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Solvable models in the scattering theory for the Aharonov–Bohm effect

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Abstract

We briefly review the scattering theory for the Schrödinger operator related to the Aharonov–Bohm effect, by exhibiting some explicitly solvable models; an infinitesimally thin solenoid in $\mathbb{R}^2$, two infinitesimally thin solenoids with the quantized magnetic fluxes in $\mathbb{R}^2$, and an infinitesimally thin ring solenoid with the quantized magnetic flux in $\mathbb{R}^3$. We give explicit formulas for the incoming wave solution and the scattering amplitude for these models in terms of some special functions, with the graphs obtained by a numerical method. We also review known high-energy asymptotic formulas related to these examples.

1 Introduction

In 1959, Y. Aharonov and D. Bohm published a remarkable paper [Ah-Bo], in which they stated that the magnetic field enclosed in an electrically shielded solenoid can affect the electron moving outside the solenoid through the magnetic vector potential, in the formulation of non-relativistic quantum mechanics. This phenomenon is called the *Aharonov–Bohm effect*, and there was a large controversy concerning the existence of the Aharonov–Bohm effect in the real world (for the detail, see e.g. Peshkin–Tonomura [Pe-To]). The decisive experiment was done by A. Tonomura et al. [To]. They prepared toroidal apparatus in which they succeeded to enclose the magnetic field completely with the aid of the super-conductive effect, and observed the interference pattern by two coherent electron beams going through inside and outside the hole of the toroidal magnet. The obtained picture clearly shows that the phase shift of the electron wave by the magnetic flux inside the torus[1].

There are also numerous works concerning the direct and the inverse scattering theory related to the Aharonov–Bohm effect. First, we list the results about the Schrödinger operator H_N in $\mathbb{R}^2$ with N pointlike magnetic fields (for the definition of H_N, see (5) below), which models N infinitely long, infinitesimally thin solenoids perpendicular to the plane. The case $N = 1$ is studied by Aharonov–Bohm [Ah-Bo], and later more mathematically rigorously by Ruijsenaars [Ru]. In the case $N \geq 2$, Šťovíček [St] constructs the Green function of H_2 by path-integral method, and the result is further developed in Kocábová–Šťovíček [Ko-St1] and Košťáková–Šťovíček [Ko-St2]. Nambu [Na] tries to calculate the scattering amplitude for H_N, but it is not successful (instead, Nambu studies the dynamical vortices). Ito–Tamura [It-Ta1, It-Ta2] obtain the asymptotic formula for the scattering amplitude for H_N, in the limit the distance between solenoids tends to infinity. Roux–Yafaev [Ro-Ya] study the essential spectrum of the scattering matrix for more general magnetic Schrödinger operator in $\mathbb{R}^2$, including the Aharonov–Bohm one. Yafaev [Ya] also studies the diagonal singularity of the scattering amplitude for the magnetic Schrödinger operators in $\mathbb{R}^d$ ($d = 2, 3$).

There are a lot of results about the inverse scattering problem for H_N or related magnetic Schrödinger operators in $\mathbb{R}^d$ ($d \geq 2$); e.g. Arians [Ar] ($d \geq 2$), Nicoleau [Ni] ($d \geq 2$), Weder [We] ($d = 2$), Eskin [Es1, Es2] ($d \geq 2$), Ballesteros–Weder [Ba-We1, Ba-We2, Ba-We3] ($d = 3$), and Eskin–Isozaki–O'Dell [Es-Is-Od] ($d = 2$) (for more comprehensive list, see Eskin [Es3]). Especially, Ballesteros–Weder [Ba-We1, Ba-We2, Ba-We3] give a detailed analysis of the scattering by toroidal magnetic fields in $\mathbb{R}^3$, and the result in [Ba-We2] is considered to be a mathematical justification of the experiment by Tonomura et al. [To]. We also note that Iwatsuka–M–Shimada [Iw-Mi-Sh] prove the norm-resolvent

[1] The picture can also be seen in Hitachi's web-site.
http://www.hitachi.com/rd/portal/highlight/quantum/aharonov-bohm/index.html

convergence of the Schrödinger operators with toroidal magnetic fields in the limit as the thickness of the torus tends to 0 with fixed magnetic flux through the section of the torus.

However, the paper by Gu–Qian [Gu-Qi] seems to be missed in the previous papers. Gu–Qian calculate the scattering amplitude for the Schrödinger operator H_2 in $\mathbb{R}^2$ with two pointlike magnetic fields, by using the *elliptic coordinate*. The result of Gu–Qian is mathematically not complete, and the author justifies Gu–Qian's result in [Mi1, Mi2] under the *magnetic quantization condition* (see (7) below), that is, the magnetic flux of each pointlike field is an integer multiple of the quantum of magnetic flux. The condition (7) comes from the covering structure of the elliptic coordinate as a map from $\mathbb{R}^2$ onto itself (see [Mi1]). Interestingly, the condition (7) matches with the experiment by Tonomura et al. [To] (the magnetic flux is quantized by a super-conductive effect), and also plays an important role in the nodal domain problem for the Schrödinger operators with AB-magnetic fields (see Helffer [He] and references therein).

In this short note, we exhibit some explicitly solvable models, in the sense that we can calculate the incoming plane wave and the scattering amplitude explicitly in terms of some special functions. We also give the graphs obtained by a numerical method, which help an intuitive understanding of the Aharonov–Bohm effect. In sections 2 and 3, we study the Schrödinger operator with N pointlike magnetic fields in $\mathbb{R}^2$, which models N infinitely long, infinitesimally thin solenoids perpendicular to the plane. It is well-known that the model is solvable when $N = 1$ ([Ah-Bo, Ru]; see also Theorem 2.1 below). When $N = 2$, the model is solvable in the elliptic coordinate, if the magnetic fluxes are quantized to be integer multiples of the quantum of magnetic flux ([Gu-Qi, Mi1, Mi2]; see also Theorem 3.2 below). We also review the asymptotic formula of the scattering amplitude by Ito–Tamura [It-Ta1], and compare their result with the explicit formula. In section 4, we consider the Schrödinger operator with a toroidal magnetic field in $\mathbb{R}^3$, which models the experiment by Tonomura et al., especially we review the result by Ballesteros–Weder [Ba-We2]. In the limiting case the torus becomes a ring of thickness 0, and the flux through the section of the ring is quantized to be an integer multiple of the quantum of magnetic flux, the model is solvable in the *oblate spheroidal coordinate* (Theorem 4.1). The last result seems to be new (there is no proof here, and the proof will be written elsewhere). All the figures except Figure 7 are obtained by using Wolfram Mathematica 10.3.

2 Scattering by single solenoid in $\mathbb{R}^2$

In the paper Aharonov–Bohm [Ah-Bo], they study the scattering of an electron by an infinitesimally thin, infinitely long magnetic solenoid, parallel to the z-axis. Because of the translational invariance along the z-axis, they consider the two-dimensional Hamiltonian[2]

$$\begin{aligned} H &= \left(\frac{1}{i}\nabla - A_\alpha\right)^2 \quad \text{on } \mathbb{R}^2, \quad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right), \\ A_\alpha &= (A_{\alpha,x}, A_{\alpha,y}) = \alpha\left(-\frac{y}{r^2}, \frac{x}{r^2}\right), \quad r = \sqrt{x^2+y^2}, \end{aligned} \tag{1}$$

with the regular boundary condition at $(x, y) = O = (0, 0)$ (we use the normalization $\hbar = 1$, $m = 1/2$, and $e = 1$ for simplicity). Here α is a real constant, and the z-component of the magnetic field corresponding to the magnetic vector potential A_α is given by

$$\operatorname{curl} A_\alpha = \frac{\partial A_{\alpha,y}}{\partial x} - \frac{\partial A_{\alpha,x}}{\partial y} = 2\pi\alpha\delta_O,$$

where δ_O is the Dirac measure at the origin. Aharonov and Bohm compute the incoming wave solution $\varphi_- = \varphi_-(\mathbf{x}; \mathbf{p})$ to the stationary Schrödinger equation $H\varphi_- = k^2\varphi_-$ ($k > 0$), which behaves like

$$\varphi_-(\mathbf{x}; \mathbf{p}) \sim e^{i\mathbf{x}\cdot\mathbf{p}} + f(\tau \to \theta; k^2)\frac{e^{ikr}}{r^{1/2}} \quad (r \to \infty),$$

[2]Here the sign of α is opposite to the one used in Aharonov–Bohm [Ah-Bo] and Ruijsenaars [Ru], rather our choice matches with the one in Ito–Tamura [It-Ta1]. So the formula (3) does not seem to coincide with [Ru, (4.11)], but it coincides with [It-Ta1, (1.2)].

where $\mathbf{x} = (x, y) = (r\cos\theta, r\sin\theta)$, and $\mathbf{p} = (k\cos\tau, k\sin\tau)$ is the momentum of the incident wave[3]. The factor $f(\tau \to \theta; k^2)$ is called the *scattering amplitude*, and $|f(\tau \to \theta; k^2)|^2$ the *differential scattering cross section*, which tells us the ratio of the particles coming from the incident direction τ and scattered into the final direction θ.

Ruijsenaars [Ru] studies this problem mathematically more rigorous way. Especially Ruijsenaars proves that the incoming wave solution φ_- is also defined as the integral kernel of the wave operator $W_- = \text{s-}\lim_{t\to-\infty} e^{itH}e^{-itH_0}$ in the time-dependent formalism, as follows.

$$W_-u(\mathbf{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \varphi_-(\mathbf{x};\mathbf{p})\hat{u}(\mathbf{p})d\mathbf{p}, \tag{2}$$

where $\hat{u}(\mathbf{p}) = (2\pi)^{-1}\int_{\mathbb{R}^2} e^{-i\mathbf{x}\cdot\mathbf{p}}u(\mathbf{x})d\mathbf{x}$ is the Fourier transform of u. Since the expression (2) is obtained by replacing $e^{i\mathbf{x}\cdot\mathbf{p}}$ in the Fourier inversion formula by $\varphi_-(\mathbf{x};\mathbf{p})$, we sometimes use the formal expression $\varphi_-(\mathbf{x};\mathbf{p}) = W_-e^{i\mathbf{x}\cdot\mathbf{p}}$. Ruijsenaars also calculates the singularity of the scattering amplitude at the forward direction $\tau = \theta$. Their results are summarized as follows.

Theorem 2.1 (Aharonov–Bohm [Ah-Bo], Ruijsenaars [Ru]). *Put* $\mathbf{x} = (x, y) = (r\cos\theta, r\sin\theta)$, *and* $\mathbf{p} = (k\cos\tau, k\sin\tau)$. *Then, the following holds.*

(i) The incoming wave solution φ_- *is given by*

$$\varphi_-(\mathbf{x};\mathbf{p}) = \sum_{m=-\infty}^{\infty} i^{|m|}e^{i\delta_{m,k}}J_{|m-\alpha|}(kr)e^{im(\theta-\tau)},$$

where J_ν *is the Bessel function of order* ν, *and* $\delta_{m,k} = (|m| - |m-\alpha|)\pi/2$.

(ii) The scattering amplitude $f(\tau \to \theta; k^2)$ *is given by*

$$\begin{aligned} & f(\tau \to \theta; k^2) \\ = & \left(\frac{2\pi}{ik}\right)^{1/2}\left((\cos\alpha\pi - 1)\delta(\theta-\tau) - \frac{i\sin\alpha\pi}{\pi}e^{i[\alpha](\theta-\tau)}\,\mathrm{P.V.}\frac{e^{i(\theta-\tau)}}{e^{i(\theta-\tau)}-1}\right), \end{aligned} \tag{3}$$

where $[x]$ *is the integer part of* x *and* P.V. *denotes the principal value. In particular, the differential scattering cross section* $|f(\tau \to \theta; k^2)|^2$ *is given by*

$$|f(\tau \to \theta; k^2)|^2 = \frac{1}{2\pi k}\frac{\sin^2(\alpha\pi)}{\sin^2((\theta-\tau)/2)} \quad (\theta \neq \tau).$$

The number $\delta_{m,k}$ is called the *scattering phase shift*, which is determined by the asymptotic formula

$$\begin{aligned} J_{|m-\alpha|}(kr) &\sim \sqrt{\frac{2}{\pi kr}}\cos\left(kr - \frac{|m-\alpha|\pi}{2} - \frac{\pi}{4}\right) \\ &= \sqrt{\frac{2}{\pi kr}}\cos\left(kr - \frac{|m|\pi}{2} - \frac{\pi}{4} + \delta_{m,k}\right) \quad (r \to \infty), \end{aligned}$$

so $\delta_{m,k}$ measures the phase difference between the partial wave solution in the perturbed system and that in the free system. Once we know all the scattering phase shifts $\delta_{m,k}$, we can calculate the scattering amplitude $f(\tau \to \theta; k^2)$ by the formula

$$f(\tau \to \theta; k^2) = (2\pi ik)^{-1/2}\sum_{m=-\infty}^{\infty}(e^{2i\delta_{m,k}} - 1)e^{im(\theta-\tau)}. \tag{4}$$

The characteristic feature in Theorem 2.1 is that the phase shift $\delta_{m,k}$ is independent of k, and the sum (4) can be calculated explicitly.

[3] More precisely, Aharonov and Bohm consider the case $\mathbf{p}$ is directed to the negative x-axis, but the argument is essentially the same for other $\mathbf{p}$.

In Figure 1, we give the density plot of the real part of φ_- in the case the momentum of the incident wave $\mathbf{p} = (1,0)$ and $\alpha = 1/2$. The arrow indicates the incident direction. We observe the difference between the phase of φ_- above and below the positive x-axis. This reflects the singular behavior of the scattering amplitude in the formula (3) at $\theta = \tau$, caused by the magnetic flux at the origin.

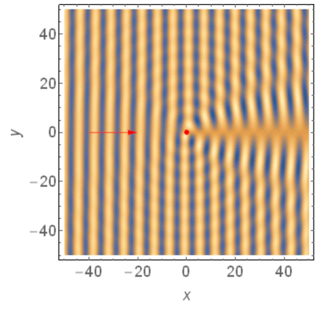

Figure 1: The density plot of $\mathrm{Re}\,\varphi_-$.

3 Scattering by N solenoids in $\mathbb{R}^2$

Next, we consider the case there are N solenoids ($N \geq 2$) perpendicular to the plane. The Hamiltonian is given by

$$H_N = \left(\frac{1}{i}\nabla - A_N\right)^2 \quad \text{on } \mathbb{R}^2, \quad A_N = \sum_{j=1}^{N} A_{\alpha_j}(x - x_j, y - y_j), \tag{5}$$

where A_{α_j} is the vector potential given in (1) with $\alpha = \alpha_j$, and $2\pi\alpha_j$ is the magnetic flux through the solenoid at $\mathbf{x}_j = (x_j, y_j)$ ($j = 1, \ldots, N$). We impose the regular boundary condition at each $\mathbf{x}_j$. Below we review the results of Ito–Tamura [It-Ta1] and the author [Mi2], concerning the incoming wave solution and the scattering amplitude for the operator H_N.

3.1 The result by Ito–Tamura

In Ito–Tamura [It-Ta1], they consider the case $N = 2$, $\mathbf{x}_1 = O$, $\mathbf{x}_2 = \mathbf{d}$ ($\mathbf{d} \in \mathbb{R}^2 \setminus \{O\}$), and calculate the asymptotics of the scattering amplitude as $|\mathbf{d}| \to \infty$. In this result, we identify an angle $\theta \in [0, 2\pi)$ with a unit vector $\omega = (\cos\theta, \sin\theta) \in S^1$ ($S_1 = \{\omega \in \mathbb{R}^2 \mid |\omega| = 1\}$), and we denote $f(\omega \to \omega'; k^2)$ instead of $f(\theta \to \theta'; k^2)$. For $\mathbf{x} \in \mathbb{R}^2 \setminus \{0\}$ and $\omega \in S^1$, let $\gamma(\mathbf{x}; \omega)$ denotes the azimuth angle of $\mathbf{x}$ from the direction ω with $0 \leq \gamma(\mathbf{x}; \omega) < 2\pi$. We use the notation

$$\exp(i\alpha\gamma(\mathbf{x};\omega)) = \begin{cases} \exp(i\alpha\gamma(\mathbf{x};\omega)) & (\mathbf{x}/|\mathbf{x}| \neq \omega), \\ (1 + \exp(2\pi i\alpha))/2 & (\mathbf{x}/|\mathbf{x}| = \omega), \end{cases}$$

$$\exp(i\alpha\tau(\mathbf{x};\omega,\omega')) = \exp(i\alpha\gamma(\mathbf{x};\omega))\overline{\exp(i\alpha\gamma(\mathbf{x};-\omega'))}.$$

Theorem 3.1 (Ito–Tamura [It-Ta1]). *Let $f_j(\omega \to \omega'; k^2)$ be the scattering amplitude by the single solenoid at $\mathbf{x}_j$ with flux $2\pi\alpha_j$ ($j = 1, 2$) (f_2 depends on $\mathbf{d}$), and $f_{\mathbf{d}}(\omega \to \omega'; k^2)$ be the scattering amplitude by the two solenoids. Then, for $\omega \neq \omega'$, $f_{\mathbf{d}}(\omega \to \omega'; k^2)$ behaves like*

$$\begin{aligned} f_{\mathbf{d}}(\omega \to \omega'; k^2) &= \exp(i\alpha_2\tau(-\mathbf{d};\omega,\omega'))f_1(\omega \to \omega'; k^2) \\ &+ \exp(i\alpha_1\tau(\mathbf{d};\omega,\omega'))f_2(\omega \to \omega'; k^2) + o(1) \end{aligned} \tag{6}$$

as $|\mathbf{d}| \to \infty$ with fixed $\hat{\mathbf{d}} = \mathbf{d}/|\mathbf{d}|$.

The phase factors in the right hand side of (6) are considered to be an appearance of the AB-effect. Actually, Kostrykin–Schrader [Ko-Sc] also study a large distance limit of the scattering amplitude for the Schrödinger operator $H_{\mathbf{d}} = -\Delta + V_1 + V_2(\cdot - \mathbf{d})$, and there is no such phase factor like (6) in their formula [Ko-Sc, (1.19)]. In Ito–Tamura [It-Ta2], they also consider the case $N \geq 3$.

3.2 Formulas in the elliptic coordinate

In the case $N = 2$, $\mathbf{x}_1 = (-a, 0)$ and $\mathbf{x}_2 = (a, 0)$ for some $a > 0$, we can analyze the operator H_2 by using the elliptic coordinate. This idea is used in Gu–Qian [Gu-Qi], and further developed by the author [Mi1, Mi2]. For the detail in this subsection, see [Mi1, Mi2]. For the Mathieu functions, see Abramowitz–Stegun [Ab-St] or McLachlan [Mc].

The *elliptic coordinate* is defined by

$$\begin{cases} x = a\cosh\xi\cos\eta, \\ y = a\sinh\xi\sin\eta, \end{cases} \qquad \xi \geq 0, \ -\pi < \eta \leq \pi,$$

where a is a positive constant. The coordinate curves $\xi =$ const. are confocal ellipses, and $\eta =$ const. confocal hyperbolas. In the elliptic coordinate, the Helmholtz equation $-\Delta u = \lambda u$ can be solved by separation of variables. The solutions are

$$u = \mathrm{Ce}_n(\xi, q)\mathrm{ce}_n(\eta, q) \quad (n = 0, 1, 2, \ldots),$$
$$u = \mathrm{Se}_n(\xi, q)\mathrm{se}_n(\eta, q) \quad (n = 1, 2, \ldots),$$

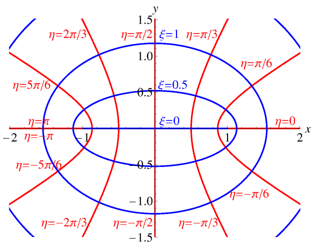

Figure 2: Elliptic coordinate.

where $q = a^2\lambda/4$. The functions ce_n and se_n are called the *Mathieu functions*, and Ce_n and Se_n are called the *modified Mathieu functions*. The angular functions ce_n and se_n are similar to the trigonometric functions, e.g., ce_n and Ce_n are even functions, and se_n and Se_n are odd functions.

When $N = 2$, $\mathbf{x}_1 = (-a, 0)$ and $\mathbf{x}_2 = (a, 0)$, the AB Hamiltonian H_2 is represented as

$$H_2 = e^{i(\alpha_1\theta_1 + \alpha_2\theta_2)}(-\Delta)e^{-i(\alpha_1\theta_1 + \alpha_2\theta_2)},$$
$$\theta_1 = \arg((x + a) + yi), \quad \theta_2 = \arg((x - a) + yi).$$

Put $v = e^{-i(\alpha_1\theta_1 + \alpha_2\theta_2)}u$. Then, the eigenequation $H_2 u = \lambda u$ is equivalent to $-\Delta v = \lambda v$, but v becomes a multi-valued function unless α_1 and α_2 are integers. Analyzing the latter equation, we conclude the eigenequation $H_2 u = \lambda u$ can be solved by separation of variables in the elliptic coordinate if and only if

$$\alpha_1, \alpha_2 \in (1/2)\mathbb{Z}. \tag{7}$$

The condition (7) is called the *magnetic quantization condition*, since $2\pi \cdot (1/2) = \pi$ is equal to the quantum of magnetic flux $h/(2e)$ under our normalization $\hbar = h/(2\pi) = 1$ and $e = 1$. In the case both α_1 and α_2 are odd multiples of $1/2$, the solutions are

$$u = e^{i(\alpha_1\theta_1 + \alpha_2\theta_2)}\mathrm{Fe}_n(\xi, q)\mathrm{ce}_n(\eta, q) \quad (n = 0, 1, 2, \ldots),$$
$$u = e^{i(\alpha_1\theta_1 + \alpha_2\theta_2)}\mathrm{Ge}_n(\xi, q)\mathrm{se}_n(\eta, q) \quad (n = 1, 2, \ldots),$$

where $q = a^2\lambda/4$. The functions Fe_n and Ce_n satisfy the same ordinary differential equation with respect to ξ, but Fe_n has the opposite parity to that of Ce_n, that is, Ce_n is even and Fe_n is odd. The same relation holds for Ge_n and Se_n.

We introduce the *normalized modified Mathieu functions* Jc_n, Js_n, Jf_n, and Jg_n, which are constant multiples of Ce_n, Se_n, Fe_n, and Ge_n, respectively, and behave like

$$\mathrm{Jc}_n(\xi, q) \sim \mathrm{Js}_n(\xi, q) \sim \sqrt{\frac{2}{\pi kr}}\cos\left(kr - \frac{n\pi}{2} - \frac{\pi}{4}\right),$$
$$\mathrm{Jf}_n(\xi, q) \sim \sqrt{\frac{2}{\pi kr}}\cos\left(kr - \frac{n\pi}{2} - \frac{\pi}{4} + \delta_{n,k}\right),$$
$$\mathrm{Jg}_n(\xi, q) \sim \sqrt{\frac{2}{\pi kr}}\cos\left(kr - \frac{n\pi}{2} - \frac{\pi}{4} + \epsilon_{n,k}\right)$$

as $\xi \to \infty$, where $q = a^2k^2/4$ and $r = a\cosh\xi \sim |\mathbf{x}|$. The first formula means Jc_n and Js_n have the same asymptotics as that of the Bessel function J_n. The constants $\delta_{n,k}$ and $\epsilon_{n,k}$ are the scattering phase shifts in this context, which can be computed at least numerically.

Theorem 3.2 ([Mi2]). *Assume $N = 2$, $\mathbf{x}_1 = (-a, 0)$, $\mathbf{x}_2 = (a, 0)$, $\alpha_1 = -1/2$, and $\alpha_2 = 1/2$. We use the elliptic coordinate in $\mathbf{x}$-space and the polar coordinate in $\mathbf{p}$-space, that is,*

$$\mathbf{x} = (a\cosh\xi\cos\eta, a\sinh\xi\sin\eta), \quad \mathbf{p} = (k\cos\tau, k\sin\tau),$$

and put $q = a^2k^2/4$. Then, the following holds.

(i) The incoming wave solution $\varphi_- = W_- e^{i\mathbf{x}\cdot\mathbf{p}}$ to $H_2\varphi_- = k^2\varphi_-$ with momentum $\mathbf{p}$ is

$$\begin{aligned}\varphi_-(\mathbf{x};\mathbf{p}) \;&=\; e^{i(-\theta_1+\theta_2)/2}\Big(\sum_{n=0}^{\infty} i^n e^{i\delta_{n,k}}\mathrm{Jf}_n(\xi,q)\mathrm{ce}_n(\eta,q)\mathrm{ce}_n(\tau,q)\\ &\qquad + \sum_{n=1}^{\infty} i^n e^{i\epsilon_{n,k}}\mathrm{Jg}_n(\xi,q)\mathrm{se}_n(\eta,q)\mathrm{se}_n(\tau,q)\Big).\end{aligned}$$

(ii) The scattering amplitude $f(\tau \to \theta; k^2)$ is given by

$$\begin{aligned}f(\tau \to \theta; k^2) \;&=\; \sqrt{\frac{2}{\pi i k}}\left(\sum_{n=0}^{\infty}(e^{2i\delta_{n,k}} - 1)\mathrm{ce}_n(\theta,q)\mathrm{ce}_n(\tau,q)\right.\\ &\qquad \left. + \sum_{n=1}^{\infty}(e^{2i\epsilon_{n,k}} - 1)\mathrm{se}_n(\theta,q)\mathrm{se}_n(\tau,q)\right). \qquad (8)\end{aligned}$$

Formally the formula (8) is obtained by replacing $\cos n\theta$ by $\mathrm{ce}_n(\theta,q)$, $\sin n\theta$ by $\mathrm{se}_n(\theta,q)$, and $\delta_{m,k}$ by $\delta_{n,k}$ or $\epsilon_{n,k}$, in the formula (4) (the term $m = 0$ needs a slight modification). Actually, both formulas can be proved by the plane wave expansion formula in each coordinate. In Figure 3-4, we give the density plots of the real part of the incoming wave solutions for $k = 4$ and $\tau = 0, \pi/2$, which describe the AB-phase shift in this model. In Figure 5 and 6, we give the graphs of the differential scattering cross sections (DSCS) $|f(\tau \to \theta; 1000)|^2$ obtained by the Ito–Tamura asymptotic formula (6) and that by the elliptic coordinate (8), respectively. A small difference is seen near $\theta = 0$, which is the direction of $\mathbf{d} = \mathbf{x}_2 - \mathbf{x}_1$, where the convergence of the Ito–Tamura formula (6) is not uniform.

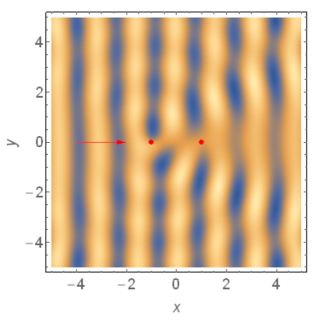

Figure 3: Plot of $\mathrm{Re}\,\varphi_-$, for $k = 4, \tau = 0$.

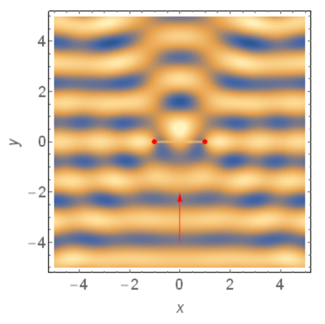

Figure 4: Plot of $\mathrm{Re}\,\varphi_-$, for $k = 4, \tau = \pi/2$.

4 Scattering by the toroidal magnetic field in $\mathbb{R}^3$

4.1 The result by Ballesteros–Weder

In Ballesteros–Weder [Ba-We2], they study the Schrödinger operator with the magnetic field enclosed inside the torus

$$T = \{(x, y, z) \in \mathbb{R}^3 \mid R_1^2 \le |x|^2 + |y|^2 \le R_2^2,\ |z| \le H\}$$

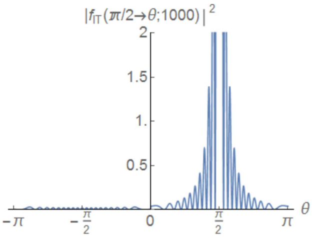

Figure 5: DSCS by the Ito–Tamura formula, $\tau = \pi/2$.

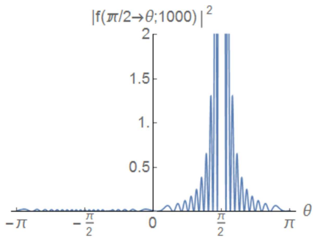

Figure 6: DSCS by the elliptic coordinate, $\tau = \pi/2$.

for some $0 < R_1 < R_2$ and $H > 0$. Put

$$\Omega = \mathbb{R}^3 \setminus (T \cup S), \quad S = \{(x, y, z) \in \mathbb{R}^3 \mid |x|^2 + |y|^2 > R_2^2,\ z = 0\}.$$

Since Ω is simply connected, in the set Ω the magnetic Schrödinger operator H is represented as $H = \Phi H_0 \Phi^{-1}$, $H_0 = -\Delta$, by some gauge function Φ (see (10) below). So, for a Gaussian wave packet $\varphi_{\mathbf{v}}$ going through the hole of the torus with velocity $\mathbf{v}$, we can use the approximation

$$e^{-itH}\varphi_{\mathbf{v}} \doteqdot \Phi e^{-itH_0}\Phi^{-1}\varphi_{\mathbf{v}}.$$

This kind of approximation is also used by Aharonov–Bohm [Ah-Bo], so Ballesteros–Weder call the approximation *Aharonov–Bohm ansatz*. Ballesteros–Weder give a very precise error estimate for this approximation under the situation of the experiment by Tonomura et al. [Ba-We2, Theorem 1.1], which justifies the experiment qualitatively.

4.2 Infinitesimally thin ring magnetic field

Here we take another approach to this problem. We consider a ring R of thickness 0 in $\mathbb{R}^3$, given by

$$R = \{\mathbf{x} = (x, y, 0) \in \mathbb{R}^3 \mid |\mathbf{x}| = a\},$$

for some $a > 0$. We consider the magnetic Schrödinger operator

$$H = \left(\frac{1}{i}\nabla - A\right)^2 \quad \text{on} \quad \mathbb{R}^3,$$

where $A \in C^\infty(\mathbb{R}^3 \setminus R; \mathbb{R}^3)$ is the magnetic vector potential satisfying

$$\nabla \times A = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus R,$$
$$\int_D (\nabla \times A)\cdot \mathbf{n}\, dS = \int_{\partial D} A \cdot d\ell = o\pi, \qquad (9)$$

for any small disc D pierced by the ring R, where o is an odd integer, and $\mathbf{n}$ is the unit normal vector on D (the direction of $\mathbf{n}$ is appropriately fixed), with $\operatorname{supp} A$ bounded set in $\mathbb{R}^3$. (9) is the magnetic quantization condition in this context.

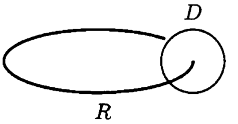

Figure 7: Zero thickness ring magnetic field.

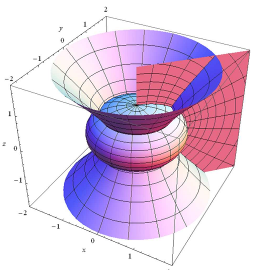

Figure 8: The oblate spheroidal coordinate.

We impose the regular boundary condition along R, or more precisely, H is the Friedrichs extension of the operator $H|_{C_0^\infty(\mathbb{R}^3\setminus R)}$. This is an idealized model to the experiment by Tonomura et al., and the advantage is that this model is explicitly solvable in the *oblate spheroidal coordinate*, defined as follows.

$$\begin{cases} x = a\cosh\xi\sin\eta\cos\phi, \\ y = a\cosh\xi\sin\eta\sin\phi, \\ z = a\sinh\xi\cos\eta, \end{cases} \qquad \begin{pmatrix} \xi \geq 0, \\ 0 \leq \eta \leq \pi, \\ -\pi < \phi \leq \pi, \end{pmatrix}$$

where a is a positive constant. The surface $\xi = const.$ is a flattened ellipsoid, $\eta = const.$ a hyperboloid of one sheet, $\phi = const.$ a half-plane (see Figure 8). In the oblate spheroidal coordinate, the equation $-\Delta u = k^2 u$ can be solved by separation of variables, and the solutions are

$$\begin{aligned} &u = S_{m\ell}(-iak, \cos\eta) R^{(1)}_{m\ell}(-iak, i\sinh\xi)\cos m\phi, \\ &u = S_{m\ell}(-iak, \cos\eta) R^{(1)}_{m\ell}(-iak, i\sinh\xi)\sin m\phi \quad (m \neq 0), \\ &(m = 0, 1, 2, \ldots, \quad \ell = m, m+1, m+2, \ldots), \end{aligned}$$

where $S_{m\ell}$ is the *angular spheroidal wave function*, and $R^{(1)}_{m\ell}$ the *radial spheroidal wave function of the first kind*. For the detail of these functions, see Abramowitz–Stegun [Ab-St] or Flammer [Fl].

Take $\mathbf{x}_0 \in \mathbb{R}^3$ sufficiently large, and put

$$\Phi(\mathbf{x}) = \exp\left(i\int_{\mathbf{x}_0}^{\mathbf{x}} A \cdot d\ell\right) \quad (\mathbf{x} \in \mathbb{R}^3),$$

where the path of the integral is in the region $\mathbb{R}^3 \setminus R$. The function Φ is two-valued, since

$$\exp\left(i\int_{\partial D} A \cdot d\ell\right) = e^{i\alpha\pi} = -1$$

for any small disc D pierced by the ring R. Moreover, we have

$$Hu = \Phi(-\Delta)\Phi^{-1}u. \tag{10}$$

Then, as in the two-dimensional case, the solutions to $Hu = k^2 u$ are given as

$$\begin{aligned} &u = \Phi \cdot S_{m\ell}(-iak, \cos\eta) R^{(5)}_{m\ell}(-iak, i\sinh\xi)\cos m\phi, \\ &u = \Phi \cdot S_{m\ell}(-iak, \cos\eta) R^{(5)}_{m\ell}(-iak, i\sinh\xi)\sin m\phi \quad (m \neq 0), \\ &(m = 0, 1, 2, \ldots, \quad \ell = m, m+1, m+2, \ldots.) \end{aligned}$$

The functions $R^{(1)}_{m\ell}(c, z)$ and $R^{(5)}_{m\ell}(c, z)$ ($c = -iak$) satisfy the same ordinary differential equation with respect to z, and have different parities. Both functions Φ and $S_{m\ell}(-iak, \cos\eta)R^{(5)}_{m\ell}(-iak, i\sinh\xi)\cos m\phi$ (or $\cdots\sin m\phi$) are two-valued with respect to the original coordinate x, and the product of these functions is single-valued. The scattering phase shift $\delta^m_{\ell,k}$ is defined by the asymptotics

$$\begin{aligned} R^{(1)}_{m\ell}(c, z) &\sim \frac{1}{cz}\cos\left(cz - \frac{\ell+1}{2}\pi\right), \\ R^{(5)}_{m\ell}(c, z) &\sim \frac{1}{cz}\cos\left(cz - \frac{\ell+1}{2}\pi + \delta^m_{\ell,k}\right) \end{aligned}$$

as $z \to i\infty$. When $c = -iak$ and $z = i\sinh\xi$, we have $cz = k \cdot a\sinh\xi \sim kr$ ($r = |\mathbf{x}|$) and the limit $z \to i\infty$ corresponds the limit $r \to \infty$. Thus $R^{(1)}_{m\ell}(c, z)$ has the same asymptotic behavior as the asymptotics of the spherical Bessel function. Our main result is formulated as follows.

Theorem 4.1. *(i) We introduce the oblate spheroidal coordinate in* $\mathbf{x}$*-space and the spherical coordinate in* $\mathbf{p}$*-space, that is,*

$$\begin{aligned} \mathbf{x} &= (a\cosh\xi\sin\eta\cos\phi,\ a\cosh\xi\sin\eta\sin\phi,\ a\sinh\xi\cos\eta), \\ \mathbf{p} &= (k\sin\tau\cos\psi,\ k\sin\tau\sin\psi,\ k\cos\tau). \end{aligned}$$

Then, the incoming wave solution $\varphi_- = W_- e^{i\mathbf{x}\cdot\mathbf{p}}$ *with momentum* $\mathbf{p}$ *for the quantized ring magnetic field is*

$$\begin{aligned}\varphi_-(\mathbf{x};\mathbf{p}) &= \Phi\cdot\sum_{m=0}^{\infty}\sum_{\ell=m}^{\infty} i^\ell c_{\ell,m}^2 e^{i\delta_{\ell,k}^m} R_{m\ell}^{(5)}(-iak, i\sinh\xi)\\ &\qquad\cdot S_{m\ell}(-iak,\cos\eta)S_{m\ell}(-iak,\cos\tau)\cos(m(\phi-\psi)),\end{aligned}$$

where $c_{\ell,m} = \sqrt{(2-\delta_{0,m})(2\ell+1)(\ell-m)!/(\ell+m)!}$.

(ii) We introduce the spherical coordinate in S^2 *as*

$$\begin{aligned}\omega &= (\sin\tau\cos\psi, \sin\tau\sin\psi, \cos\tau),\\ \omega' &= (\sin\tau'\cos\psi', \sin\tau'\sin\psi', \cos\tau')\end{aligned}$$

for $\omega, \omega' \in S^2$. *Then, the scattering amplitude with energy* k^2 $(k > 0)$ *for the quantized ring magnetic field is*

$$\begin{aligned}f(\omega'\to\omega;k^2) &= \frac{1}{2ik}\sum_{m=0}^{\infty}\sum_{\ell=m}^{\infty}(e^{2i\delta_{\ell,k}^m}-1)c_{\ell,m}^2\\ &\qquad\cdot S_{m\ell}(-iak,\cos\tau)S_{m\ell}(-iak,\cos\tau')\cos(m(\psi-\psi')),\end{aligned}$$

where $c_{\ell,m}$ *is as in (i).*

Let us try to reproduce the interference pattern in the experiment by Tonomura et al. by using the above formula. We put $a = 1$, and consider the object wave $\varphi_{-,1} = W_- e^{i\mathbf{p}_1\cdot\mathbf{x}}$ with the incident momentum $\mathbf{p}_1 = (0, 0, 8)$, which goes through the hole of the ring solenoid vertically. We also consider the reference wave $\varphi_{-,2} = W_- e^{i\mathbf{p}_2\cdot\mathbf{x}}$ with $\mathbf{p}_2 = (0, 4\sqrt{3}, 4)$. The two vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ have the same length, and make the angle $\pi/3$. The density plot of the real part of $\varphi_{-,1}$ and that of $\varphi_{-,2}$ on the yz-plane are given in Figure 9, 10, respectively. The central segment represents the ring R. We observe the phase shift occurs behind the ring, depending on the incident direction of the electron beam. In these figures, we take $\Phi = 1$ for simplicity, so $\varphi_{-,j}$ are discontinuous on the disc enclosed by the ring.

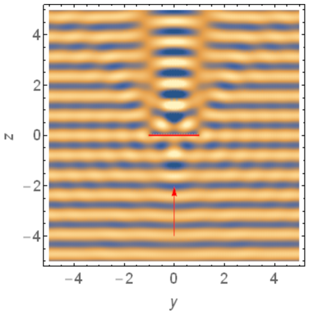

Figure 9: The density plot of $\mathrm{Re}\,\varphi_{-,1}$.

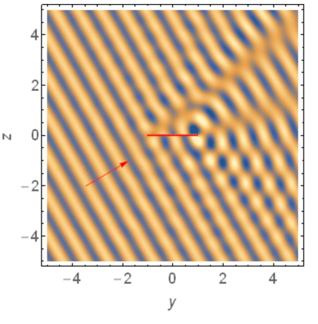

Figure 10: The density plot of $\mathrm{Re}\,\varphi_{-,2}$.

Of course, the real part of the wave function is gauge-dependent, and not an observable quantity. The interference pattern caused by the superposition of these two waves is defined by $|\varphi_{-,1} + \varphi_{-,2}|^2$, which is a gauge-independent quantity. In Figure 11-14, we give the density plots of the interference pattern on several planes. Especially, Figure 12, the interference pattern behind the ring magnet, is similar to the picture by Tonomura et al. We also give the spherical density plots of the differential scattering cross section $|f(\omega' \to \omega; k^2)|^2$ in Figure 15 and 16.

At present we do not have an asymptotic formula for the scattering amplitude in the high energy limit in terms of elementary functions, like the Ito–Tamura formula (6). To our knowledge, the only known result is the Aharonov–Bohm ansatz by Ballesteros–Weder [Ba-We2]; when the velocity $\mathbf{v}$ is directed to the z-axis, the scattering operator acts on the Gaussian wave packet $\varphi_{\mathbf{v}}$ as the multiplication by $\exp(2\pi\alpha i)$ ($2\pi\alpha$ is the magnetic flux through the section of the torus) in the high energy limit. Finding more general asymptotic formula, which explains Figure 9 and 10 more clearly, is an interesting problem.

In the computational side, the rapid computation of the spheroidal wave function is desired recently, mainly in application to the signal processing analysis (see e.g. Kirby [Ki1, Ki2]). We hope the progress in this area, which will give us more detailed information about our scattering problem.

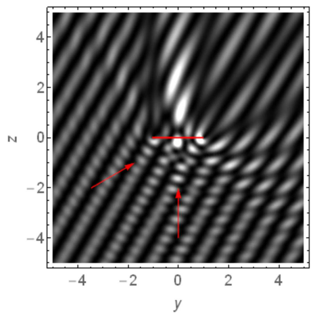

Figure 11: Interference pattern on yz-plane.

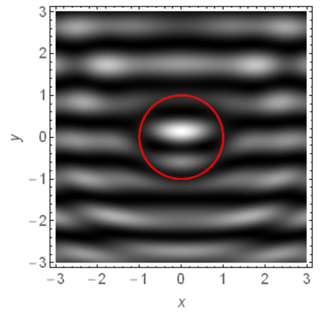

Figure 12: Interference pattern on $z = 3$.

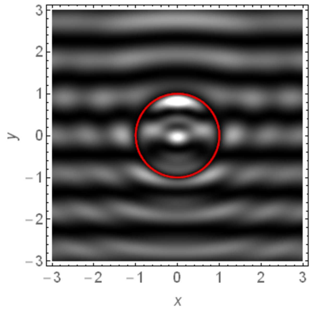

Figure 13: Interference pattern on $z = 0$.

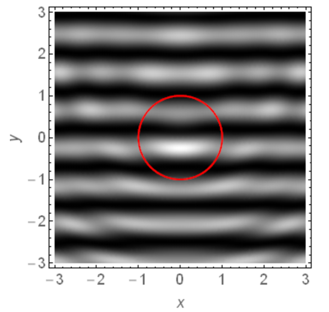

Figure 14: Interference pattern on $z = -2$.

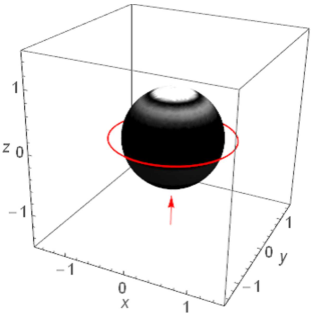

Figure 15: DSCS for $\tau' = 0$, $\psi' = 0$, $k = 8$.

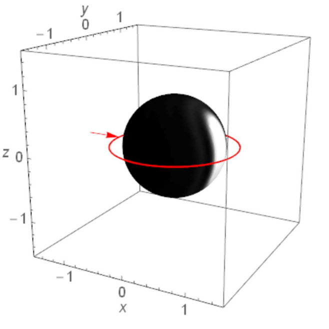

Figure 16: DSCS for $\tau' = \pi/2$, $\psi' = 0$, $k = 8$.

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