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RESONANCES FOR OBSTACLES IN HYPERBOLIC SPACE

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We consider scattering by star-shaped obstacles in hyperbolic space and show that for the Dirichlet problem resonances satisfy a universal bound

$$|\operatorname{Im}\lambda| \geq \frac{1}{2}$$

which is optimal in dimension 2. In odd dimensions we also show that

$$|\operatorname{Im}\lambda| \geq \frac{\mu}{\rho},$$

for a universal constant μ, where ρ is the radius of a ball containing the obstacle; this gives an improvement for small obstacles. That gives lower bounds on the rate of exponential decay of waves outside of the obstacle.

In dimensions 3 and higher the proofs follow the classical vector field approach of Morawetz, while in dimension 2 we obtain our bound by working with spaces coming from general relativity. The lattter approach is inspired by the works of Vasy [Va13] and Hintz–Vasy [HiVa15]. We also show that in odd dimensions resonances of small obstacles are close, in a suitable sense, to Euclidean resonances. The full account of the results in presented in [HiZw17a].

For $\kappa > 0$ we define hyperbolic n-space with constant curvature $-\kappa^2$ as

$$(\mathbb{H}^n_\kappa, g_\kappa) = (\mathbb{R}^n, dr^2 + s_\kappa^2 h), \tag{1}$$

where (r,ω) are polar coordinates on $\mathbb{R}^n$, $h = h(\omega, d\omega)$ is the round metric on $\mathbb{S}^{n-1}$, and $s_\kappa(r) = \kappa^{-1}\sinh(\kappa r)$. We include Euclidean space as the case of $\kappa = 0$, $s_0(r) = r$.

Suppose that $\mathcal{O} \subset \mathbb{R}^n \simeq \mathbb{H}^n_\kappa$ is a bounded open set with smooth boundary, and denote by

$$P_\kappa := -\Delta_{g_\kappa} - \left(\tfrac{n-1}{2}\right)^2 \kappa^2 \tag{2}$$

the self-adjoint operator on $L^2(\mathbb{H}^n_\kappa \setminus \mathcal{O}, d\operatorname{vol}_{g_\kappa})$ with domain

$$\mathcal{D}(P_\kappa) := H^2(\mathbb{H}^n_\kappa \setminus \mathcal{O}) \cap H^1_0(\mathbb{H}^n_\kappa \setminus \mathcal{O}).$$

The resolvent of P_κ, $\kappa > 0$,

$$R_\kappa(\lambda) := (P_\kappa - \lambda^2)^{-1} \colon L^2(\mathbb{H}^n_\kappa \setminus \mathcal{O}) \to L^2(\mathbb{H}^n_\kappa \setminus \mathcal{O}), \quad \operatorname{Im}\lambda > 0, \tag{3}$$

continues meromorphically to a family of operators defined on $\mathbb{C}$:

$$R_\kappa(\lambda) \colon L^2_{\mathrm{comp}}(\mathbb{H}^n_\kappa \setminus \mathcal{O}) \to L^2_{\mathrm{loc}}(\mathbb{H}^n_\kappa \setminus \mathcal{O}).$$

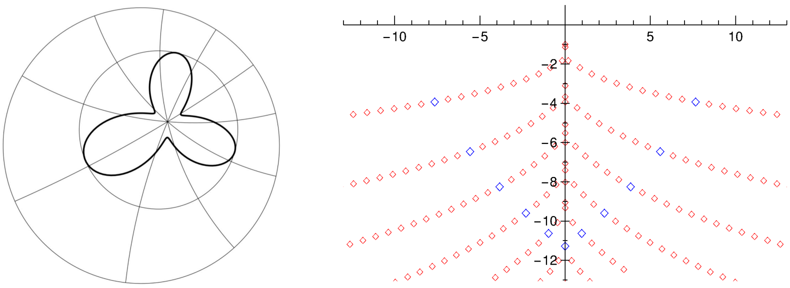

FIGURE 1. *Left:* a star-shaped obstacle in the Poincaré disc with resonances satisfying a universal bound $\operatorname{Im}\lambda \le -\frac{1}{2}$. *Right:* resonances of a disk with radius $R = 1$ in $\mathbb{H}^2$. Highlighted are resonances corresponding to the angular momentum $\ell = 12$.

For $\kappa = 0$, the same result is true when n is odd; in even dimensions the continuation takes place on the logarithmic plane.

We denote the set of poles of $R_\kappa(\lambda)$ (included according to their multiplicities) by $\operatorname{Res}(\mathcal{O}, \kappa)$. The elements of $\operatorname{Res}(\mathcal{O}, \kappa)$ are called *scattering resonances* and they determine decay and oscillations of reflected waves outside of $\mathcal{O}$ – see [Zw17] for a recent survey and references. In the odd-dimensional Euclidean case their study goes back to classical works of Lax–Phillips [LaPh68] and Morawetz [Mo66a], and the relation between the distribution of resonances and the geometry of obstacles has been much studied, especially for high energies ($|\operatorname{Re}\lambda| \to \infty$) – see [Zw17, §2.4].

When the obstacle is star-shaped, a universal lower bound on *resonance widths*, $|\operatorname{Im}\lambda|$, can be given in terms of the radius of the support of the obstacle. Following earlier contributions of Morawetz [Mo66a],[Mo66b],[Mo72] and using Lax–Phillips theory [LaPh68], Ralston [Ra78] obtained the bound

$$\mathcal{O} \subset B_{\mathbb{R}^n}(x_0, \rho) \implies \inf_{\lambda \in \operatorname{Res}(\mathcal{O}, 0)} |\operatorname{Im}\lambda| \ge \rho^{-1} \tag{4}$$

for odd $n \ge 3$. Remarkably this bound is optimal in dimensions three and five – see Fig. 2 and [HiZw17b] for a discussion of this result.

In this paper we investigate analogues of (4) for $\mathcal{O} \subset B_{\mathbb{H}^n_\kappa}(x_0, \rho)$. The first result shows that the resonance widths have a universal lower bound independent of the diameter of the obstacle. Intuitively this is due to the fact that infinity is much "larger" in the hyperbolic case.

Theorem 1. *Suppose that $\mathcal{O} \subset \mathbb{H}^n_\kappa$ is a star-shaped obstacle. Then*

$$\inf_{\lambda \in \operatorname{Res}(\mathcal{O}, \kappa)} |\operatorname{Im}\lambda| \ge \kappa/2. \tag{5}$$

When $n \geq 3$ the proof is based on the vector field method of Morawetz; to obtain an argument valid also when $n = 2$ (where the estimate is sharp when $\mathcal{O} = \emptyset$) we use an approach based on ideas from general relativity and estimates on resonant states. The hyperbolic space version of Morawetz's estimate for $n \geq 3$ and a slight refinement of the argument from [Mo66a] gives an improvement for small obstacles in odd dimensions; this is due to the sharp Huyghens principle.

Theorem 2. *Suppose that $\mathcal{O} \subset \mathbb{H}^n_\kappa$ is a star-shaped obstacle and that $n \geq 3$ is* odd. *Then*

$$\mathcal{O} \subset B_{\mathbb{H}^n_\kappa}(x_0, \rho) \implies \inf_{\lambda \in \mathrm{Res}(\mathcal{O},\kappa)} |\operatorname{Im} \lambda| \geq \mu \rho^{-1} \tag{6}$$

for a universal constant μ.

Remark. Jens Marklof suggested a formulation of Theorems 1 and 2 which does not depend on κ: there exist constants c_n such that for star-shaped obstacles $\mathcal{O} \subset \mathbb{H}^n_\kappa$, n odd,

$$\mathcal{O} \subset B_{\mathbb{H}^n_\kappa}(x_0, \rho) \implies \inf_{\lambda \in \mathrm{Res}(\mathcal{O},\kappa)} |\operatorname{Im} \lambda| \geq c_n \frac{\mathrm{vol}(\partial B_{\mathbb{H}^n_\kappa}(0,\rho))}{\mathrm{vol}(B_{\mathbb{H}^n_\kappa}(0,\rho)))}.$$

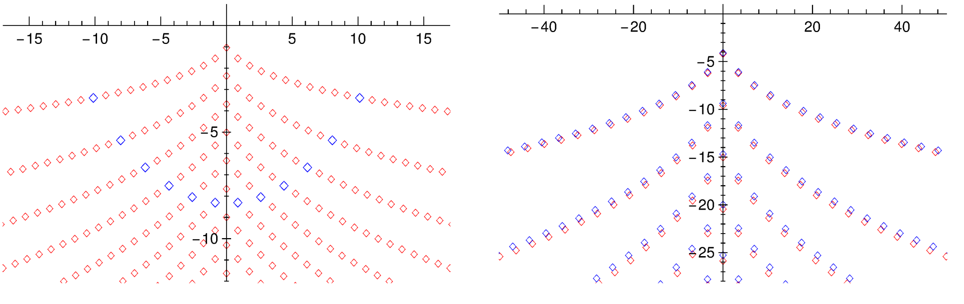

FIGURE 2. *Left:* resonances for the ball of radius one in $\mathbb{R}^3$. For each spherical momentum ℓ they are given by solutions of $H^{(2)}_{\ell+1/2}(\lambda) = 0$ where $H^{(2)}_\nu$ is the Hankel function of the second kind and order ν. Each zero appears as a resonance of multiplicity $2\ell + 1$; highlighted are resonances corresponding to $\ell = 12$. *Right:* resonances of the ball with radius $R = 0.25$ in $\mathbb{H}^3$ (red) and in $\mathbb{R}^3$ (blue); this illustrates Theorem 3.

We expect that $\mu = 1$ in (6). (An adaptation of Ralston's argument [Ra78] should work but would require some buildup of scattering theory; for a proof of his crucial estimate without using Lax–Phillips theory, see [DyZw, Exercise 3.5].) That the estimate (6) is independent of κ is related to rescaling: identifying an obstacle with a subset of $\mathbb{R}^n$ and denoting by $x \mapsto \varepsilon x$ the Euclidean dilation, we see that if $\sigma \in \mathrm{Res}(\varepsilon\mathcal{O}, 1)$ then $\varepsilon\sigma \in \mathrm{Res}(\mathcal{O}, \varepsilon)$, and $\varepsilon\sigma$ should be close to a resonance in $\mathrm{Res}(\mathcal{O}, 0)$. So even though the bound (5) gets worse for small κ, the bound in odd dimensions is close to (4) and improves for small diameters. This is illustrated by Fig. 2 and confirmed by the following theorem:

Theorem 3. *Suppose that $\mathcal{O} \subset \mathbb{H}^n_\kappa \simeq \mathbb{R}^n$ is an arbitrary bounded obstacle with smooth boundary and that $n \geq 3$ is odd. Then*

$$\mathrm{Res}(\mathcal{O}, \kappa) \to \mathrm{Res}(\mathcal{O}, 0), \quad \kappa \to 0,$$

locally uniformly and with multiplicities.

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