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Note on Generalized Root Systems and Generalized Quantum Groups

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Abstract: We introduce applications of generalized root systems to generalized quantum groups.

1 Introduction (History)

In [HY08], the axiomatic definition of generalized root systems is introduced. It is improved in [CH09]. Those definitions use semigroup terminology, or categorical terminology. In [Y15] (see also [BY18]), those were rewritten without using categorical terminology. Weyl groupoids are naturally defined associated to the generalized root systems. Recall the Matsumoto therorem for the Coxeter group $\langle s_i | i \in I\rangle$, which tells that two reduced expression of the same element of the Coxeter group can be transformed from one to the other be repetition of changing expression by

$$(M2)\ \underbrace{s_i s_i s_j \cdots}_{m_{ij}} = \underbrace{s_j s_j s_i \cdots}_{m_{ij}} \text{ with } i, j \in I,\ i \neq j \text{ and } m_{ij} := |\{(s_i s_i)^k | k \in \mathbb{Z}\}| < \infty.$$

Note that the defining relations of the Coxeter group are composed of $(M2)$ and $(M1)$ $s_i^2 = i$ $(i \in I)$, called the Coxeter relations. In [HY08], it is shown that Matsumoto-type theorem for the Weyl groupoids holds and that the defining relations of the Weyl groupoids are formed by the Coxeter-type relations. In AY18, we introduced Nil-Hecke algeberas and a Bruhat order of the Weyl groupoids (see Theorem 3 of this paper). As applications of the Weyl groupoids, wa have achieved:

(1) Although Weyl groupoids had not been introduced, the Serre-type defining relations of the affine Lie superalgebras are obtained in [Y99]. This reproved the Serre-type defining relations of the finite dimensional simple Lie superalgebras of type A-G obtained in [Y94]. In [Y94] and [Y99], we also got the Serre-type defining relations of the finite and affine type quantum superalgebras in some way. In [AAY11], we got the Serre-type defining relations of finite-super-$ABCD$-type Nichols algebras of diagonal-type (including multi-parameter finite-$ABCD$-type quantum superalgebras).

(2) We got the Drinfeld second realizations of the $A^{(1)}(m,n)$-type (resp, $D^{(1)}(2,1;x)$-type) quantum superalgebras in [Y99] (resp. [HSTY08]).

(3) We got the Shapovalop determinant formula

$$\det \mathrm{Shap}_\lambda^{\chi,\pi} = \prod_{\alpha\in R^+(\chi,\pi)} \prod_{t_\alpha=1}^{\infty} (-\hat{\rho}^{\chi,\pi}(\alpha)\chi(\alpha,\alpha)^{-t_\alpha}K_\alpha + L_\alpha)^{|\mathfrak{P}_\lambda^{\chi,\pi}(\alpha;t_\alpha)|}.$$

of the finite-type generalized quantum groups $U(\chi,\pi)$ in [HY10, Theorem 7.3], where $\mathrm{Shap}_\lambda^{\chi,\pi}$ means the Shapovalop matrix of the weight λ and where see [BY18, Theorem 7.5] for detail of the right hand side; we needed the condition that $\chi(\alpha,\alpha)\neq 0$ for $\alpha\in R^+(\chi,\pi)$. Finite-type $U(\chi,\pi)$ can be finite-type quantum groups and finite-type quantum superalgebras and their Lusztig's small quantum groups.

(4) We got the classification theorem of the finite-dimensional irreducible representations of the finite-type non-finite dimensional $U(\chi,\pi)$ in [AYY15] over zero-characteristic field, and we also recover in [AYY15] the Kac's list of the classification theorem of the finite-dimensional simple modules of the finite-dimensional simple Lie superalgebras of type A-G.

(5) We got the explicit formula of the universal R-matrix of the finite-type $U(\chi,\pi)$ in [AY15].

(6) We got the Harish-Chandra theorem for $U(\chi,\pi)$ in (3) in [BY18], see Section 3 of this paper.

(7) We got the Kostant-Lusztig $\mathbb{A}$-form of the finite-type multi parametet quantum groups in [JMY17].

2 Generalized Root Systems

Let I be a non-empty finite set. Let V be a $\mathbb{R}$-linear space with a $\mathbb{R}$-basis $\{v_i|i\in I\}$, so $\dim_{\mathbb{R}} V = |I|$ and $V =: \oplus_{i\in I}\mathbb{R}v_i$. Let $V_{\mathbb{Z}} =: \oplus_{i\in I}\mathbb{Z}v_i$. Then $V_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module with a $\mathbb{Z}$-basis $\{v_i|i\in I\}$, and $\mathrm{rank}_{\mathbb{Z}}V_{\mathbb{Z}} = |I|$. Let $\mathcal{P}(V_{\mathbb{Z}})$ be a power set of $V_{\mathbb{Z}}$. Let $\mathcal{B}(V_{\mathbb{Z}})$ be a set of all $\mathbb{Z}$-bases of $V_{\mathbb{Z}}$, so $\mathcal{B}(V_{\mathbb{Z}})\subset\mathcal{P}(V_{\mathbb{Z}})$.

Let $R\in\mathcal{P}(V_{\mathbb{Z}})$. Assume $R\neq\emptyset$. For $B\in\mathcal{B}(V_{\mathbb{Z}})$, let $R^{B,+} := R\cap \mathrm{Span}_{\mathbb{Z}_{\geq 0}}B$ and $R^{B,-} := R\cap\mathrm{Span}_{\mathbb{Z}_{\leq 0}}B$. We say that B is *a base* of R if the following (x) and

(y) hold:

(x) $R = R^{B,+} \cup R^{B,-}$.
(y) $\forall \alpha \in B,\ \mathbb{Z}\alpha \cap R = \{\alpha, -\alpha\}$.

Let $\mathcal{B}(R)$ be a set of all bases of R, so $\mathcal{B}(R) \subset \mathcal{B}(V_{\mathbb{Z}})$.

Let $R \in \mathcal{P}(V_{\mathbb{Z}})$. Assume $R \neq \emptyset$. Let $\mathbb{B}$ be a non-empty subset of $\mathcal{B}(R)$. We say that $(R, \mathbb{B})$ is *a generalized root system* [HY08] (see also [BY18, Y15]) if

$$\forall B \in \mathbb{B},\ \forall \alpha \in B,\ \exists \tau_\alpha(B) \in \mathbb{B},\ R^{\tau_\alpha(B),+} \cap R^{B,-} = \{-\alpha\}.$$

For $R \in \mathcal{P}(V_{\mathbb{Z}})$, let $\mathcal{M}(R)$ be the set of all maps from I to R.

Theorem 2.1. [BY18, HY08, Y15]. *Let $(R, \mathbb{B})$ be a generalized root system. Then there exist a non-empty subset $\check{\mathbb{B}}$ of $\mathcal{M}(R)$ and bijections $\tau_i : \check{\mathbb{B}} \to \check{\mathbb{B}}$ $(i \in I)$ satisfying* (a)-(c) *below.*

(a) *The map $\varphi : \check{\mathbb{B}} \to \mathcal{P}(R)$ defined by $\varphi(\pi) := \pi(I)$ is injection, where $\mathcal{P}(R)$ is the power set of R.*

(b) $\varphi(\check{\mathbb{B}}) = \mathbb{B}$.

(c) $\forall \pi \in \check{\mathbb{B}},\ \forall i \in I,\ (\tau_i(\pi))(I) = \tau_{\pi(i)}(\pi(I))$.

In particular, for $\pi \in \check{\mathbb{B}}$ and $i, j \in I$, there exist $N^\pi_{ij} \in \mathbb{Z}$ such that $(\tau_i(\pi))(j) = \pi(j) + N^\pi_{ij}\pi(i)$, which implies that $N^\pi_{ii} = -2$ and $N^\pi_{ij} \in \mathbb{Z}_{\geq 0}$ $(j \neq i)$. Moreover, $(\tau_i)^2 = \mathrm{id}_{\check{\mathbb{B}}}$ and $N^{\tau_i(\pi)}_{ij} = N^\pi_{ij}$.

Let $\pi \in \check{\mathbb{B}}$. For a map $f : \mathbb{N} \to I$ and $t \in \mathbb{Z}_{\geq 0}$, define $\pi_{f,t} \in \check{\mathbb{B}}$ by $\pi_{f,0} := \pi$ and $\pi_{f,t} := \tau_{f(t)}(\pi_{f,t-1})$ $(t \in \mathbb{N})$.

Lemma 2.2. *Assume $|R| < \infty$. Let $k := \frac{|R|}{2}$. Then $R^{\pi(I),+} = \{\pi_{f,t-1}(f(t)) | t \in \mathbb{N}, 1 \leq t \leq k\}$ and $R^{\pi(I),-} = -R^{\pi(I),+}$.*

Let $\mathbb{J}$ be the set of all maps $f : \mathbb{N} \to I$.

Let $(R, \mathbb{B})$ be a generalized root system. For $\pi \in \check{\mathbb{B}}$ and $i \in I$, define the $\mathbb{Z}$-module automorphism $s^\pi_i : V \to V$ by $s^\pi_i(v_j) := v_j + N^\pi_{ij} v_i$. For $\pi \in \check{\mathbb{B}}$ and $f \in \mathbb{J}$, let $\pi_{f,0} := \pi$ and $\pi_{f,t} := \tau_{f(t)}(\pi_{f,t-1})$, and let $1^\pi s_{f,0} := \mathrm{id}_V$ and $1^\pi s_{f,t} = (1^\pi s_{f,t-1}) \circ s^{\pi_{f,t}}_{f(t)}$. Let

$$\ell^\pi_{f,t} := \min\{r \in \mathbb{Z}_{\geq 0} | \exists g \in \mathbb{J},\ 1^\pi s_{g,r} = 1^\pi s_{f,t}\}.$$

Theorem 2 [HY08]. *We have*

$$\ell^\pi_{f,t} = |R^{\pi_{f,t}(I),+} \cap R^{\pi,-}|.$$

In particular, if $1^\pi s_{f,t} = 1^\pi s_{g,r}$*, then* $\pi_{f,t} = \pi_{g,r}$*.*

Let $(R, \mathbb{B})$ be a generalized root system. Let $\check{\mathbb{B}}_\lambda$ $(\lambda \in \Lambda)$ be non-empty subsets of $\check{\mathbb{B}}$ such that

(i) $\cup_{\lambda\in\Lambda}\check{\mathbb{B}}_\lambda = \mathbb{B}$ and $\check{\mathbb{B}}_\lambda \cap \check{\mathbb{B}}_\mu = \emptyset$ $(\lambda \neq \mu)$.

(ii) For $\lambda \in \Lambda$ and $\pi, \pi' \in \check{\mathbb{B}}_\lambda$, defining the $\mathbb{Z}$-module automorphism $p : V \to V$ by $p(\pi(i)) := \pi'(i)$ $(i \in I)$, we have $p(R^{\pi(I),+}) = R^{\pi'(I),+}$.

(iii) $\forall\lambda \in \Lambda$, $\forall i \in I$, $\exists\mu \in \Lambda$, $\tau_i(\check{\mathbb{B}}_\lambda) = \check{\mathbb{B}}_\mu$.

For π, $\pi' \in \check{\mathbb{B}}$, we write $\pi \equiv \pi'$ if $\{\pi, \pi'\} \subset \check{\mathbb{B}}_\lambda$ for some $\lambda \in \Lambda$.

Theorem 3 [AY18]. *Assume that* $\ell^\pi_{f,k} = k$ *and* $1^\pi s_{f,k} = 1^\pi s_{g,k}$*. Assume that there exists a non-empty proper subset* $S = \{i_1, \ldots, i_x\}$ $(i_{t-1} < i_t)$ *of* $\{1, \ldots, k\}$ *such that* $\pi_{f,t-1} \equiv \pi_{f,t}$ $(t \in \{1, \ldots, k\} \setminus S)$ *and* $\ell^\pi_{f',x} = x$*, where* $f' \in \mathbb{J}$ *is such that* $f'(1) := i_1$*, …,* $f'(x) := i_x$*. Then there exists a non-empty proper subset* $T = \{j_1, \ldots j_x\}$ $(j_{t-1} < j_t)$ *of* $\{1, \ldots, k\}$ *such that* $\pi_{g,t-1} \equiv \pi_{g,t}$ $(t \in \{1, \ldots, k\} \setminus T)$ *and* $1^\pi s_{f',x} = 1^\pi s_{g',x}$*, where* $g' \in \mathbb{J}$ *is such that* $g'(1) := j_1$*, …,* $g'(x) := j_x$*. (Note that* $|S| = x = |T|$*.)*

3 Generalized Quantum Groups

Let $\mathbb{K}$ be an algebraically closed field. Let $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$. Let $\chi : V_\mathbb{Z} \times V_\mathbb{Z} \to \mathbb{K}^\times$ be a map such that $\chi(\lambda, \mu+\nu) = \chi(\lambda, \mu)\chi(\lambda, \nu)$ and $\chi(\lambda+\mu, \nu) = \chi(\lambda, \nu)\chi(\mu, \nu)$ for all λ, μ, $\nu \in V_\mathbb{Z}$. Let $\pi \in \check{\mathbb{B}}$. Let $U = U(\chi, \pi)$ be the $\mathbb{K}$-algebra (with 1) satisfying the following conditions $(U1)$-$(U5)$. Existence and uniqueness of U is well-known.

$(U1)$ U is generated by the elements K_λ, L_λ $(\lambda \in V_\mathbb{Z})$ and E_i, F_i $(i \in I)$ satisfying the equations $K_0 = L_0 = 1$, $K_\lambda K_\mu = K_{\lambda+\mu}$, $L_\lambda L_\mu = L_{\lambda+\mu}$, $K_\lambda L_\mu = L_\mu K_\lambda$, $K_\lambda E_i K_{-\lambda} = \chi(\lambda, \pi(i))E_i$, $K_\lambda F_i K_{-\lambda} = \chi(\lambda, -\pi(i))F_i$, $L_\lambda E_i L_{-\lambda} = \chi(-\pi(i), \lambda)E_i$, $L_\lambda F_i L_{-\lambda} = \chi(\pi(i), \lambda)F_i$, $E_i F_j - F_j E_i = \delta_{ij}(-K_{\pi(i)} + L_{\pi(i)})$.

$(U2)$ There exist the $\mathbb{K}$-subspaces $U_\lambda = U(\chi, \pi)_\lambda$ $(\lambda \in V_\mathbb{Z})$ of U such that $U = \oplus_{\lambda\in V_\mathbb{Z}} U_\lambda$, $U_\lambda U_\mu \subset U_{\lambda+\mu}$, $K_\lambda \in U_0$, $L_\lambda \in U_0$, $E_i \in U_{\pi(i)}$, $F_i \in U_{-\pi(i)}$.

$(U3)$ Let $U^0 = U^0(\chi, \pi)$ (resp. $U^+ = U^+(\chi, \pi)$), resp. $U^- = U^-(\chi, \pi)$) be the $\mathbb{K}$-subalgebra (with 1) generated by $K_\lambda L_\mu$ $(\lambda, \mu \in V_\mathbb{Z})$ (resp. E_i $(i \in I)$, resp. F_i $(i \in I)$). Define the $\mathbb{K}$-linear map $\jmath_1 : U^- \otimes U^0 \otimes U^+ \to U$ by $\jmath_1(Y \otimes Z \otimes X) := YZX$. Then $\jmath_1$ is a $\mathbb{K}$-linear isomorphism.

$(U4)$ Define the $\mathbb{K}$-linear map $\jmath_2 : V_\mathbb{Z} \times V_\mathbb{Z} \to U^0$ by $\jmath_2(\lambda, \mu) := K_\lambda L_\mu$. Then $\jmath_2$ is injective, and $\jmath_2(V_{\mathbb{Z}} \times V_{\mathbb{Z}})$ is a $\mathbb{K}$-basis of U^0.

$(U5)$ We have $\{X \in U^+ | E_i X = X E_i (\forall i \in I)\} = U^+ \cap U_0$ and $\dim_{\mathbb{K}}\{Y \in U^- | F_i Y = Y F_i (\forall i \in I)\} = U^- \cap U_0$.

Let $V_{\mathbb{Z}}^{\pi,+} := \oplus_{i \in I} \mathbb{Z}_{\geq 0}\pi(i) (\subset V_{\mathbb{Z}})$. For $\lambda \in V_{\mathbb{Z}}$, let $U_\lambda^+ = U^+(\chi,\pi)_\lambda := U^+ \cap U_\lambda$ and $U_\lambda^- = U^-(\chi,\pi)_\lambda := U^- \cap U_\lambda$. Then $U^+ = \oplus_{\lambda \in V_{\mathbb{Z}}^{\pi,+}} U_\lambda^+$, $U^- = \oplus_{\lambda \in V_{\mathbb{Z}}^{\pi,+}} U_{-\lambda}^-$, and $U_0^+ = U_0^- = \mathbb{K}1$. We also have $U_{\pi(i)}^+ = \mathbb{K}E_i$ and $U_{-\pi(i)}^- = \mathbb{K}F_i$ $(i \in I)$.

For $n \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{K}^\times$, let $(n)_t := \sum_{x=1}^n t^{x-1}$ and $(n)_t! := \prod_{m=1}^n (m)_t$. For a set X and a map $f : X \to \mathbb{N}$, let $\mathfrak{M}(X,f) := \{(x,k) \in X \times \mathbb{N} | \forall x \in X, 1 \leq k \leq f(x)\}$ and define the map $p^{\langle X,f \rangle} : \mathfrak{M}(X,f) \to X$ by $p^{\langle X,f \rangle}(x,k) := x$. For $Y \in \mathcal{P}(V_{\mathbb{Z}})$ and a map $f : Y \to \mathbb{N}$, let $\mathfrak{P}^{\langle Y,f \rangle}$ be the set of maps $g : \mathfrak{M}(Y,f) \to \mathbb{Z}_{\geq 0}$ such that $(g(y))_{\chi(p^{\langle Y,f \rangle}(y), p^{\langle Y,f \rangle}(y))}! \neq 0$ for all $y \in \mathfrak{M}(Y,f)$ and such that $|g^{-1}(\mathbb{N})| < \infty$. For $\lambda \in V_{\mathbb{Z}}^{\pi,+}$, $Z \in \mathcal{P}(V_{\mathbb{Z}}^{\pi,+})$ and a map $f : Z \to \mathbb{N}$, let $\mathfrak{P}_\lambda^{\langle Z,f \rangle} := \{g \in \mathfrak{P}^{\langle Z,f \rangle} | \sum_{z \in \mathfrak{M}(Z,f)} g(z) p^{\langle Z,f \rangle}(z) = \lambda\}$.

Theorem 3.1. (*Kharchenko's PBW theorem* [Kha99]) *There exists a unique pair* $((R^+(\chi,\pi), \varphi_+^{\chi,\pi})$ *of* $R^+(\chi,\pi) \in \mathcal{P}(V_{\mathbb{Z}}^{\pi,+})$ *and a map* $\varphi_+^{\chi,\pi} : Z \to \mathbb{N}$ *such that* $\dim U^+(\chi,\pi)_\lambda = \dim U^-(\chi,\pi)_{-\lambda} = |\mathfrak{P}_\lambda^{\langle R^+(\chi,\pi), \varphi_+^{\chi,\pi} \rangle}|$ *for all* $\lambda \in V_{\mathbb{Z}}^{\pi,+}$.

Let $R(\chi,\pi) := R^+(\chi,\pi) \cup (-R^+(\chi,\pi)) (\in \mathcal{P}(V_{\mathbb{Z}}))$.

Theorem 3.2. (*Heckenberger's Weyl groupoid theorem* [Hec06]) *If* $|R^+(\chi,\pi)| < \infty$, *then* $R(\chi,\pi)$ *is a generalized root system and* $\varphi_+^{\chi,\pi}(\alpha) = 1$ *for all* $\alpha \in R^+(\chi,\pi)$.

Define the $\mathbb{K}$-linear map $\mathfrak{Sh}^{\chi,\pi} : U(\chi,\pi) \to U^0(\chi,\pi)$ by $\mathfrak{Sh}^{\chi,\pi}{}_{|U(\chi,\pi)^0} = \mathrm{id}_{U(\chi,\pi)^0}$ and $\mathfrak{Sh}^{\chi,\pi}(\mathrm{Span}_{\mathbb{K}}(U^-(\chi,\pi)_{-\lambda} U^0(\chi,\pi) U^+(\chi,\pi)_\mu) = \{0\}$ for λ, $\mu \in V_{\mathbb{Z}}^{\pi,+}$ with $\lambda + \mu \neq 0$.

Let $\omega : V_{\mathbb{Z}} \to \mathbb{K}^\times$ be a map such that $\omega(\lambda+\mu) = \omega(\lambda)\omega(\mu)$ for all λ, $\mu \in V_{\mathbb{Z}}$. Let $\mathfrak{Z}_\omega(\chi,\pi) := \{Z \in U(\chi,\pi)_0 | \forall \lambda \in V_{\mathbb{Z}}, \forall X \in U(\chi,\pi)_\lambda, ZX = \omega(\lambda) XZ\}$. Let $\mathfrak{HC}_\omega^{\chi,\pi} := \mathfrak{Sh}^{\chi,\pi}{}_{|\mathfrak{Z}_\omega(\chi,\pi)}$. Define the map $\hat{\rho}^{\chi,\pi} : V_{\mathbb{Z}} \to \mathbb{K}^\times$ by $\hat{\rho}^{\chi,\pi}(\sum_{i \in I} k_i \pi(i)) := \prod_{i \in I} \chi(\pi(i),\pi(i))^{k_i}$, where $k_i \in \mathbb{Z}$.

Lemma 3.3. ([BY18, Lemma 9.2]) $\mathfrak{HC}_\omega^{\chi,\pi}$ *is injective.*

Assume $|R^+(\chi,\pi)| < \infty$ and assume $\chi(\alpha,\alpha) \neq 1$ for all $\alpha \in R^+(\chi,\pi)$. Let $\beta \in R^+(\chi,\pi)$. Let $q_\beta := \chi(\beta,\beta)$. Let $c_\beta := 0$ if $q_\beta^n \neq 1$ for all $n \in \mathbb{N}$, and let $c_\beta := \min\{m \in \mathbb{N} | q_\beta^m = 1\}$ if $q_\beta^m = 1$ for some $m \in \mathbb{N}$. Let $\mathfrak{B}_\omega^{\chi,\pi}(\beta)$ be the $\mathbb{K}$-subspace of $U^0(\chi,\pi)$ formed by the elements $\sum_{\lambda,\mu \in V_{\mathbb{Z}}} a_{(\lambda,\mu)} K_\lambda L_\mu$ with $a_{(\lambda,\mu)} \in \mathbb{K}$ satisfying the following conditions $(e1)$-$(e4)$. For λ, $\mu \in V_{\mathbb{Z}}$, let $\omega_{\lambda,\mu;\beta}^{\chi,\pi} := \frac{\omega(\beta)\chi(\beta,\mu)}{\chi(\lambda,\beta)}$.

$(e1)$ For λ, $\mu \in V_{\mathbb{Z}}$ and $t \in \mathbb{Z}$, if $c_\beta = 0$ and $q_\beta^t = \omega_{\lambda,\mu;\beta}^{\chi,\pi}$, then $a_{(\lambda+t\beta,\mu-t\beta)} = \hat{\rho}^{\chi,\pi}(\beta)^t a_{(\lambda,\mu)}$.

$(e2)$ For λ, $\mu \in V_{\mathbb{Z}}$, if $c_\beta = 0$ and $q_\beta^t \neq \omega^{\chi,\pi}_{\lambda,\mu;\beta}$ for all $t \in \mathbb{Z}$, then $a_{(\lambda,\mu)} = 0$.

$(e3)$ For λ, $\mu \in V_{\mathbb{Z}}$ and $t \in \mathbb{N}$ with $1 \leq t \leq c_\beta - 1$, if $c_\beta \geq 2$ and $q_\beta^t = \omega^{\chi,\pi}_{\lambda,\mu;\beta}$, then $\sum_{x\in\mathbb{Z}} a_{(\lambda+(c_\beta x+t)\beta,\mu-(c_\beta x+t)\beta)}\hat{\rho}^{\chi,\pi}(\beta)^{-(c_\beta x+t)} = \sum_{y\in\mathbb{Z}} a_{(\lambda+c_\beta y\beta,\mu-c_\beta y\beta)}\hat{\rho}^{\chi,\pi}(\beta)^{-c_\beta y}$.

$(e4)$ For λ, $\mu \in V_{\mathbb{Z}}$, if $c_\beta \geq 2$ and $q_\beta^t \neq \omega^{\chi,\pi}_{\lambda,\mu;\beta}$ for all $t \in \mathbb{Z}$, then for all $t \in \mathbb{N}$ with $1 \leq t \leq c_\beta - 1$, $\sum_{x\in\mathbb{Z}} a_{(\lambda+(c_\beta x+t)\beta,\mu-(c_\beta x+t)\beta)}\hat{\rho}^{\chi,\pi}(\beta)^{-(c_\beta x+t)} = \sum_{y\in\mathbb{Z}} a_{(\lambda+c_\beta y\beta,\mu-c_\beta y\beta)}\hat{\rho}^{\chi,\pi}(\beta)^{-c_\beta y}$.

Let $\mathfrak{B}^{\chi,\pi}_\omega := \cap_{\beta\in R^+(\chi,\pi)}\mathfrak{B}^{\chi,\pi}_\omega(\beta)$.

Theorem 3.4. ([BY18, Theorem 10.4]) *Assume* $|R^+(\chi,\pi)| < \infty$ *and assume* $\chi(\alpha,\alpha) \neq 1$ *for all* $\alpha \in R^+(\chi,\pi)$. *Then* $\mathrm{Im}\mathfrak{H}\mathfrak{C}^{\chi,\pi}_\omega = \mathfrak{B}^{\chi,\pi}_\omega$.

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