot\|_{\varphi_1,\psi_1},$$

where $m = \min\{m_{\varphi_1,\varphi_2}, m_{\psi_1,\psi_2}\}$ and $M = \max\{M_{\varphi_1,\varphi_2}, M_{\psi_1,\psi_2}\}$.

Combining this with the following simple fact, we obtain an estimation of the Banach-Mazur distance between two Day-James spaces.

Lemma 2.3. *Let $\|\cdot\|$ and $\|\cdot\|'$ be two norms on the same underlying vector space X. Suppose that*

$$m\|\cdot\| \leq \|\cdot\|' \leq M\|\cdot\|.$$

Then

$$\delta((X,\|\cdot\|),(X,\|\cdot\|')) \leq \log(M/m).$$

Corollary 2.4. *Let $\varphi_i, \psi_i \in \Psi_2$, $i = 1, 2$. Then*

$$\delta([\ell_{\varphi_1,\psi_1}],[\ell_{\varphi_2,\psi_2}]) \leq \log(\max\{M_{\varphi_1,\varphi_2}, M_{\psi_1,\psi_2}\}/\min\{m_{\varphi_1,\varphi_2}, m_{\psi_1,\psi_2}\}).$$

On the other hand, we have the following simple lemma.

Lemma 2.5. *Let $\psi \in \Psi_2$. If a sequence (ψ_n) in Ψ_2 converges uniformly to ψ, then* $\lim_n m_{\psi,\psi_n} = \lim_n M_{\psi,\psi_n} = 1$.

Combining this with Corollary 2.4, we have the following result. We assume that the set $\Psi_2 \times \Psi_2$ is equipped with the product topology induced by the usual supremum norm.

Proposition 2.6. *The mapping $(\varphi,\psi) \to [\ell_{\varphi,\psi}]$ from $\Psi_2 \times \Psi_2$ into $Q(2)$ is continuous and surjective.*

From this and the fact that Ψ_2 is compact in the supremum norm topology [15], we have a well-known fact.

Corollary 2.7. *$Q(2)$ is a compact metric space.*

3 A partial estimation of $F(\sqrt{2})$

Recall that the function F given in the introduction is defined by

$$F(s) := \sup\{C_{NJ}(X) : X \text{ is a normed space with } J(X) = s\}.$$

for each $s \in [\sqrt{2}, 2]$. In this section, we first give a general estimation for the function F. For this, we recall the following result of Takahashi and Kato [12, Corollary 3]

Lemma 3.1. *Let X be a normed space. Then*

$$0 \leq J(X) - C_{NJ}(X) \leq \sqrt{2} - 1.$$

Proposition 3.2. *The inequality*

$$s + 1 - \sqrt{2} \leq F(s) \leq s$$

holds for each $s \in [\sqrt{2}, 2]$.

Proof. This easily follows from the preceding lemma and the definition of F. □

In what follows, we consider the value of $F(\sqrt{2})$, and give a partial estimation. As was mentioned in the introduction, if $\dim X \geq 3$, then $J(X) = \sqrt{2}$ if and only if $C_{NJ}(X) = 1$, which happens if and only if X is an inner product space. Thus, in the case of $s = \sqrt{2}$, we may assume that $\dim X = 2$.

Now we recall the following result.

Theorem 3.3 (Kato, Maligranda and Takahashi [4]). *Let X and Y be normed spaces isomorphic to each other. Then*

$$e^{-\delta(X,Y)} J(X) \leq J(Y) \leq e^{\delta(X,Y)} J(X)$$

and

$$e^{-2\delta(X,Y)} C_{NJ}(X) \leq C_{NJ}(Y) \leq e^{2\delta(X,Y)} C_{NJ}(X).$$

Since normed spaces isometrically isomorphic to each other have the same James and von Neumann-Jordan constants, we can define the functions $J : Q(2) \to [\sqrt{2}, 2]$ and $C_{NJ} : Q(2) \to [1, 2]$ by $J([X]) := J(X)$ and $C_{NJ}([X]) = C_{NJ}(X)$ for each $[X] \in Q(2)$, respectively. The following fact immediately follows from the preceding theorem.

Corollary 3.4. *The functions $J : [X] \mapsto J(X)$ and $C_{NJ} : [X] \to C_{NJ}(X)$ are continuous on $Q(2)$.*

From this, the set $\{[X] \in Q(2) : J([X]) = \sqrt{2}\}$ is closed (and hence, it is compact). Moreover, since the function $C_{NJ} : [X] \to C_{NJ}(X)$ is continuous on $Q(2)$, there exists a two-dimensional normed space X such that $J(X) = \sqrt{2}$ and $C_{NJ}(X) = C_{NJ}([X]) = F(\sqrt{2})$. On the other hand, since $C_{NJ}(X) = J(X)$ if and only if $C_{NJ}(X) = J(X) = 2$, at least, we have

$$F(\sqrt{2}) = C_{NJ}(X) < J(X) = \sqrt{2}.$$

Unfortunately, there is no effective characterization of two-dimensional normed spaces with James constant $\sqrt{2}$; and so it is difficult to consider the general case. However, for the class of $\pi/2$-rotation invariant norms on $\mathbb{R}^2$, we can compute the exact value of

$$\sup\{C_{NJ}([X]) : X = (\mathbb{R}^2, \|\cdot\|) \text{ is } \pi/2\text{-rotation invariant}, \ J(X) = \sqrt{2}\}.$$

Recall that a norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be *θ-rotation invariant* if the θ-rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is an isometry on $(\mathbb{R}^2, \|\cdot\|)$. As was shown in [6], each $\pi/2$-rotation invariant normed space is isometrically isomorphic to some Day-James space of the form $\ell_{\psi,\widetilde{\psi}}$, where $\widetilde{\psi}$ is an element of Ψ_2 given by $\widetilde{\psi}(t) = \psi(1-t)$ for each $t \in [0,1]$. So we may assume that $X = \ell_{\psi,\widetilde{\psi}}$ for some $\psi \in \Psi_2$.

In [6], it was shown that if $\|\cdot\|$ is $\pi/2$-rotation invariant, then $J((\mathbb{R}^2, \|\cdot\|)) = \sqrt{2}$ if and only if $\|\cdot\|$ is $\pi/4$-rotation invariant. Furthermore, the class of $\pi/4$-rotation invariant Day-James spaces are in a one-to-one correspondence with a certain collection of convex functions on $[0,1]$; see [6]. Namely, let

$$\Gamma = \left\{\psi \in \Psi_2 : \max\left\{1 - \left(1 - \frac{1}{\sqrt{2}}\right)t, \frac{1}{\sqrt{2}} + \left(1 - \frac{1}{\sqrt{2}}\right)t\right\} \le \psi(t)\right\}.$$

For each $\psi \in \Psi_2$ such that $\|\cdot\|_{\psi,\widetilde{\psi}}$ is $\pi/4$-rotation invariant, let

$$\psi^\flat(t) = (1 + (\sqrt{2}-1)t)\psi\left(\frac{t}{\sqrt{2} + (2-\sqrt{2})t}\right) \tag{2}$$

for each $t \in [0,1]$. Then $\psi^\flat \in \Gamma$. Conversely, if $\psi \in \Gamma$, then

$$\psi^\sharp(t) = \begin{cases} (1-(2-\sqrt{2})t)\psi\left(\dfrac{\sqrt{2}t}{1-(2-\sqrt{2})t}\right) & (t \in [0,1/2]), \\ (\sqrt{2}-1)(1+\sqrt{2}t)\psi\left(\dfrac{2t-1}{(\sqrt{2}-1)(1+\sqrt{2}t)}\right) & (t \in [1/2,1]) \end{cases}$$

defines an element of Ψ_2, and $\|\cdot\|_{\psi^\sharp,\widetilde{\psi^\sharp}}$ is $\pi/4$-rotation invariant. Obviously, these constructions preserve the usual order of functions. Moreover, $(\psi^\flat)^\sharp = \psi$ for each $\psi \in \Psi_2$; and $(\psi^\sharp)^\flat = \psi$ for each $\psi \in \Gamma$. Combining these facts with [5, Example 3.7] and [6, Remark 4.6], we have

$$\max\{1-t, t, 1/\sqrt{2}\} \le \psi(t) \le \max\{1-(2-\sqrt{2})t, \sqrt{2}-1+(2-\sqrt{2})t\} \tag{3}$$

for each $\psi \in \Psi_2$ such that $\|\cdot\|_{\psi,\widetilde{\psi}}$ is $\pi/4$-rotation invariant. We here note that the functions

$$\psi_{\mathrm{cro}}(t) = \max\{1-t, t, 1/\sqrt{2}\}$$

and

$$\psi_{\mathrm{iro}}(t) = \max\{1-(2-\sqrt{2})t, \sqrt{2}-1+(2-\sqrt{2})t\}$$

are corresponding to the norms on $\mathbb{R}^2$ whose unit spheres are, respectively, the circumscribed and inscribed regular octagons of the unit circle. In other words, if $\|\cdot\|_{\psi,\tilde{\psi}}$ is $\pi/4$-rotation invariant, then its unit sphere lies "between" the circumscribed and inscribed regular octagons of the unit circle.

In what follows, the symbol ψ_2 denotes the fixed function given by

$$\psi_2(t) = ((1-t)^2 + t^2)^{1/2}$$

for each $t \in [0,1]$. Clearly, $\|\cdot\|_{\psi_2} = \|\cdot\|_2$, the Euclidean norm on $\mathbb{R}^2$. Using the above correspondence, we have the following lemma.

Lemma 3.5. *Let $\psi \in \Psi_2$. Suppose that $\|\cdot\|_{\psi,\tilde{\psi}}$ is $\pi/4$-rotation invariant. Then*

$$m_{\psi_2,\psi}\|\cdot\|_2 \le \|\cdot\|_{\psi,\tilde{\psi}} \le M_{\psi_2,\psi}\|\cdot\|_2.$$

Moreover, the equalities

$$m_{\psi_2,\psi} = \min_{t\in[0,1/2]} \frac{\psi(t)}{\psi_2(t)} = \min_{t\in[0,1]} \frac{\psi^\flat(t)}{\psi_2^\flat(t)}$$

and

$$M_{\psi_2,\psi} = \max_{t\in[0,1/2]} \frac{\psi(t)}{\psi_2(t)} = \max_{t\in[0,1]} \frac{\psi^\flat(t)}{\psi_2^\flat(t)}$$

hold.

From this, we have the main result in this note.

Theorem 3.6.

$$\sup\{C_{NJ}([X]) : X = (\mathbb{R}^2, \|\cdot\|) \textit{ is } \pi/2\textit{-rotation invariant},\ J(X) = \sqrt{2}\}$$
$$= 4 - 2\sqrt{2}.$$

Since each $\pi/4$-rotation invariant Day-James space can be constructed by an element of Γ, there are various examples of such spaces. So $4 - 2\sqrt{2}$ is seemed to be the upper bound also in the general case. However, the authors do not know whether this is true. The problem is still open.

Conjecture 3.7. $F(\sqrt{2}) = 4 - 2\sqrt{2}$.

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