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MULTI-PARAMETER ASYMPTOTICS FOR TRUNCATED WIENER-HOPF OPERATORS

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1. Introduction

This note is devoted to the study of (bounded, self-adjoint) operators of the form

$$W(a;\alpha\Lambda) := \chi_{\alpha\Lambda}\mathcal{F}^* a \mathcal{F}\chi_{\alpha\Lambda},\ \ \alpha > 0, \tag{1.1}$$

on $\mathsf{L}^2(\mathbb{R}^d)$, $d\ge 1$, where χ_Λ is the indicator function of a set $\Lambda\subset\mathbb{R}^d$, and $\alpha\Lambda = \{\alpha\mathbf{x} : \mathbf{x}\in\Lambda\}$. The notation $\mathcal{F}$ stands for the unitary Fourier transform in $\mathsf{L}^2(\mathbb{R}^d)$. The real-valued function a, called *symbol* is assumed to be bounded and smooth. We call the operator (1.1) a (truncated) Wiener–Hopf operator. We are interested in the asymptotics of the trace of the following operator difference

$$D(a,\alpha\Lambda;f) := \chi_{\alpha\Lambda} f(W(a;\alpha\Lambda))\chi_{\alpha\Lambda} - W(f\circ a;\alpha\Lambda), \tag{1.2}$$

as $\alpha\to\infty$, with some suitably chosen functions f. The second operator on the right-hand side of (1.2) can be viewed as a regularizing term: it makes the operator (1.2) trace class even if $f(0)\neq 0$ and Λ is unbounded, under some extra mild conditions on Λ and f. On the other hand, if $f(0)=0$, Λ is bounded and the symbol a decays fast at infinity, then both operators on the right-hand side of (1.2) are easily shown to be trace class.

Asymptotic properties of $D(a,\alpha\Lambda;f)$ depend strongly on the smoothness of the symbol a. For the full asymptotic expansion of $\operatorname{tr} D(a,\alpha\Lambda;f)$ in powers of α^{-1} with smooth symbols a, smooth functions f and smooth bounded domains Λ, we refer to A. Budylin–V. Buslaev [1] and H. Widom [15]. The leading term of this expansion is of order α^{d-1}. For symbols a with jump discontinuities we only mention the papers by H. Landau–H. Widom [3], H. Widom [13] (for $d=1$) and by A.V. Sobolev [8], [9] (for arbitrary $d\ge 1$). Compared to the smooth case, the leading asymptotic term acquires an extra log-factor. For example, for the symbol $a=\chi_\Omega$ with a bounded piece-wise smooth region Ω, the trace $\operatorname{tr} D(\chi_\Omega,\alpha\Lambda;f)$ is of order $\alpha^{d-1}\log\alpha$.

The mentioned asymptotic results find their applications in the study of large-scale behaviour of the spatially bipartite entanglement entropy of free fermions in thermal equilibrium, see [2], [4], [5]. For this application the symbol is taken to be the *Fermi*

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symbol

$$a_T(\boldsymbol{\xi}) = a_{T,\mu}(\boldsymbol{\xi}) := \frac{1}{1+\exp\frac{h(\boldsymbol{\xi})-\mu}{T}}, \quad \boldsymbol{\xi}\in\mathbb{R}^d, \tag{1.3}$$

where $T > 0$ is the temperature, and $\mu \in \mathbb{R}$ is the (fixed) chemical potential. The real-valued *free Hamiltonian* $h \in \mathsf{C}^\infty$ is assumed to satisfy $h(\boldsymbol{\xi}) \geq c|\boldsymbol{\xi}|^\beta$, $\beta > 0$, as $|\boldsymbol{\xi}| \to \infty$, and is such that the level set $\{\boldsymbol{\xi} \in \mathbb{R}^d : h(\boldsymbol{\xi}) = \mu\}$ is a smooth surface with finitely many connected components. Thus the *Fermi sea*

$$\Omega = \{\boldsymbol{\xi} \in \mathbb{R}^d : h(\boldsymbol{\xi}) < \mu\} \tag{1.4}$$

is a smooth bounded region. It is natural to define this symbol for $T = 0$ as the point-wise limit $a_0(\boldsymbol{\xi}) = \chi_\Omega(\boldsymbol{\xi}) = \lim_{T\to 0} a_T(\boldsymbol{\xi})$. Since the symbol a_0 is discontinuous, it is not surprising that the nature of large-scale entropy asymptotics is different for $T = 0$ and $T > 0$, see [4], [5].

Partly motivated by the above example, in this note we concentrate on the transition from smooth to discontinuous symbols. The precise statements, proofs and detailed discussions are found in the paper [11]. In this note we illustrate the results of [11] by considering just one example of such a "transitional" symbol, the Fermi symbol (1.3). The parameter μ is kept fixed, but the temperature T is allowed to vary simultaneously with the scaling parameter α.

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2. The results

2.1. Main results. For a function $f : \mathbb{R} \to \mathbb{C}$ and any $s_1, s_2 \in \mathbb{R}$ define the integral

$$U(s_1, s_2; f) = \int_0^1 \frac{f\big((1-t)s_1 + ts_2\big) - [(1-t)f(s_1) + tf(s_2)]}{t(1-t)} dt. \tag{2.1}$$

It is clear that $U(s_1, s_2; 1) = U(s_1, s_2; t) = 0$, for all $s_1, s_2 \in \mathbb{R}$. This integral is finite for any Hölder function f. For a smooth symbol $a = a(\xi)$, $\xi \in \mathbb{R}$, define

$$\mathcal{B}(a; f) := \frac{1}{8\pi^2} \lim_{\varepsilon\to 0} \iint\limits_{|\xi_1-\xi_2|>\varepsilon} \frac{U\big(a(\xi_1), a(\xi_2); f\big)}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2. \tag{2.2}$$

If f is smooth, then this definition coincides with the standard double integral. The principal value definition becomes necessary for functions f featuring in the theorems below, see [10] for details.

As shown in [14], in the case $d = 1$, for smooth f and a we have $\operatorname{tr} D(a, \mathbb{R}_+; f) = \mathcal{B}(a; f)$. For the multi-dimensional case the asymptotic coefficient is defined as follows. For a unit vector $\mathbf{e} \in \mathbb{R}^d$, $d \geq 2$, introduce the hyperplane

$$\Pi_{\mathbf{e}} := \{\boldsymbol{\xi} \in \mathbb{R}^d : \mathbf{e}\cdot\boldsymbol{\xi} = 0\}.$$

Introduce the orthogonal coordinates $\boldsymbol{\xi} = (\mathring{\boldsymbol{\xi}}, t)$ such that $\mathring{\boldsymbol{\xi}} \in \Pi_{\mathbf{e}}$ and $t \in \mathbb{R}$. Then we set

$$(2.3)\quad \mathcal{B}_d(a; \partial\Lambda, f) := \frac{1}{(2\pi)^{d-1}} \int_{\partial\Lambda} \mathcal{A}_d(a, \mathbf{n}_{\mathbf{x}}; f) dS_{\mathbf{x}}, \quad \mathcal{A}_d(a, \mathrm{e}; f) := \int_{\Pi_{\mathbf{e}}} \mathcal{B}\big(a(\mathring{\boldsymbol{\xi}}, \,\cdot\,); f\big) d\mathring{\boldsymbol{\xi}}.$$

As illustrated in Theorem 2.2 below, this coefficient describes the large-scale behaviour for smooth symbols. Here and henceforth we assume that

$$(2.4)\qquad \begin{cases} \Lambda \text{ is a region with finitely many connected components} \\ \text{such that the boundary } \partial\Lambda \text{ is a union} \\ \text{of bounded piece-wise smooth surfaces.} \end{cases}$$

Thus the integral (2.3) is well-defined.

For discontinuous symbols we need a different asymptotic coefficient. Define the quantity

$$(2.5)\qquad \mathfrak{V}(\partial\Lambda, \partial\Omega) = \frac{1}{(2\pi)^{d+1}} \int\limits_{\partial\Lambda} \int\limits_{\partial\Omega} |\mathbf{n}_{\boldsymbol{\xi}} \cdot \mathbf{n}_{\mathbf{x}}| dS_{\boldsymbol{\xi}} dS_{\mathbf{x}},$$

where $\mathbf{n}_{\mathbf{x}}, \mathbf{n}_{\boldsymbol{\xi}}$ are the exterior unit normals to the surfaces $\partial\Lambda$ and $\partial\Omega$ at the points $\mathbf{x}$ and $\boldsymbol{\xi}$ respectively. Now we can describe the asymptotics of $\operatorname{tr} D(a_T, \alpha\Lambda; f)$.

In the following theorems Ω is the Fermi sea defined in (1.4).

Theorem 2.1. [See [11]] *Let $d \geq 2$. Suppose that Λ satisfies (2.4), and let $X = \{z_1, z_2, \dots, z_N\} \subset \mathbb{R}$, $N < \infty$, be a collection of points on the real line. Suppose that $f \in \mathsf{C}^2(\mathbb{R} \setminus X)$ is a function such that in a neighbourhood of each point $z \in X$ it satisfies the bound*

$$(2.6)\qquad |f^{(k)}(t)| \leq C_k |t - z|^{\gamma - k}, \ k = 0, 1, 2,$$

with some $\gamma > 0$.

Let a_T be as defined in (1.3), $0 < T \leq T_0$, with a fixed $T_0 > 0$. Then

$$(2.7)\qquad \lim_{\substack{T \to 0 \\ \alpha T \geq 1}} \frac{1}{\alpha^{d-1} \log \frac{1}{T}} \operatorname{tr} D(a_T, \alpha\Lambda; f) = U(0, 1; f) \mathfrak{V}(\partial\Lambda, \partial\Omega),$$

and

$$(2.8)\qquad \lim_{\substack{\alpha \to \infty \\ \alpha T \leq 1}} \frac{1}{\alpha^{d-1} \log \alpha} \operatorname{tr} D(a_T, \alpha\Lambda; f) = U(0, 1; f) \mathfrak{V}(\partial\Lambda, \partial\Omega).$$

Note that both fomulas (2.7) and (2.8) require that $T \to 0$. The next theorem treats the case $T = const$, $\alpha \to \infty$.

Theorem 2.2. [See [7]] *Suppose that the region Λ and function f are as in Theorem 2.1. Then*

$$(2.9)\qquad \lim_{\alpha \to \infty} \alpha^{1-d} \operatorname{tr} D(a_T, \alpha\Lambda; f) = \mathcal{B}_d(a_T, \partial\Lambda; f),$$

for each $T > 0$.

The formula (2.9) is proved in [7] for much more general smooth symbols. At this point one should recall that this formula was established first by H. Widom in [12] even in the matrix case, but for smooth domains Λ and smooth functions f. We emphasize that the result of [7] (just as that of [11]) holds for non-smooth functions f and piece-wise smooth Λ.

As we see from the next theorem, the asymptotic formulas in Theorems 2.1 and 2.2 are in agreement with each other. We show this by comparing the asymptotic coefficients in (2.7) and (2.8) with the one in (2.9).

Theorem 2.3. [See [11]] *Suppose that the region* Λ *and function* f *are as in Theorem 2.1. Then*

$$\lim_{T\to 0} \frac{1}{\log\frac{1}{T}} \mathcal{B}_d(a_T; \partial\Lambda, f) = U(0,1;f)\mathfrak{W}(\partial\Lambda, \partial\Omega). \tag{2.10}$$

Therefore the formula (2.7) can be rewritten in the form (2.9), and can be viewed as an extension of (2.9) to the asymptotics in two parameters, α and T, as $\alpha \to \infty$ and $\alpha T \ge 1$. Note that (2.8) cannot be rewritten in the same way.

2.2. **Entropy: large-scale behaviour.** The regions Λ and Ω are the same as before.

In order to study the entropy we use Theorems 2.1 and 2.2 with the γ*-Rényi entropy function* $\eta_\gamma : \mathbb{R} \mapsto [0,\infty)$, defined for all $\gamma > 0$ as follows. If $\gamma \ne 1$, then

$$\eta_\gamma(t) := \begin{cases} \frac{1}{1-\gamma}\log\left[t^\gamma + (1-t)^\gamma\right] & \text{for } t \in (0,1), \\ 0 & \text{for } t \notin (0,1), \end{cases} \tag{2.11}$$

and for $\gamma = 1$ (the von Neumann case) it is defined as the limit

$$\eta_1(t) := \lim_{\gamma\to 1} \eta_\gamma(t) = \begin{cases} -t\log(t) - (1-t)\log(1-t) & \text{for } t \in (0,1), \\ 0 & \text{for } t \notin (0,1). \end{cases} \tag{2.12}$$

For $\gamma \ne 1$ the function η_γ satisfies condition (2.6) with γ replaced with $\varkappa = \min\{\gamma, 1\}$, and with $X = \{0,1\}$. The function η_1 satisfies (2.6) with an arbitrary $\gamma \in (0,1)$, and the same set X.

Various entropies were studied in [4], [5] and [6]. For the sake of illustration we discuss only *the* γ*-Rényi entanglement entropy (EE)* with respect to the bipartition $\mathbb{R}^d = \Lambda \cup (\mathbb{R}^d \setminus \Lambda)$, as defined in [6, Section 10]:

$$\mathrm{H}_\gamma(T,\mu;\alpha\Lambda) := \operatorname{tr} D(a_{T,\mu}, \alpha\Lambda; \eta_\gamma) + \operatorname{tr} D(a_{T,\mu}, \mathbb{R}^d \setminus \alpha\Lambda; \eta_\gamma). \tag{2.13}$$

We are interested in the behaviour of this quantity when $T \to 0$ and $\alpha \to \infty$.

Theorem 2.4. *Let* $d \ge 2$. *The EE satisfies*

$$\lim_{\substack{T\to 0\\ \alpha T\ge 1}} \frac{1}{\alpha^{d-1}\log\frac{1}{T}} \mathrm{H}_\gamma(T,\mu;\alpha\Lambda) = \pi^2\frac{1+\gamma}{3\gamma}\mathfrak{W}(\partial\Lambda, \partial\Omega), \tag{2.14}$$

and

$$\lim_{\substack{\alpha\to\infty \\ \alpha T\le 1}} \frac{1}{\alpha^{d-1}\log\alpha}\mathrm{H}_\gamma(T,\mu;\alpha\Lambda) = \pi^2\frac{1+\gamma}{3\gamma}\mathfrak{W}(\partial\Lambda,\partial\Omega). \tag{2.15}$$

If $T>0$ is fixed, then

$$\lim_{\alpha\to\infty} \alpha^{1-d}\mathrm{H}_\gamma(T,\mu;\alpha\Lambda) = 2\mathfrak{B}_d(a_{T,\mu},\partial\Lambda;\eta_\gamma). \tag{2.16}$$

Proof. Formulas (2.14) and (2.15) follow from (2.7) and (2.8) respectively upon observing (cf. [4]) that

$$U(0,1;\eta_\gamma) = \int_0^1 \frac{\eta_\gamma(t)}{t(1-t)}dt = \pi^2\frac{1+\gamma}{6\gamma}.$$

The formula (2.16) is a direct consequence of (2.9). □

For $d=1$ and $\alpha T\ge 1$ Theorem 2.4 was proved in [6]. We also stress that the formula (2.15) agrees with the large-scale asymptotics of the entropy H_γ for the zero temperature case, which were found in [4].

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