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On quantum scattering in time-dependent electromagnetic fields

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In this note, we study the quantum dynamics of a charged particle moving in the plane in the presence of some time-dependent electromagnetic fields. The results mentioned here have been obtained in Adachi-Kawamoto [1] and [2].

1 Case 1

We consider a quantum system of a charged particle moving in the plane $\boldsymbol{R}^2$ in the presence of the constant magnetic field $\boldsymbol{B}$ which is perpendicular to the plane, and the time-dependent electric field $E(t)$ which always lies in the plane. We set $\boldsymbol{B}=(0,0,B)\in\boldsymbol{R}^3$ with $B>0$, and $E(t)=(E_1(t),E_2(t))\in\boldsymbol{R}^2$. Then the free Hamiltonian acting on $L^2(\boldsymbol{R}^2)$ is defined by

$$H_0(t)=H_0^B-qE(t)\cdot x, \quad H_0^B=(p-qA(x))^2/(2m), \tag{1.1}$$

where $m>0$, $q\in\boldsymbol{R}\setminus\{0\}$, $x=(x_1,x_2)$ and $p=(p_1,p_2)=(-i\partial_1,-i\partial_2)$ are the mass, the charge, the position, and the canonical momentum of the charged particle, respectively, and $A(x)=(-Bx_2/2,Bx_1/2)$ is the vector potential in the symmetric gauge. $p-qA(x)$ is called the kinetic momentum of the charged particle, and H_0^B is called the free Landau Hamiltonian.

The first result which we would like to mention in this case is concerned with the factorization of the propagator $U_0(t,s)$ generated by $H_0(t)$:

Theorem 1.1 (Adachi-Kawamoto [1]). *The following Avron-Herbst type formula for $U_0(t,0)$*

$$U_0(t,0)=e^{-ia(t)}e^{ib(t)\cdot x}T(c(t))e^{-itH_0^B}, \quad T(c(t))=e^{-ic(t)\cdot qA(x)}e^{-ic(t)\cdot p} \tag{1.2}$$

holds, where $b(t) = (b_1(t), b_2(t))$, $c(t) = (c_1(t), c_2(t))$ *and* $a(t)$ *are given by*

$$
\begin{aligned}
b(t) &= \int_0^t \hat{R}(-\omega(t-s))(qE(s))\,ds, \quad c(t) = \int_0^t b(s)/m\,ds, \\
a(t) &= \int_0^t \{b(s)^2/(2m) + b(s)\cdot qA(c(s))/m\}\,ds, \\
(\hat{R}(\eta)v)^{\mathrm{T}} &= \begin{pmatrix} \cos\eta & -\sin\eta \\ \sin\eta & \cos\eta \end{pmatrix} v^{\mathrm{T}}, \quad \omega = qB/m.
\end{aligned} \tag{1.3}
$$

$|\omega|$ is called the Larmor frequency, and $T(\xi) = e^{-i\xi\cdot k}$ is called the magnetic translation generated by the pseudomomentum $k = p + qA(x)$ of the charged particle. It is well known that

$$
\sigma(H_0^B) = \sigma_{\mathrm{pp}}(H_0^B) = \big\{|\omega|(n+1/2) \bigm| n \in \boldsymbol{N} \cup \{0\}\big\}
$$

holds, which implies that there is no scattering state in the system governed by H_0^B. However, as is also well known, if the constant electric field $E = (E_1, E_2)$ is switched on, then the guiding center of the charged particle drifts with the drift velocity $\alpha = (E_2/B, -E_1/B)$. This phenomenon implies the existence of scattering states in the system governed by $H_0(t)$ with $E(t) \equiv E$. Such a drift phenomenon implies that $|c(t)|$ is growing as $t \to \pm\infty$.

Now we will consider the case where $E(t) = E_0(\cos(\nu t + \theta), \sin(\nu t + \theta))$ with $E_0 > 0$, $\nu \in \boldsymbol{R}$ and $\theta \in [0, 2\pi)$. For the sake of brevity, we set $\tilde{e}(\eta) = (\cos\eta, \sin\eta)$. Then $E(t)$ is written as $E_0\tilde{e}(\nu t + \theta)$, and the instantaneous drift velocity $(E_2(t)/B, -E_1(t)/B) = \hat{R}(-\pi/2)E(t)/B$ is written as $(E_0/B)\tilde{e}(\nu t + \theta - \pi/2)$. Since $\hat{R}(\omega s)(qE(s)) = qE_0\tilde{e}(\tilde{\nu}s + \theta)$ with $\tilde{\nu} = \nu + \omega$, we have

$$
b(t) = \begin{cases} (qE_0/\tilde{\nu})(\tilde{e}(\nu t + \theta - \pi/2) - \tilde{e}(-\omega t + \theta - \pi/2)), & \tilde{\nu} \neq 0, \\ qE_0 t\tilde{e}(-\omega t + \theta), & \tilde{\nu} = 0, \end{cases}
$$

$$
c(t) = \begin{cases} (E_0/B)((\delta\tilde{e})(\tilde{\nu}t; -\omega t + \theta)/\tilde{\nu} - (\delta\tilde{e})(\nu t; \theta)/\nu), & \nu\tilde{\nu} \neq 0, \\ (E_0/B)((\delta\tilde{e})(-\omega t; \theta)/(-\omega) + t\tilde{e}(\theta - \pi/2)), & \nu = 0, \\ (E_0/B)(-t\tilde{e}(-\omega t + \theta - \pi/2) - (\delta\tilde{e})(-\omega t; \theta)/(-\omega)), & \tilde{\nu} = 0, \end{cases}
$$

where we put $(\delta\tilde{e})(\eta; \zeta) = \tilde{e}(\eta + \zeta) - \tilde{e}(\zeta)$ for the sake of brevity. Since $|c(t)|$ is growing like $(E_0/B)|t|$ as $t \to \pm\infty$ when $\nu\tilde{\nu} = 0$, then one can expect the existence of scattering states even if the system under consideration is governed by the perturbed Hamiltonian $H(t) = H_0(t) + V$. The case where $\tilde{\nu} = 0$, that is, $\nu = -\omega$, is closely related to the cyclotron resonance. Here we pose the following assumption $(V1)_\rho$ with $\rho > 0$ on the time-independent potential V, which we make simpler than in [1] for the sake of brevity:

$(V1)_\rho$ V is a real-valued function belonging to $C^2(\boldsymbol{R}^2)$, and satisfies the decaying condition $|(\partial^\alpha V)(x)| \le C_\alpha \langle x\rangle^{-\rho-|\alpha|}$ $(|\alpha| \le 2)$. Here $\langle x\rangle = \sqrt{1+|x|^2}$.

Then we obtain the following result about the existence of (modified) wave operators:

Theorem 1.2 ([1]). *Suppose that V satisfies $(V1)_\rho$ for some $\rho > 0$, and that $E(t) = E_0\tilde{e}(\nu t + \theta)$ with $\nu \in \{0, -\omega\}$ and $\theta \in [0, 2\pi)$. If $\rho > 1$, then the wave operators*

$$W^\pm = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^* U_0(t,0) \tag{1.4}$$

exist. If $0 < \rho \le 1$, then the modified wave operators

$$W_G^\pm = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^* U_0(t,0) e^{-i\int_0^t V(c(s))\,ds} \tag{1.5}$$

exist. Here $U(t,s)$ stands for the propagator generated by $H(t)$.

Next we will consider the problem of the asymptotic completeness of wave operators. Here we need the additional assumptions that $\nu = 0$ and that V is of short-range, that is, $\rho > 1$. Since the Hamiltonians under consideration are independent of t in this case, we write $H_0(t)$ and $H(t)$ as H_0 and H, respectively. Then we obtain the following result:

Theorem 1.3 ([1]). *Suppose that V satisfies $(V1)_\rho$ for some $\rho > 1$, and that $E(t) \equiv E_0\tilde{e}(\theta)$ with $\theta \in [0, 2\pi)$. Then $W^\pm$ are asymptotically complete, that is,*

$$\operatorname{Ran} W^\pm = L^2_{\mathrm{c}}(H), \tag{1.6}$$

where $L^2_{\mathrm{c}}(H)$ is the continuous spectral subspace of the Hamiltonian H.

In the study in the case where $\nu = -\omega$, the rotating frame is useful: The Schrödinger equation under consideration is

$$i\partial_t \Psi(t) = H(t)\Psi(t), \quad H(t) = H_0^B - qE_0\tilde{e}(-\omega t + \theta)\cdot x + V(x).$$

$\tilde{e}(-\omega t + \theta)\cdot x$ can be written as $(\hat{R}(-\omega t)\tilde{e}(\theta))\cdot x = \tilde{e}(\theta)\cdot(\hat{R}(\omega t)x)$. Now we introduce the angular momentum $\tilde{L} = x_1p_2 - x_2p_1$. $e^{i\eta\tilde{L}}$ is a unitary operator on $L^2(\boldsymbol{R}^2)$ given by

$$(e^{i\eta\tilde{L}}u)(x) = u(\hat{R}(\eta)x).$$

For $\Psi(t) = \Psi(t,x)$, we introduce

$$\Phi(t,x) = (e^{-i\omega t\tilde{L}}\Psi(t))(x) = \Psi(t, \hat{R}(-\omega t)x).$$

Then $\Phi(t) = \Phi(t, x)$ satisfies the Schrödinger equation

$$i\partial_t \Phi(t) = \hat{H}(t)\Phi(t), \quad \hat{H}(t) = \omega\tilde{L} + e^{-i\omega t\tilde{L}} H(t) e^{i\omega t\tilde{L}}.$$

We emphasis that $\hat{H}(t)$ can be written as

$$\hat{H}(t) = H_0^{-B} - qE_0\tilde{e}(\theta)\cdot x + V(\hat{R}(-\omega t)x), \quad H_0^{-B} = (p + qA(x))^2/(2m)$$

(see [1]). By using such a rotating frame, the problem under consideration can be reduced to the one in the case where $\nu = 0$, the magnetic field is given by $-\boldsymbol{B}$, and the potential is given as the rotating potential $V(\hat{R}(-\omega t)x)$, which is periodic in time. In particular, if V is radial, that is, V depends on $|x|$ only, then $V(\hat{R}(-\omega t)x) \equiv V(x)$. Therefore the asymptotic completeness can be guaranteed by virtue of Theorem 1.3 if V is of short-range. In the same way as above, the scattering problems for the time-periodic Hamiltonian

$$\tilde{H}(t) = H_0^B - qE_0\tilde{e}(-\omega t + \theta)\cdot x + V(\hat{R}(\omega t)x)$$

can be reduced to the ones for the time-independent Hamiltonian

$$\hat{H} = H_0^{-B} - qE_0\tilde{e}(\theta)\cdot x + V(x).$$

Then the asymptotic completeness can be guaranteed by virtue of Theorem 1.3, even if the short-range potential V is not radial.

2 Case 2

We consider a quantum system of a charged particle moving in the plane $\boldsymbol{R}^2$ in the presence of a periodically pulsed magnetic field $\boldsymbol{B}(t)$ which is perpendicular to the plane. We suppose that $\boldsymbol{B}(t) = (0, 0, B(t)) \in \boldsymbol{R}^3$ is given by

$$\begin{aligned} B(t) &= \begin{cases} B, & t \in \bigcup_{n\in\boldsymbol{Z}} I_{B,n} =: I_B, \\ 0, & t \in \bigcup_{n\in\boldsymbol{Z}} I_{0,n} =: I_0, \end{cases} \\ I_{B,n} &= [nT, nT + T_B), \quad I_{0,n} = [nT + T_B, (n+1)T), \end{aligned} \tag{2.1}$$

where $B > 0$ and $0 < T_B < T$. T is the period of $\boldsymbol{B}(t)$. We put $T_0 = T - T_B > 0$. The free Hamiltonian acting on $L^2(\boldsymbol{R}^2)$ is defined by

$$H_0(t) = (p - qA(t, x))^2/(2m), \tag{2.2}$$

where

$$\begin{aligned} A(t, x) &= (-B(t)x_2/2, B(t)x_1/2) \\ &= \begin{cases} (-Bx_2/2, Bx_1/2) = A(x), & t \in I_B, \\ (0, 0), & t \in I_0, \end{cases} \end{aligned} \tag{2.3}$$

is the vector potential in the symmetric gauge. Then $H_0(t)$ is represented as

$$H_0(t) = \begin{cases} H_0^B, & t \in I_B, \\ H_0^0, & t \in I_0, \end{cases} \tag{2.4}$$

where $H_0^0 = p^2/(2m)$ is the free Schrödinger operator. Let $U_0(t,s)$ be the propagator generated by $H_0(t)$. By (2.4) and the self-adjointness of H_0^B and H_0^0, $U_0(t,0)$ is represented as

$$U_0(t,0) = \begin{cases} e^{-i(t-nT)H_0^B} U_0(T,0)^n, & t \in I_{B,n}, \\ e^{-i(t-(nT+T_B))H_0^0} e^{-iT_B H_0^B} U_0(T,0)^n, & t \in I_{0,n}, \end{cases} \tag{2.5}$$

with $n \in \boldsymbol{Z}$, where

$$U_0(T,0) = e^{-iT_0 H_0^0} e^{-iT_B H_0^B} \tag{2.6}$$

is the Floquet operator associated with $H_0(t)$, $U_0(T,0)^0 = \mathrm{Id}$, and $U_0(T,0)^n = (U_0(T,0)^*)^{-n}$ when $-n \in \boldsymbol{N}$. Put

$$\omega = qB/m, \quad \bar{\omega} = \omega/2, \quad \bar{\bar{\omega}} = \bar{\omega}/2 = \omega/4. \tag{2.7}$$

Taking account of $e^{-i(2\pi/|\omega|)H_0^B} = e^{-i(\pi/|\bar{\omega}|)H_0^B} = -\mathrm{Id}$, we always assume

$$0 < |\bar{\omega}| T_B < \pi \tag{2.8}$$

for the sake of simplicity.

Let $\tilde{S}_0^0(t;x,y)$ and $\tilde{S}_0^B(t;x,y)$ be integral kernels of $e^{-itH_0^0}$ and $e^{-itH_0^B}$, respectively. As is well known, these are represented as

$$\begin{aligned} \tilde{S}_0^0(t;x,y) &= \frac{m}{2\pi i t} e^{im(x-y)^2/(2t)}, \\ \tilde{S}_0^B(t;x,y) &= \frac{m|\bar{\omega}|}{2\pi i \sin(|\bar{\omega}|t)} e^{im|\bar{\omega}|x^2/(2\tan(|\bar{\omega}|t))} \\ &\quad \times e^{-im|\bar{\omega}|(\hat{R}(\bar{\omega}t)x)\cdot y/\sin(|\bar{\omega}|t)} e^{im|\bar{\omega}|y^2/(2\tan(|\bar{\omega}|t))}. \end{aligned} \tag{2.9}$$

By using these formulas, we obtained the representation of the intergral kernel $\tilde{S}_0(t;x,y)$ of $U_0(t,0)$ (see Adachi-Kawamoto [2]). Here, for the sake of simplicity, we give it with $t = nT$ $(n \in \boldsymbol{N})$ only:

$$\tilde{S}_0(nT;x,y) = \frac{1}{2\pi i c_n \theta_n} e^{ix^2/(2\theta_n)} e^{-i(\hat{R}(\phi_n)x)\cdot y/(c_n\theta_n)} e^{i\sigma_n y^2/(2\theta_n)}, \tag{2.10}$$

where $\{\theta_n\}$, $\{c_n\}$, $\{\sigma_n\}$ and $\{\phi_n\}$ satisfy the recurrence relations

$$\frac{1}{\theta_{n+1}} = \left(1 - \frac{1}{c_1^2\sigma_1}\right)\frac{1}{\theta_1} + \frac{1}{(c_1\sigma_1)^2(\theta_1/\sigma_1 + \theta_n)},$$

$$\frac{1}{c_{n+1}\theta_{n+1}} = \frac{1}{c_1\sigma_1 c_n(\theta_1/\sigma_1+\theta_n)},$$
$$\frac{\sigma_{n+1}}{\theta_{n+1}} = \Big(\sigma_n - \frac{1}{c_n^2}\Big)\frac{1}{\theta_n} + \frac{1}{c_n^2(\theta_1/\sigma_1+\theta_n)},$$
$$\phi_{n+1} = \phi_1 + \phi_n$$

with

$$\theta_1 = \frac{L_{12}}{L_{22}}, \quad c_1 = L_{22}, \quad \phi_1 = \bar{\omega}T_B, \quad \sigma_1 = \sigma_0(T) = \frac{L_{11}}{L_{22}},$$
$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} \cos(\bar{\omega}T_B) - \bar{\omega}T_0\sin(\bar{\omega}T_B) & \bar{\omega}T_0\cos(\bar{\omega}T_B) + \sin(\bar{\omega}T_B) \\ -\sin(\bar{\omega}T_B) & \cos(\bar{\omega}T_B) \end{pmatrix}.$$

One can obtain $\phi_n = n\bar{\omega}T_B$ immediately. We note that $L \in \mathrm{SL}(2, \boldsymbol{R})$ and that the recurrence relation of $\{\theta_n\}$ can be written by L as follows:

$$\theta_{n+1} = \frac{L_{11}\theta_n + L_{12}}{L_{21}\theta_n + L_{22}}.$$

If $T_0 \neq T_{0,\mathrm{cr}} = 1/(|\bar{\omega}|\tan(|\bar{\omega}|T_B)) > 0$, then θ_n, c_n and σ_n are represented as

$$\theta_n = \frac{L_{12}\mu_n}{L_{22}\mu_n - \mu_{n-1}}, \quad c_n = L_{22}\mu_n - \mu_{n-1}, \quad \sigma_n = \frac{L_{11}\mu_n - \mu_{n-1}}{L_{22}\mu_n - \mu_{n-1}},$$
$$\mu_n = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}, \quad \lambda_\pm = \lambda_0 \pm \sqrt{\lambda_0^2 - 1}, \quad \lambda_0 = (L_{11} + L_{22})/2.$$

$\lambda_\pm$ are the eigenvalues of L. From now on we always assume $T_0 \neq T_{0,\mathrm{cr}}$ and $L_{12} \neq 0$, that is, $T_0 \neq T_{0,\mathrm{res}} = -\tan(|\bar{\omega}|T_B))/|\bar{\omega}|$. By (2.10), the following factorization of $U_0(nT, 0)$ can be given:

$$U_0(nT, 0) = e^{i\phi_n \tilde{L}} M(\theta_n) D(c_n\theta_n) \mathscr{F} M\Big(\frac{\theta_n}{\sigma_n}\Big). \tag{2.11}$$

Here $M(\tau)$, $D(\tau)$ and $\mathscr{F}$ are unitary operators on $L^2(\boldsymbol{R}^2)$ given by

$$\begin{aligned} &(M(\tau)\varphi)(x) = e^{ix^2/(2\tau)}\varphi(x), \quad (D(\tau)\varphi)(x) = \frac{1}{i\tau}\varphi\Big(\frac{x}{\tau}\Big), \\ &\mathscr{F}[\varphi](\xi) = \frac{1}{2\pi}\int_{\boldsymbol{R}^2} e^{-ix\xi}\varphi(x)\,dx. \end{aligned} \tag{2.12}$$

In the study of some scattering problems for this system, the growing order of the argument $c_n\theta_n = L_{12}\mu_n$ of the dilation operator $D(c_n\theta_n)$ in (2.11) as $n \to \infty$

is an important factor. If $|\lambda_0| < 1$, then $|\lambda_\pm| = 1$, which implies that $|c_n\theta_n|$ is bounded with respect to n; while if $|\lambda_0| > 1$, then $\lambda_- < -1 < \lambda_+ < 0$ holds. Thus $|c_n\theta_n|$ is growing exponentially like $|\lambda_-|^n = e^{n\log|\lambda_-|}$ as $n \to \infty$. Such a phenomenon is called a parametric resonance. We note that $|\lambda_0| > 1$ is equivalent to $T_0 > T_{0,\mathrm{cr}}$.

In the case where $T_0 > T_{0,\mathrm{cr}}$, we will consider the problem of the asymptotic completeness of wave operators as in Case 1. We pose the following assumption $(V2)_\rho$ with $\rho > 0$ on the time-independent potential V:

$(V2)_\rho$ V is a real-valued function belonging to $C(\boldsymbol{R}^2)$, and satisfies the decaying condition $|V(x)| \le C\langle x\rangle^{-\rho}$.

Then we obtain the following result:

Theorem 2.1 ([2]). *Suppose that T_0 satisfies $T_0 > T_{0,\mathrm{cr}}$. When $\pi/2 < |\tilde{\omega}|T_B < \pi$, assume that T_0 satisfies $T_0 \ne T_{0,\mathrm{res}}$ additionally. Assume that V satisfies the condition $(V2)_\rho$ for some $\rho > 0$. Then the wave operators*

$$W^\pm = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^* U_0(t,0) \tag{2.13}$$

exist, and are asymptotically complete:

$$\mathrm{Ran}(W^\pm) = \mathscr{H}_{\mathrm{ac}}(U(T,0)). \tag{2.14}$$

Here $U(t,s)$ stands for the propagator generated by $H(t) = H_0(t) + V$, and $\mathscr{H}_{\mathrm{ac}}(U(T,0))$ is the absolutely continuous spectral subspace associated with the Floquet operator $U(T,0)$.

Since we assume that V is time-independent, the existence of $U(t,0)$ can be guaranteed as follows: Since $H(t)$ is represented as

$$H(t) = \begin{cases} H_0^B + V = H^B, & t \in I_B, \\ H_0^0 + V = H^0, & t \in I_0, \end{cases} \tag{2.15}$$

$U(t,0)$ is represented as

$$U(t,0) = \begin{cases} e^{-i(t-nT)H^B}U(T,0)^n, & t \in I_{B,n}, \\ e^{-i(t-(nT+T_B))H^0}e^{-iT_BH^B}U_0(T,0)^n, & t \in I_{0,n}, \end{cases} \tag{2.16}$$

with $n \in \boldsymbol{Z}$, where

$$U(T,0) = e^{-iT_0H^0}e^{-iT_BH^B} \tag{2.17}$$

is the Floquet operator associated with $H(t)$.

References

[1] T. Adachi and M. Kawamoto, Avron-Herbst type formula in crossed constant magnetic and time-dependent electric fields, Lett. Math. Phys. **102** (2012), 65–90.

[2] T. Adachi and M. Kawamoto, Quantum scattering in a periodically pulsed magnetic field, Ann. Henri Poincaré 17 (2016), 2409–2438.