数理解析研究所講究録
第2075巻 2018年 67-72

On the expansion coefficients of Tau-functions of the KP and BKP hierarchies

Yoko Shigyo
(Department of Mathematics, Tsuda University)

1 KP hierarchy

For the function $\tau(x)$ of $x=(x_1, x_2, \dots)$ the KP hierarchy [4] is the bilinear equation given by

$$\int \tau(x-y-[k^{-1}])\tau(x+y+[k^{-1}]) \exp\left(-2\sum_{j=1}^{\infty} y_j k^j\right) dk = 0, \tag{1}$$

where $[k^{-1}] = (k^{-1}, k^{-1}/2, k^{-3}/3, \dots)$, $y=(y_1, y_2, \dots)$. The integral denotes taking the coefficient of k^{-1} in the Laurent expansion.

Any formal power series $\tau(x)$ can be expanded as

$$\tau(x) = \sum_{\lambda} \xi_\lambda s_\lambda(x), \tag{2}$$

where λ runs over all partitions.

A subset $M \subset \mathbb{Z}$ is called a Maya diagram of charge c if M satisfies the following conditions:
(i) $\mathbb{Z}_{\geq 0} \cap M$ and $\mathbb{Z}_{<0} \setminus M$ are finite,
(ii) $\#(\mathbb{Z}_{\geq 0} \cap M) - \#(\mathbb{Z}_{<0} \setminus M) = c$.

A partition is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ of nonnegative integers such that $|\lambda| = \sum_{i\geq 1} \lambda_i$ is finite. We identify a partition λ with its Young diagram, which is a left-justified array of $|\lambda|$ cells with λ_i cells in the ith row. Given a partition λ, we put

$$p(\lambda) = \#\{i : \lambda_i \geq i\}, \quad \alpha_i = \lambda_i - i, \quad \beta_i = \lambda'_i - i \quad (1 \leq i \leq p(\lambda)),$$

where λ'_i is the number of cells in the jth column of the Young diagram of λ. Then we write $\lambda = (\alpha_1, \dots, \alpha_{p(\lambda)} | \beta_1, \dots, \beta_{p(\lambda)})$ and call it the Frobenius notation of λ.

Example 1 *If* $\lambda = (3,2,1)$ *the Frobenius notation of* λ *is* $(2,0|2,0)$.

We can identify Maya diagrams M of charge 0 with partitions $\lambda = (\lambda_1, \lambda_2, \cdots) = (\alpha_1, \dots, \alpha_r | \beta_1, \dots \beta_r)$ by way of the following conditions:
(i)

$$M = (\lambda_1 - 1, \lambda_2 - 2, \lambda_3 - 3, \dots),$$

(ii)

$$\mathbb{Z}_{\geq 0} \cap M = \{\alpha_1, \dots, \alpha_r\}, \quad \mathbb{Z}_{<0} \setminus M = \{-\beta_1 - 1, \dots, -\beta_r - 1\}.$$

Example 2 *If $\lambda = (3,2,1) = (2,0|2,0)$, Maya diagram M corresponded to λ becomes*

$$M = (3-1, 2-2, 1-3, -4, -5, \cdots) = (2, 0, -2, -4, -5, \cdots).$$

Proposition 1 *[17] The function $\tau(x)$ given as (2) is a solution of the KP hierarchy if and only if the coefficients $\{\xi[M]\}_M$ satisfy the Plücker relations*

$$\sum_{i\geq 1} (-1)^i \xi[m_1, m_2, \dots, \hat{m}_i, \dots] \xi[m_i, n_1, n_2, \dots] = 0, \tag{3}$$

where $M = (m_1, m_2, \dots)$ is Maya diagram of charge 1 and $N = (n_1, n_2, \dots)$ is Maya diagram of charge -1. The $\hat{m}_i$ means removing m_i from the sequence.

Proposition 2 *The function $\tau(x)$ is a solution of the KP hierarchy if and only if the coefficients ξ_λ satisfy the following Plücker relations:*

$$\sum_{i=1}^{p+1} (-1)^i \xi \begin{pmatrix} m_1, \dots, \widehat{m_i}, \dots, m_{p+1} \\ m'_1, \dots, m'_p \end{pmatrix} \xi \begin{pmatrix} m_i, n_1, \dots, n_q \\ n'_1, \dots, n'_{q+1} \end{pmatrix}$$
$$= \sum_{j=1}^{q+1} (-1)^{p+j} \xi \begin{pmatrix} m_1, \dots, m_{p+1} \\ m'_1, \dots, m'_p, n'_j \end{pmatrix} \xi \begin{pmatrix} n_1, \dots, n_q \\ n'_1, \dots, \widehat{n'_j}, \dots, n'_{q+1} \end{pmatrix}, \tag{4}$$

for any sequences $m_1, \dots, m_{p+1}, m'_1, \dots, m'_p, n_1, \dots, n_q, n'_1, \dots, n'_{q+1}$ of nonnegative integers.

Corollary 1 *The function $\tau(x)$ is a solution of the KP hierarchy if and only if the coefficients ξ_λ satisfy the following Plücker relations:*

$$\xi \begin{pmatrix} a_1, \dots, a_r \\ b_1, \dots, b_r \end{pmatrix} \xi \begin{pmatrix} c_1, \dots, c_s \\ d_1, \dots, d_s \end{pmatrix}$$
$$= \sum_{k=1}^{r} (-1)^{r-k} \xi \begin{pmatrix} a_1, \dots, \widehat{a_k}, \dots, a_r \\ b_1, \dots, b_{r-1} \end{pmatrix} \xi \begin{pmatrix} a_k, c_1, \dots, c_s \\ b_r, d_1, \dots, d_s \end{pmatrix}$$
$$+ \sum_{l=1}^{s} (-1)^{l-1} \xi \begin{pmatrix} a_1, \dots, a_r \\ b_1, \dots, b_{r-1}, d_l \end{pmatrix} \xi \begin{pmatrix} c_1, \dots, c_s \\ b_r, d_1, \dots, \widehat{d_l}, \dots, d_s \end{pmatrix}, \tag{5}$$

and

$$\xi \begin{pmatrix} a_1, \dots, a_r \\ b_1, \dots, b_r \end{pmatrix} \xi \begin{pmatrix} c_1, \dots, c_s \\ d_1, \dots, d_s \end{pmatrix}$$
$$= \sum_{k=1}^{r} (-1)^{r-k} \xi \begin{pmatrix} a_1, \dots, a_{r-1} \\ b_1, \dots, \widehat{b_k}, \dots, b_r \end{pmatrix} \xi \begin{pmatrix} a_r, c_1, \dots, c_s \\ b_k, d_1, \dots, d_s \end{pmatrix}$$
$$+ \sum_{l=1}^{s} (-1)^{l-1} \xi \begin{pmatrix} a_1, \dots, a_{r-1}, c_l \\ b_1, \dots, b_r \end{pmatrix} \xi \begin{pmatrix} a_r, c_1, \dots, \widehat{c_l}, \dots, c_s \\ d_1, \dots, d_s \end{pmatrix}, \tag{6}$$

for any sequence of nonnegative integers $(a_1, \dots, a_r)$, $(b_1, \dots, b_r)$, $(c_1, \dots, c_s)$ and $(d_1, \dots, d_s)$.

2 Main theorem

Fix a partition $\mu = (\gamma_1, \ldots, \gamma_s | \delta_1, \ldots, \delta_s)$. We assume that $\tau(x)$ has the following expansion:

$$\tau(x) = s_\mu(x) + \sum_{\lambda \supset \mu} \xi_\lambda s_\lambda(x). \tag{7}$$

Theorem 1 *[14] The function $\tau(x)$ gievn by (7) is a solution of the KP hierarchy if and only if the expansion coefficitnes $\{\xi_\lambda\}_\lambda$ is the following formulae for a partition $\lambda = (\alpha_1, \ldots, \alpha_r | \beta_1, \ldots, \beta_r)$:*

$$\xi_\lambda = (-1)^s \det \begin{pmatrix} (z_{\alpha_i, \beta_j})_{1 \le i,j \le r} & (u_{\alpha_i}^{(j)})_{1 \le i \le r, 1 \le j \le s} \\ (v_{\beta_j}^{(i)})_{1 \le i \le s, 1 \le j \le r} & O \end{pmatrix}, \tag{8}$$

where $z_{\alpha,\beta}$, $u_\alpha^{(j)}$, $v_\beta^{(i)}$ satisfy

$$\begin{cases} z_{\alpha,\beta} = \xi \begin{pmatrix} \alpha, \gamma_1, \ldots, \gamma_s \\ \beta, \delta_1, \ldots, \delta_s \end{pmatrix}, \\ u_\alpha^{(j)} = \xi \begin{pmatrix} \alpha, \gamma_1, \ldots, \hat{\gamma}_j, \ldots, \gamma_s \\ \delta_1, \ldots, \delta_s \end{pmatrix}, \\ v_\beta^{(i)} = \xi \begin{pmatrix} \gamma_1, \ldots, \gamma_s \\ b, \delta_1, \ldots, \hat{\delta}_i, \ldots, \delta_s \end{pmatrix}. \end{cases} \tag{9}$$

To derive the determinant formulae (8) we need the following lemma.

Lemma 1 *Fix a partition μ. Suppose that $\tau(x)$ given by (7) is a solution of the KP hierarchy. Then ξ_λ can be expressed as a polynomial in*

$$\begin{aligned} I_\mu = &\left\{ \xi \begin{pmatrix} a, \gamma_1, \ldots, \gamma_s \\ b, \delta_1, \ldots, \delta_s \end{pmatrix} : a, b \in \mathbb{Z}_{\ge 0} \right\} \\ &\cup \left\{ \xi \begin{pmatrix} a, \gamma_1, \ldots, \hat{\gamma}_j, \ldots, \gamma_s \\ \delta_1, \ldots, \delta_s \end{pmatrix} : a \in \mathbb{Z}_{\ge 0}, 1 \le j \le s \right\} \\ &\cup \left\{ \xi \begin{pmatrix} \gamma_1, \ldots, \gamma_s \\ b, \delta_1, \ldots, \hat{\delta}_i, \ldots, \delta_s \end{pmatrix} : b \in \mathbb{Z}_{\ge 0}, 1 \le i \le s \right\}. \end{aligned}$$

Example 3 *We consider the case of $\mu = (\gamma | \delta)$ and $\lambda = (\alpha_1, \alpha_2 | \beta_1, \beta_2)$. The set I_μ becomes*

$$I_\mu = \left\{ \xi \begin{pmatrix} \alpha_i, \gamma \\ \beta_j, \delta \end{pmatrix} : i, j = 1, 2 \right\} \cup \left\{ \xi \begin{pmatrix} \alpha_i \\ \delta \end{pmatrix} : i = 1, 2 \right\} \cup \left\{ \xi \begin{pmatrix} \gamma \\ \beta_j \end{pmatrix} : j = 1, 2 \right\}$$

Using(5) we have

$$\xi \begin{pmatrix} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \end{pmatrix} \xi \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = -\xi \begin{pmatrix} \alpha_2 \\ \beta_1 \end{pmatrix} \xi \begin{pmatrix} \alpha_1, \gamma \\ \beta_2, \delta \end{pmatrix} + \xi \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \xi \begin{pmatrix} \alpha_2, \gamma \\ \beta_2, \delta \end{pmatrix} + \xi \begin{pmatrix} \alpha_1, \alpha_2 \\ \beta_1, \delta \end{pmatrix} \xi \begin{pmatrix} \gamma \\ \beta_2 \end{pmatrix}.$$

Similarly using (5) and (6) we have

$$\xi\binom{\alpha_i}{\beta_j} = \xi\binom{\alpha_i}{\delta}\xi\binom{\gamma}{\beta_j}, \xi\binom{\alpha_1,\alpha_2}{\beta_1,\delta} = -\xi\binom{\alpha_1}{\delta}\xi\binom{\alpha_2,\gamma}{\beta_1,\delta} + \xi\binom{\alpha_1,\gamma}{\beta_1,\delta}\xi\binom{\alpha_2}{\delta}$$

Then we have

$$\xi\binom{\alpha_1,\alpha_2}{\beta_1,\beta_2} = -\det\begin{pmatrix} \xi\binom{\alpha_1,\gamma}{\beta_1,\delta} & \xi\binom{\alpha_1,\gamma}{\beta_2,\delta} & \xi\binom{\alpha_1}{\delta} \\ \xi\binom{\alpha_2,\gamma}{\beta_1,\delta} & \xi\binom{\alpha_2,\gamma}{\beta_2,\delta} & \xi\binom{\alpha_2}{\delta} \\ \xi\binom{\gamma}{\beta_1} & \xi\binom{\gamma}{\beta_2} & O \end{pmatrix}.$$

3 BKP hierarchy

The BKP hierarchy [4] is a system of non-linear equations for $\tau(x)$ given by

$$\oint e^{-2\tilde{\xi}(y,k)}\tau(x-y-2[k^{-1}]_o)\tau(x+y+2[k^{-1}]_o)\frac{dk}{2\pi ik} = \tau(x-y)\tau(x+y),$$

where the integral means taking the coefficient of k^{-1} in the expansion of the integrand in the series of k.

A formal power series $\tau(x)$, $x = (x_1, x_3, \cdots)$ can be expanded in terms of Schur's Q-function as

$$\tau(x) = \sum_{\mu} \xi_\mu Q_\mu\left(\frac{x}{2}\right), \tag{10}$$

where μ runs over all strict partitions.

For a skew summetric matrix $A = (a_{i,j})_{1\geq i,j\geq 2m}$ Pfaffian $\mathrm{Pf}(a_{i,j})$ [8] is defined by

$$\mathrm{Pf}(a_{ij}) = \sum \mathrm{sgn}(i_1, \ldots, i_{2m}) \cdot a_{i_1,i_2}a_{i_3,i_4}\cdots a_{i_{2m-1},i_{2m}}, \tag{11}$$

where the sum is over all permutations of (1,...,2m) such that

$$i_1 < i_3 < \cdots < i_{2m-1}, \quad i_1 < i_2, \cdots, i_{2m-1} < i_{2m},$$

and $\mathrm{sgn}(i_1, \ldots, i_{2m})$ is the signature of the permutation $(i_1, \ldots, i_{2m})$. In order to describe $\mathrm{Pf}(a_{ij})$ more conveniently we use some set of symbols X_i, $1 \leq i \leq 2m$. Set $(X_i, X_j) = a_{ij}$ and define $\mathrm{Pf}((X_i, X_j))$ as

$$\mathrm{Pf}((X_i, X_j)) = (X_1, \cdots, X_{2m}).$$

The Pfaffian can be expanded as

$$(X_1, \cdots, X_{2m}) = \sum_{j=2}^{2m}(-1)^j (X_1, X_j)(X_2, \cdots, \hat{X}_j, \cdots, X_{2m})$$

For a strict partition $\lambda = (\lambda_1, \cdots, \lambda_M)$ we assume that $\tau(x)$ is expanded as

$$\tau(x) = Q_\lambda(\frac{x}{2}) + \sum_{|\mu|>|\lambda|} \xi_\mu Q_\mu(\frac{x}{2}), \tag{12}$$

where $\mu = (\mu_1, \cdots, \mu_k)$ is a strict partition.

Theorem 2 *[19] Suppose that $\tau(x)$ has the expansion (12). Then $\tau(x)$ is a solution of the BKP hierarchy if and only if the coefficients ξ_μ, $\mu = (\mu_1, \cdots, \mu_k)$, $l(\mu) = k$ are given by the following formulae.*

(i) $M = 2L - 1$,

$$\xi_\mu = \begin{cases} (\Lambda^{(1)}, \cdots, \Lambda^{(2L-1)}, \mu_1, \cdots, \mu_{2l-1}), & \text{if } k = 2l-1, \\ (\Lambda, \Lambda^{(1)}, \cdots, \Lambda^{(2L-1)}, \mu_1, \cdots, \mu_{2l}), & \text{if } k = 2l. \end{cases} \tag{13}$$

(ii) $M = 2L$,

$$\xi_\mu = \begin{cases} (\Lambda, \Lambda^{(1)}, \cdots, \Lambda^{(2L)}, \mu_1, \cdots, \mu_{2l-1}), & \text{if } k = 2l-1, \\ (\Lambda^{(1)}, \cdots, \Lambda^{(2L)}, \mu_1, \cdots, \mu_{2l}), & \text{if } k = 2l, \end{cases} \tag{14}$$

where the elements of the Pfaffian are

$$\begin{aligned} (\Lambda^{(i)}, n) &= \xi_{(\lambda_1, \cdots, \hat{\lambda_i}, \cdots, \lambda_L, n)}, \\ (\Lambda, n) &= \xi_{(\lambda_1, \cdots, \lambda_L, n)}, \\ (n_i, n_j) &= \xi_{(\lambda_1, \cdots, \lambda_L, n_i, n_j)}, \\ (\Lambda, \Lambda^{(i)}) &= (\Lambda^{(i)}, \Lambda^{(j)}) = 0. \end{aligned}$$

Example 4 *We consider the case of $\lambda = (\lambda_1)$ and $\mu = (\mu_1, \mu_2, \mu_3)$. Then*

$$\begin{aligned} \xi_\mu &= (\Lambda^{(1)}, \mu_1, \mu_2, \mu_3) \\ &= (\Lambda^{(1)}, \mu_1)(\mu_2, \mu_3) - (\Lambda^{(1)}, \mu_2)(\mu_1, \mu_3) + (\Lambda^{(1)}, \mu_3)(\mu_1, \mu_2) \\ &= \xi_{(\mu_1)}\xi_{(\lambda_1, \mu_2, \mu_3)} - \xi_{(\mu_2)}\xi_{(\lambda_1, \mu_1, \mu_3)} + \xi_{(\mu_3)}\xi_{(\lambda_1, \mu_1, \mu_2)} \end{aligned}$$

References

[1] A. Alexandrov, V. Kazakov, S. Leurent, Z. Tsuboi, A. Zabrodin, Classical tau function for quantum spin chains, *J. High Energy Phys.* **2013** (2013), issue 9, Article:64.

[2] A. Alexandrov, S. Leurent, Z. Tsuboi, A. Zabrodin, The master T-operator for the Gaudin model and the KP-hierarchy, *Nuclear Phys. B* **833** (2014), 173–223.

[3] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Transformation groups for soliton equations IV. A new hierarchy of soliton equation KP-type, *Phys. D* **4** (1982), 343–365.

[4] E. Date, M. Kashiwara, M. Jimbo, and T. Miwa, Transformation groups for soliton equations, in "Nonlinear Integrable Systems — Classical Theory and Quantum Theory", M. Jimbo and T. Miwa (eds.), World Sci., Singapore, 1983, pp.39–119.

[5] J. C. Eilbeck, V. Z. Enolski and J. Gibbons, Sigma, tau and Abelian functions of algebraic curves, *J. Phys. A* **43** (2010), 455216.

[6] V. Z. Enolski and J. Harnad, Schur function expansions of KP tau functions associated to algebraic curves, *Russian Math. Surveys* **66** (2011), 767–807.

[7] G. Z. Giambelli, Alcuna propreita delle funzioni simmetriche charatteristiche, *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **38** (1903), 323–344.

[8] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, 2004 (Cambridge tracts in mathematics:155).

[9] M. Ishikawa and S. Okada, Identities for determinants and Pfafians, and their applications, *Sugaku Expositions* **27** (2014), 85–116.

[10] D. E. Knuth, Overlapping Pfaffians, *Electron. J. Combin.* **3** (no. 2, The Foata Festschrift) (1996), #R5.

[11] I. G. Macdonald, *Symmetric Functions and Hall Polynomials, second edition*, Oxford University Press, 1995.

[12] A. Nakayashiki, Sigma function as a tau function, *Int. Math. Res. Not. IMRN* **2010-3** (2010), 373–394.

[13] A. Nakayashiki, Tau function approach to theta functions, *Int. Math. Res. Not. IMRN* **2016-17** (2016), 5202–5248.

[14] A. Nakayashiki, S. Okada and Y. Shigyo, On the Expansion Coefficients of KP Tau Function, Journal of Integrable Systems, Vol.2, Issue 1 (2017).

[15] Y. Ohta, Bilinear Theory of Solitons, Doctoral Theses, Graduate School of Engineering, University of Tokyo, 1992.

[16] S. Okada, Generalized Sylvester formulas and skew Giambelli identities, in preparation.

[17] M. Sato and Y. Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifold, in "Nolinear Partial Differential Equations in Applied Sciences", P. D. Lax, H. Fujita and G. Strang (eds.), North-Holland, Amsterdam, and Kinokuniya, Tokyo, 1982, pp.259–271.

[18] Y. Shigyo, On addition formulae of KP, mKP and BKP hierarchies, *SIGMA Symmetry Integrability Geom. Methods Appl.* **9** (2013), 035.

[19] Y. Shigyo, On the expansion coefficients of tau-function of the BKP-hierarchy, *J. Phys. A* **49** (2016), 295201.

[20] W. Wenzel, Pfaffian forms and Δ-matroids, *Discrete Math.* **115** (1993), 253–266.