数理解析研究所講究録
第2073巻 2018年 125-128

作用素平均と一般化逆行列

Operator means and generalized inverse

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The Karcher mean $X = \mathsf{G}(\omega_j; A_j)$ for invertible $A_j \geq 0$ with a weight $\{\omega_j\}$ is defined as a unique solution of the *Karcher equation* [7, 9, 10]:

$$\sum_j \omega_j S(X|A_j) = \sum_j \omega_j X^{\frac{1}{2}} \log\left(X^{-\frac{1}{2}} A_j X^{-\frac{1}{2}}\right) X^{\frac{1}{2}} = 0.$$

Also we extend the Karcher mean to non-invertible case in [7], which is an extension of the weighted geometric mean $A \#_t B = \mathsf{G}((1-t), t; A, B)$. We showed

$$\ker A \vee \ker B = \ker A \# B = \ker A \#_{\frac{1}{2}} B.$$

Moreover we extended such multi-variable *operator mean* $\mathsf{M}(A_j) = \mathsf{M}(\omega_j; A_1, ..., A_n)$ including the Karcher mean satisfying

(M1) **transformer equality**: $T^*\mathsf{M}(\omega_j; A_j)T = \mathsf{M}(\omega_j; T^*A_jT)$ for all invertible T.

(M1') **homogeneity**: $\mathsf{M}(\omega_j; tA_j) = t\mathsf{M}(\omega_j; A_j)$ for $t > 0$.

(M2) **normalization**: $\mathsf{M}(\omega_j; A) = A$.

(M3) **monotonicity**: $A_j \leq B_j$ implies $\mathsf{M}(\omega_j; A_j) \leq \mathsf{M}(\omega_j; B_j)$.

(M4) **sub-additivity**: $\mathsf{M}(\omega_j; A_j + B_j) \geq \mathsf{M}(\omega_j; A_j) + \mathsf{M}(\omega_j; B_j)$.

(M5) **adjoint sub-additivity**: $\mathsf{M}(\omega_j; A_j : B_j) \leq \mathsf{M}(\omega_j; A_j) : \mathsf{M}(\omega_j; B_j)$.

(M6) **orthogonality**: $\mathsf{M}(\omega_j; \bigoplus_m A_j^{(m)}) = \bigoplus_m \mathsf{M}(\omega_j; A_j^{(m)})$.

Here : stands for the parallel sum defined by $A : B = (A^{-1} + B^{-1})^{-1}$. In addition, we can define

$$\mathsf{M}(\omega_j; A_j) = \operatorname*{s-lim}_{\varepsilon \to 0} \mathsf{M}(\omega_j; (A_j + \varepsilon))$$

for (non-invertible) positive operators A_j where the above properties preserve, which includes our extended Karcher mean. In this extension, note that the transformer inequality $T^*\mathsf{M}(\omega_j; A_j)T \leq \mathsf{M}(\omega_j; T^*A_jT)$ holds for all operator T as we showed in [7].

For the parallel sum, rephrasing them into the harmonic mean, we have

$$A \,\mathrm{h}\, B = 2(A : B) = A\left(\frac{A+B}{2}\right)^{\dagger} B$$

if $A+B$ has the the generalized inverse [1]. Incidentally the Moore-Penrose generalized inverse † for operators was discussed in [5, 11]: It is known that if $\operatorname{ran} X$ is closed, then $\operatorname{ran} X^*$, $\operatorname{ran} XX^*$ and $\operatorname{ran} X^*X$ are also closed, and $(X^*X)^{\dagger} = \left(X^*X|_{\operatorname{ran} X^*}\right)^{-1} \oplus 0_{(\operatorname{ran} X^*)^{\perp}}$ and $X^{\dagger} = (X^*X)^{\dagger}X^* = X^*(XX^*)^{\dagger}$. Similarly we discuss the constructing formulae for operator means using the Moore-Penrose inverses if they exist:

$$A^{\frac{1}{2}}(I \,\mathrm{m}\, A^{\dagger\frac{1}{2}} B A^{\dagger\frac{1}{2}}) A^{\frac{1}{2}} \qquad \text{or} \qquad B^{\frac{1}{2}}(B^{\dagger\frac{1}{2}} A B^{\dagger\frac{1}{2}} \,\mathrm{m}\, I) B^{\frac{1}{2}}.$$

Here we recall an equality condition for transformer inequality for certain means [2]:

Theorem F . *If* $\ker T^* \subset \ker A \cap \ker B$, *then* $T^*(A \,\mathrm{m}\, B)T = (T^*AT) \,\mathrm{m}\, (T^*BT)$ *for an operator mean* m.

But the original proof of the above was based on the integral representation of operator means, so that we cannot extend the equality in Theorem F to multi-variable means. Under the closedness of the ranges for operators, we show the equality for our extended (multi-variable) operator means including the Karcher operator mean:

Theorem 1. *Let* $\mathsf{M}(A_j) = \mathsf{M}(\omega_j; A_1, .., A_n)$ *be an operator mean (satisfying the orthogonality). If an operator* T *on* H *satisfies* $\ker T^* \subset \bigcap_j \ker A_j$ *and* $\operatorname{ran} T$ *is closed, then the transformer equality holds:*

$$T^*\mathsf{M}(A_j)T = \mathsf{M}(T^*A_jT).$$

Proof. Note that $\operatorname{ran} T^*$ is also closed. Recall that $P = TT^{\dagger}$ and $Q = T^{\dagger}T$ are projections onto $\operatorname{ran} T$ and $\operatorname{ran} T^*$ respectively, see e.g. [5, 11]. By the assumption $\operatorname{ran} T^{\perp} = \ker T^* \subset \ker A_j$, we have $PA_jP = A_j$ for all j. Also $QT^*A_jTQ = T^*A_jT$ implies $QM(T^*A_jT)Q = M(T^*A_jT)$ for all j by the orthogonality. Then we have

$$\begin{aligned} T^*\mathsf{M}(A_j)T \le \mathsf{M}(T^*A_jT) &= Q\mathsf{M}(TA_jT)Q = T^*T^{\dagger *}\mathsf{M}(T^*A_jT)T^{\dagger}T \\ &\le T^*\mathsf{M}(T^{\dagger *}T^*A_jTT^{\dagger})T = T^*\mathsf{M}(PA_jP)T = T^*\mathsf{M}(A_j)T, \end{aligned}$$

which shows the required equality. □

Corollary 2. *Let* m *be an (2-variable) operator mean. If* $\ker A \subset \ker B$ *and* $\operatorname{ran} A$ *is closed, then*

$$A \,\mathrm{m}\, B = A^{\frac{1}{2}}(I \,\mathrm{m}\, A^{\dagger\frac{1}{2}} B A^{\dagger\frac{1}{2}}) A^{\frac{1}{2}}.$$

We once observed the kernel conditions for operator means, see also [3, 4]:

$$\ker A \,\mathrm{m}\, B \supset \ker A \vee \ker B \tag{1}$$

if and only if $1 \,\mathrm{m}\, 0 = 0 \,\mathrm{m}\, 1 = 0$.

Theorem 3. *Let* m *be an operator mean satisfying the above kernel condition* (1). *If* $\operatorname{ran} A$ *(resp.* $\operatorname{ran} B$*) is closed, then*

$$A \operatorname{m} B \le A^{\frac{1}{2}}(I \operatorname{m} A^{\dagger\frac{1}{2}} B A^{\dagger\frac{1}{2}}) A^{\frac{1}{2}} \qquad \left(\textit{resp.} \ \le B^{\frac{1}{2}}(B^{\dagger\frac{1}{2}} A B^{\dagger\frac{1}{2}} \operatorname{m} I) B^{\frac{1}{2}}\right).$$

Proof. Let P be the projections onto $(\ker A)^{\perp}$, that is, $P = A^{\dagger}A = A^{\dagger\frac{1}{2}}A^{\frac{1}{2}}$. The kernel condition shows $\operatorname{ran} A \operatorname{m} B \subset \operatorname{ran} P$ and hence Theorem 1 implies

$$\begin{aligned} A \operatorname{m} B &= P(A \operatorname{m} B)P \\ &\le (PAP) \operatorname{m} (PBP) = A \operatorname{m} (A^{\frac{1}{2}} A^{\dagger\frac{1}{2}} B A^{\dagger\frac{1}{2}} A^{\frac{1}{2}}) \\ &= A^{\frac{1}{2}} \left(P \operatorname{m} (A^{\dagger\frac{1}{2}} A A^{\dagger\frac{1}{2}})\right) A^{\frac{1}{2}} \le A^{\frac{1}{2}} \left(I \operatorname{m} (A^{\dagger\frac{1}{2}} B A^{\dagger\frac{1}{2}})\right) A^{\frac{1}{2}}. \end{aligned}$$

Similarly we have the other case. □

Here we observe the differences by the following examples:

Example. For $0 < a < 1$, we define a positive-definite matrix A and a projection P : Put

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}^2 = \begin{pmatrix} 1+a^2 & 2a \\ 2a & 1+a^2 \end{pmatrix}.$$

Then we have $A^{-\frac{1}{2}} = A^{\dagger\frac{1}{2}} = \dfrac{1}{1-a^2}\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix}$ and

$$P^{\frac{1}{2}}\sqrt{P^{\dagger\frac{1}{2}} A P^{\dagger\frac{1}{2}}}\,P^{\frac{1}{2}} = P\sqrt{PAP}P = \sqrt{1+a^2}\,P \,(\ge P).$$

On the other hand,

$$A^{\dagger\frac{1}{2}} P A^{\dagger\frac{1}{2}} = \frac{1}{(1-a^2)^2}\begin{pmatrix} 1 & -a \\ -a & a^2 \end{pmatrix} = \frac{1+a^2}{(1-a^2)^2} Q,$$

where $Q = \dfrac{1}{1+a^2}\begin{pmatrix} 1 & -a \\ -a & a^2 \end{pmatrix}$ is a rank 1 projection. Hence we have

$$A \# P = P \# A = A^{\frac{1}{2}}\sqrt{A^{\dagger\frac{1}{2}} P A^{\dagger\frac{1}{2}}}\,A^{\frac{1}{2}} = \frac{\sqrt{1+a^2}}{1-a^2} A^{\frac{1}{2}} Q A^{\frac{1}{2}} = \frac{1-a^2}{\sqrt{1+a^2}} P \ (\le P).$$

These differences are under the kernel inclusion as in Corollary 2.

To see a general case, we put $B = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}^2$ for $0 < b < 1$. For $X = P \oplus B$, $Y = A \oplus P$, the orthogonality shows

$$X \# Y = (P \# A) \oplus (B \# P) = \frac{1-a^2}{\sqrt{1+a^2}} P \oplus \frac{1-b^2}{\sqrt{1+b^2}} P.$$

Thus we have

$$X\#Y \le P \oplus \frac{1-b^2}{\sqrt{1+b^2}}P \equiv M_1 \quad \text{and} \quad X\#Y \le \frac{1-a^2}{\sqrt{1+a^2}}P \oplus P \equiv M_2,$$

while

$$X^{\frac{1}{2}}\sqrt{X^{\dagger\frac{1}{2}}YX^{\dagger\frac{1}{2}}}X^{\frac{1}{2}} = \sqrt{1+a^2}P \oplus \frac{1-b^2}{\sqrt{1+b^2}}P \ge M_1 \quad \text{and}$$

$$Y^{\frac{1}{2}}\sqrt{Y^{\dagger\frac{1}{2}}XY^{\dagger\frac{1}{2}}}Y^{\frac{1}{2}} = \frac{1-a^2}{\sqrt{1+a^2}}P \oplus \sqrt{1+b^2}P \ge M_2.$$

Acknowledgement. This study is partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research (C), JSPS KAKENHI Grant Number JP 16K05253.

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