

ON DS FUNCTOR FOR AFFINE LIE SUPERALGEBRAS

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ABSTRACT. We study Duflo-Serganova functor for non-twisted affine Lie superalgebras and affine vertex superalgebras.

0. INTRODUCTION

Let \mathfrak{g} be a Lie superalgebra and let $x \in \mathfrak{g}_{\bar{1}}$ satisfy the condition $[x, x] = 0$. The operator ad_x defines an odd square zero endomorphism of any \mathfrak{g} -module. This yields a functor $N \mapsto \text{DS}_x(N) := \text{Ker}_{Nx} / \text{Im}_{Nx}$ from the category of \mathfrak{g} -modules to the category of modules over $\mathfrak{g}_x := \text{DS}_x(\mathfrak{g})$.

The functor DS_x was introduced in [DS] (see also [S2]) as a means to assign an analog of singular support to representations of Lie superalgebras. This functor preserves superdimension and tensor product of representations.

Recall that the *defect* of a finite-dimensional Lie superalgebra \mathfrak{g} is the dimension of a maximal isotropic subspace in $\mathbb{Q}\Delta$; for $A(m-1, n-1), B(m, n), D(m, n)$ the defect is equal to $\min(m, n)$; for other cases of non Lie algebras it is one. It is well-known that the defect is equal to the maximal number of mutually orthogonal isotropic simple roots. A finite-dimensional simple Lie superalgebra of zero defect is either a simple Lie algebra or $\mathfrak{osp}(1|2l)$; the finite-dimensional modules over these Lie superalgebras are completely reducible (and these are the only simple Lie superalgebras with this property).

If \mathfrak{g} is a finite-dimensional Lie superalgebra, then $\text{DS}_x(\mathfrak{g})$ is a finite-dimensional Lie superalgebra of a smaller defect. If \mathfrak{g} is the affinization of $\dot{\mathfrak{g}}$ and $x \in \dot{\mathfrak{g}}_{\bar{1}}$, then \mathfrak{g}_x is the affinization of $\dot{\mathfrak{g}}_x$, see [GS].

In this paper we consider the DS functors for affine Lie superalgebras $\mathfrak{g} = \dot{\mathfrak{g}}^{(1)}$ and affine vertex superalgebras $V_k(\mathfrak{g})$; we always assume that $x \in \dot{\mathfrak{g}}_{\bar{1}}$.

Let $Vac^k(\mathfrak{g})$ be a vacuum \mathfrak{g} -module of level k and $Vac_k(\mathfrak{g})$ be its simple quotient. It is easy to see that $\text{DS}_x(Vac^k(\mathfrak{g})) = Vac^k(\text{DS}_x(\mathfrak{g}))$. We prove that for a non-negative integral k one has $\text{DS}_x(Vac_k(\mathfrak{g})) = Vac_k(\text{DS}_x(\mathfrak{g}))$ if $\dot{\mathfrak{g}}_x$ has zero defect and $\dot{\mathfrak{g}}_x \neq \mathbb{C}$, see Theorem 2.2. As a result, the corresponding vertex algebras are isomorphic, see Corollary 3.4.2.

The principal admissible modules for an affine Lie algebra \mathfrak{t} were classified in [KW5]. A level k is called principal admissible if $Vac_k(\mathfrak{t})$ is principal admissible. From Theorem

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of Arakawa [A] it follows that for a principal admissible level k the $V_k(\mathfrak{t})$ -modules in the category \mathcal{O} are completely reducible and the irreducible modules are the principal admissible modules of level k .

We introduce the principal admissible levels for an affine Lie superalgebra \mathfrak{g} using Kac-Wakimoto definition for Lie algebra case. We prove that if \mathfrak{g}_x is a simple Lie algebra and $\mathfrak{g} \neq B(n+1|n)$, then for a principal admissible level k one has $DS_x(Vac_k(\mathfrak{g})) = Vac_k(\mathfrak{g}_x)$. This implies the isomorphism of the corresponding vertex algebras. The proof is based on Arakawa's Theorem and the fact that the maximal proper submodule in $Vac^k(\mathfrak{g}_x)$ is generated by a singular vector (if \mathfrak{g}_x is a Lie algebra, this can be easily deduced from [F]). We believe that the statement holds for $\mathfrak{g}_x = \mathfrak{osp}(1|2n)^{(1)}$, however both Arakawa's and Fiebig's results are not established in this case.

In Section 1 we recall the construction of Duflo-Serganova functor DS_x and summarize the results which we use later.

In Section 2 we study DS functor for integrable vacuum modules and prove Theorem 2.2. Since integrable vacuum modules have principal admissible levels, this theorem for the case, when \mathfrak{g}_x is a simple Lie algebra and $\mathfrak{g} \neq B(n+1|n)$, is a particular case of Theorem 4.4.2. However, the proof of Theorem 2.2 is different: it does not use vertex algebras and Arakawa's Theorem. In § 2.4 we give an example when $DS_x(Vac_k(\mathfrak{g})) \neq Vac_k(\mathfrak{g}_x)$ ($\mathfrak{g} = \mathfrak{sl}(1|2)^{(1)}$, k is critical).

In Section 3 we introduce the DS functor for vertex superalgebras. In particular, we prove that if $DS_x(Vac_k(\mathfrak{g})) = Vac_k(\mathfrak{g}_x)$, then DS_x maps the simple affine vertex superalgebra $V_k(\mathfrak{g})$ to the simple affine vertex superalgebra $V_k(\mathfrak{g}_x)$. As a result, for any $V_k(\mathfrak{g})$ -module N the image $DS_x(N)$ is a $V_k(\mathfrak{g}_x)$ -module.

In Section 4 we study $Vac_k(\mathfrak{g})$ if k is a principal admissible level, (this notion we define in § 4.2 similarly to the Lie algebra case). In § 4.4 we prove that $DS_x(Vac_k(\mathfrak{g})) = Vac_k(\mathfrak{g}_x)$ if \mathfrak{g}_x is a simple Lie algebra and $\mathfrak{g} \neq B(n+1|n)$.

Let $\dot{\Sigma}$ be a set of simple roots which contains a maximal isotropic subset $S = \{\beta_1, \dots, \beta_r\}$ (r is the defect of \mathfrak{g}). We consider DS_x for $x = \sum_{i=1}^r x_i$, where x_i is a non-zero vector in \mathfrak{g}_{β_i} . In this case $\mathfrak{g}_x = DS_x(\mathfrak{g})$ has zero defect. All our results are valid also for the composition $DS_S := DS_{x_1} \circ DS_{x_2} \circ \dots \circ DS_{x_r}$. Note that $DS_S(\mathfrak{g}) \cong DS_x(\mathfrak{g})$.

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1. PRELIMINARIES

Throughout the paper $\mathfrak{g} = \mathfrak{g}^{(1)}$, where $\mathfrak{g} \neq D(2|1, a)$ is a finite-dimensional Kac-Moody superalgebra with a set of simple roots (a base) $\check{\Sigma}$ and a Cartan subalgebra \mathfrak{h} . We denote by \mathfrak{h} the Cartan subalgebra of \mathfrak{g} : $\mathfrak{h} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$.

Let $\Delta \subset \mathfrak{h}^*$ (resp., $\dot{\Delta}$) be the set of roots of \mathfrak{g} (resp., of \mathfrak{g}). We denote by $\Delta_{\bar{0}}$ and $\Delta_{\bar{1}}$ the subsets of even and odd roots. We denote by W the Weyl group of $\mathfrak{g}_{\bar{0}}$. Recall that $\Delta_{\bar{0}}$ is a union of a finite number of root systems of affine Lie algebras with the same minimal imaginary root δ . Throughout the paper we fix $\dot{\Delta}$ and denote by Λ_0 the corresponding fundamental weight, i.e. $(\Lambda_0, \delta) = 1$ and $(\Lambda_0, \dot{\Delta}) = (\Lambda_0, \Lambda_0) = 0$.

We assume that $\dot{\Delta}$ is indecomposable (i.e., $\mathfrak{g}, \mathfrak{g}$ are quasisimple in the sense of [S3]). We fix Δ_0^{\pm} and consider the subsets of positive roots Δ^+ which contain Δ_0^{\pm} . The choice of Δ^+ gives a triangular decomposition of \mathfrak{g} , compatible with the triangular decomposition of $\mathfrak{g}_{\bar{0}}$, corresponding to Δ_0^{\pm} . For a fixed subset of positive roots Δ^+ we denote by Σ the corresponding base (i.e., the set of simple roots) and by ρ the corresponding Weyl vector. For a fixed base Σ we denote by α_0 the affine root, i.e. $\Sigma = \check{\Sigma} \cup \{\alpha_0\}$, where $\check{\Sigma}$ is the base of Δ^+ .

We denote by \mathcal{O} the BGG category of finitely generated \mathfrak{g} -modules with a diagonal action of \mathfrak{h} and a locally finite action of \mathfrak{g}_{α} with $\alpha \in \Delta_0^{\pm}$. The category \mathcal{O} is equipped by a duality functor $N \mapsto N^{\#}$ and the simple modules are self-dual.

We normalize the form $(-, -)$ on \mathfrak{g} as in [KW3] and set

$$\dot{\Delta}^{\#} := \{\alpha \in \dot{\Delta}_{\bar{0}} \mid (\alpha, \alpha) > 0\}.$$

The corresponding algebra $\mathfrak{g}^{\#}$ is a simple Lie algebra; we denote its highest root by θ . We will use bases $\check{\Sigma}$ such that θ is the highest root in $\Delta^+(\check{\Sigma})$; then

$$\alpha_0 = \delta - \theta.$$

Let $\dot{\Omega}$ be the Casimir operator for \mathfrak{g} which corresponds to the invariant bilinear form $(-, -)$, see [K1], Ch. II. Recall that the dual Coxeter number h^{\vee} is half of the eigenvalue of the Casimir operator $\dot{\Omega}$ on the adjoint representation \mathfrak{g} and $h^{\vee} = (\rho, \delta)$. We always choose the Weyl vector ρ in the form $\rho = h^{\vee}\Lambda_0 + \dot{\rho}$.

We say that $k \in \mathbb{C}$ is non-critical if $k \neq -h^{\vee}$ and $\lambda \in \mathfrak{h}^*$ is non-critical if $K(\lambda) \neq -h^{\vee}$, i.e. $(\lambda + \rho, \delta) \neq 0$.

We use the following notations: if $X, Y \subset \mathfrak{h}^*$ we set $(X, Y) = \{(x, y) \mid x \in X, y \in Y\}$; for a vector space V , $X \subset V$ and $R \subset \mathbb{C}$ we use the notation $RX = \{\sum_{i=1}^s r_i x_i \mid r_i \in R, x_i \in X\}$ (for instance, $\mathbb{Z}\dot{\Delta}$ is the root lattice). For $S \subset \mathfrak{h}^*$ we set

$$S^{\perp} := \{\nu \in \mathfrak{h}^* \mid \forall \beta \in S (\beta, \nu) = 0\}.$$

If $\alpha \in \Delta$ is a non-isotropic root, we say that a \mathfrak{g} -module N is α -integrable if $\mathfrak{g}_{\pm\alpha}$ act locally nilpotently on N . We use conventions of [GK]. We say that a \mathfrak{g} -module $N \in \mathcal{O}$ is *integrable* if N is integrable as a $\dot{\mathfrak{g}}_0$ -module and N is α -integrable for each $\alpha \in \Delta$ satisfying $\|\alpha\|^2 > 0$.

For $A(m|1)^{(1)}, C(m)^{(1)}$, $N \in \mathcal{O}$ is integrable if N is integrable as a \mathfrak{g}_0 -module and \mathfrak{h} acts diagonally.

1.1. DS functor for affine Lie superalgebras. Take $x \in \mathfrak{g}_{\bar{1}}$ satisfying $[x, x] = 0$. Recall that Duflo-Serganova functor DS_x is defined by

$$DS_x(N) := Ker_N x / Im_N x;$$

we view $DS_x(N)$ as a module over \mathfrak{g}^x (where \mathfrak{g}^x is the centralizer of x in \mathfrak{g}). Note that $[x, \mathfrak{g}] \subset \mathfrak{g}^x$ acts trivially on $DS_x(N)$ and that $\mathfrak{g}_x := DS_x(\mathfrak{g}) = \mathfrak{g}^x/[x, \mathfrak{g}]$ is a Lie superalgebra. Thus $DS_x(N)$ is a \mathfrak{g}_x -module and DS_x is a functor from the category of \mathfrak{g} -modules to the category of \mathfrak{g}_x -modules. This is a tensor functor ($DS_x(N \otimes N') = DS_x(N) \otimes DS_x(N')$, see [DS]).

An exact sequence of \mathfrak{g} -modules

$$0 \rightarrow N_1 \rightarrow N \rightarrow N_2 \rightarrow 0$$

induces the exact sequence of \mathfrak{g}_x -modules

$$0 \rightarrow E \rightarrow DS_x(N_1) \rightarrow DS_x(N) \rightarrow DS_x(N_2) \rightarrow \Pi(E) \rightarrow 0.$$

Recall that for a \mathfrak{g} -module N with a diagonal action of \mathfrak{h} one has

$$sch N := \sum_{\nu \in \mathfrak{h}^*} sdim N_{\nu} e^{\nu}.$$

If $0 \rightarrow N_1 \rightarrow N \rightarrow N_2 \rightarrow 0$ is exact, then $sch DS_x(N) = sch DS_x(N_1) + sch DS_x(N_2)$.

1.2. Choice of x . In this paper we consider DS_x for $x \in \dot{\mathfrak{g}}$:

$$(1) \quad x \in \dot{\mathfrak{g}}_{\bar{1}}, \quad [x, x] = 0.$$

1.2.1. Definition. For $a \in \mathfrak{g}$ we denote by $\text{supp}(a)$ the subset of $\Delta \cup \{0\}$ such that

$$a = \sum_{\beta \in \text{supp}(a)} a_{\beta},$$

where a_{β} is a non-zero vector in \mathfrak{g}_{β} .

1.2.2. Definition. We call $S \subset \dot{\Delta}_{\bar{1}}$ an *isotropic set* if S is a basis of an isotropic subspace in \mathfrak{h}^* .

Note that if $\text{supp}(x)$ is an isotropic set, then x satisfies (1).

1.2.3. Let \dot{G} be the Lie group of $\dot{\mathfrak{g}}_{\bar{0}}$. By [DS], Thm. 4.2 each x satisfying (1) is \dot{G} -conjugate to x' , where $\text{supp}(x')$ is an isotropic set; this gives a one-to-one correspondence between the \dot{G} -orbits for x satisfying (1) and \dot{W} -orbit of isotropic sets in $\dot{\Delta}$. In particular, for each x satisfying (1) there exists a base $\dot{\Sigma}$ such that x is \dot{G} -conjugate to x' such that $\text{supp}(x')$ is an isotropic set and $\text{supp}(x') \subset \dot{\Sigma}$. We call the cardinality of $\text{supp}(x')$ the *rank* of x ; 0 has the zero rank and the maximal rank is equal to the defect of $\dot{\mathfrak{g}}$.

Let \mathfrak{t} be a Lie subalgebra of $\dot{\mathfrak{g}}_{\bar{0}}$ and N be a \mathfrak{g} -module which is \mathfrak{t} -finite (i.e., $\mathcal{U}(\mathfrak{t})v$ is finite-dimensional for each $v \in N$). Then the Lie group of \mathfrak{t} acts on N . Moreover, any element g in this Lie group induces an isomorphism between the algebras \mathfrak{g}_x and $\mathfrak{g}_{Ad_g(x)}$ and the corresponding modules $DS_x(N)$ and $DS_{Ad_g(x)}(N)$.

In particular, for a $\dot{\mathfrak{g}}$ -integrable \mathfrak{g} -module N , this construction gives an isomorphism between $DS_x(N)$ and $DS_{x'}(N)$ with x' as above ($\text{supp}(x')$ is an isotropic subset of a certain base $\dot{\Sigma}$).

1.2.4. Assume that $S := \text{supp}(x)$ is an isotropic set.

It is shown in [DS], Lemma 6.3 that $\dot{\mathfrak{g}}_x$ a finite-dimensional Kac-Moody superalgebra with the roots

$$\dot{\Delta}_x := (S^\perp \cap \dot{\Delta}) \setminus (S \cup (-S));$$

$\dot{\mathfrak{g}}_x$ can be identified with a subalgebra of $\dot{\mathfrak{g}}$ generated by the root spaces \mathfrak{g}_α with $\alpha \in \dot{\Delta}_x$ and $\dot{\mathfrak{h}}_x \subset \dot{\mathfrak{h}}^x = \{h \in \dot{\mathfrak{h}} \mid S(h) = 0\}$ such that

$$\dot{\mathfrak{h}}_x \oplus \left(\sum_{\beta \in S} \mathbb{C}h_\beta \right) = \dot{\mathfrak{h}}^x, \quad [\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] \subset \dot{\mathfrak{h}}_x \quad \forall \alpha \in \dot{\Delta}_x.$$

Moreover, $\dot{\mathfrak{h}}_x$ is a Cartan subalgebra of $\dot{\mathfrak{g}}_x$ and $\dot{\mathfrak{g}}^x = \dot{\mathfrak{g}}_x \oplus [x, \dot{\mathfrak{g}}]$. Note that $\dot{\mathfrak{h}}_x^*$ is identified with a subspace in $\dot{\mathfrak{h}}^*$ and $S^\perp = \mathbb{C}S \oplus \dot{\mathfrak{h}}_x^*$.

If $\dot{\Delta}_x$ is not empty, then $\dot{\Delta}_x$ is the root system of the Lie superalgebra $\dot{\mathfrak{g}}_x$. One can choose a set of simple roots $\dot{\Sigma}_x$ such that $\Delta^+(\dot{\Sigma}_x) = \Delta^+ \cap \dot{\Delta}_x$.

Let r be the rank of x (i.e., $|S| = r$). If $\dot{\mathfrak{g}} = A(m|n), B(m|n)$ or $D(m|n)$, then $\dot{\mathfrak{g}}_x = A(m-r|n-r), B(m-r|n-r)$ or $D(m-r|n-r)$ respectively. If $\dot{\mathfrak{g}} = C(n), G_3$ or F_4 , then $r = 1$ and $\dot{\mathfrak{g}}_x$ is the Lie algebra of type C_{n-2}, A_1 and A_2 respectively. If $\dot{\mathfrak{g}} = D(2|1; \dot{a})$, then $r = 1$ and $\dot{\mathfrak{g}}_x = \mathbb{C}$. One has

$$\text{defect } \dot{\mathfrak{g}}_x = \text{defect } \dot{\mathfrak{g}} - \text{rank } x.$$

It is easy to show (see [GS]) that $DS_x(\mathfrak{g}) = \mathfrak{g}_x$ is the affinization of $\dot{\mathfrak{g}}_x$; we identify this algebra with

$$\mathfrak{g}_x = \sum_{s=-\infty}^{\infty} (\dot{\mathfrak{g}}_x t^s) \oplus \mathbb{C}K \oplus \mathbb{C}d, \quad \mathfrak{h}_x := \dot{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}d;$$

then $\Delta_x := \Delta(\mathfrak{g}_x)$ is the affinization of $\dot{\Delta}_x$. One has

$$\mathfrak{h}_x^* = \dot{\mathfrak{h}}^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0 \subset \mathfrak{h}^*, \quad S^\perp = \mathfrak{h}_x^* \oplus \mathbb{C}S.$$

Set $\Delta_x^+ := \Delta^+(\Sigma) \cap \Delta_x$ and consider the corresponding triangular decomposition of \mathfrak{g}_x . We will describe the base Σ_x which corresponds to Δ_x^+ below.

If $\dot{\Delta}_x$ is empty, then $\dot{\mathfrak{g}}_x = 0$ or $\dot{\mathfrak{g}}_x = \mathfrak{gl}_1$. If $\dot{\mathfrak{g}}_x = 0$ (i.e., $\dot{\mathfrak{g}} = A(n|n), A(n+1|n), B(n|n), D(n|n), C(2)$), then $\mathfrak{g}_x = \mathbb{C}K \times \mathbb{C}d$. If $\dot{\mathfrak{g}}_x = \mathfrak{gl}_1$ (i.e., $\dot{\mathfrak{g}} = D(n+1|n)$ or $D(2|1, a)$), then $\mathfrak{g}_x = \mathfrak{gl}_1^{(1)}$, $\Delta^+(\mathfrak{g}_x) = \mathbb{Z}_{>0}\delta$ and $\Sigma_x = \{\delta\}$.

If $\dot{\Sigma}_x$ is connected, then $\Sigma_x := \dot{\Sigma}_x \cup \{\delta - \theta_x\}$, where θ_x is the maximal root in $\Delta^+(\dot{\Sigma}_x)$.

If $\dot{\Sigma}_x$ is not connected, then $\dot{\Delta}_x = D_2$ (i.e., $\dot{\mathfrak{g}} = D(n+2|n)$ with x of the maximal rank). In this case $\Delta_x = D_2^{(1)}$ is a union of two copies of $A_1^{(1)}$ with the same imaginary roots, that is $\Sigma_x := \dot{\Sigma}_x \cup \{\delta - \theta_x^i\}_{i=1}^2$, where $\dot{\Delta}_x^+ = \{\theta_x^1, \theta_x^2\}$.

1.3. Casimir operator. Take x as in (1). The bilinear form $(-, -)$ induces an invariant bilinear form $(-, -)_x$ on \mathfrak{g}_x . If N is an integrable \mathfrak{g} -module, then $DS_x(N)$ is an integrable \mathfrak{g}_x -module.

Let Ω be the Casimir operator for \mathfrak{g} which corresponds to the invariant bilinear form $(-, -)$, see [K1], Ch. II. Let $\dot{\Delta}_x \neq \emptyset$. By [GS], the image of Ω is the Casimir operator for \mathfrak{g}_x . This implies $\|\rho\|^2 = \|\rho_x\|_x^2$ and

$$(2) \quad [DS_x(L_{\mathfrak{g}}(\lambda)) : L_{\mathfrak{g}_x}(\lambda')] \neq 0 \implies (\lambda + 2\rho, \lambda) = (\lambda' + 2\rho_x, \lambda')_x,$$

where $\lambda' \in \mathfrak{h}_x^* = S^\perp/\mathbb{C}S$.

1.4. Duality. The duality in \mathcal{O} is defined by an anti-automorphism σ of \mathfrak{g} which stabilizes the elements of \mathfrak{h} . By above, $\mathfrak{g}_x, \mathfrak{g}_{\sigma(x)}$ are identified with a subalgebra of \mathfrak{g} which is σ -stable (in particular, $\mathfrak{g}_x = \mathfrak{g}_{\sigma(x)}$). It is not hard to see that the map $\Psi : DS_x(N^\#) \rightarrow (DS_{\sigma(x)}(N))^\#$ defined by $\Psi(f)(v) := f(v)$ is an isomorphism of \mathfrak{g}_x -modules if $N \in \mathcal{O}$.

2. INTEGRABLE VACUUM MODULES

In this section $\dot{\mathfrak{g}}$ is a finite-dimensional Kac-Moody algebra and $x \in \dot{\mathfrak{g}}_{\bar{1}}$ is such that $\text{supp}(x)$ has a maximal rank, that is $\dot{\mathfrak{g}}_x$ has zero defect. Recall that \mathfrak{g}_x is the affinization of $\dot{\mathfrak{g}}_x$.

2.1. Vacuum modules. If \mathfrak{p} is a Kac-Moody superalgebra with a Cartan subalgebra \mathfrak{t} , we denote by $L_{\mathfrak{p}}(\lambda)$ a simple highest \mathfrak{p} -module with the highest weight $\lambda \in \mathfrak{t}^*$. For an affine Kac-Moody superalgebra \mathfrak{p} we denote by $Vac_{\mathfrak{p}}^k$ the vacuum module of level k and by $|0\rangle$ the vacuum vector. For $\mathfrak{p} = \mathfrak{g}$ we write simply $L(\lambda), Vac^k$. Note that $L(k\Lambda_0) = Vac_k$ is the simple quotient of Vac^k and so it does not depend on the choice of Σ ; we call $L(k\Lambda_0)$ a *simple vacuum module*; if $L(k\Lambda_0)$ is integrable, we call it an *integrable vacuum module*.

2.1.1. The character of an integrable vacuum module is given by the Kac-Wakimoto character formula, see [GK]. From the proof it follows that an integrable vacuum module is a unique integrable quotient of Vac^k (since the proof uses only the fact that $L(k\Lambda_0)$ is a $\mathfrak{g}^\#$ -integrable quotient of Vac^k , so all integrable quotients have the same character and thus such quotient is unique).

Recall that $\mathfrak{g}^\# \neq D_2$. Let θ be the highest root of $\dot{\Delta}^\#$ and $e \in \mathfrak{g}^\#$ be the corresponding root vector. From [K1], Lem. 3.4 it follows that a quotient Vac^k/I is $\mathfrak{g}^\#$ -integrable if and only if $k \in \mathbb{Z}_{\geq 0}$ and I contains $f_0^{k+1}|0\rangle$ for $f_0 := et^{-1}$. Therefore $L(k\Lambda_0)$ is integrable if and only if $k \in \mathbb{Z}_{\geq 0}$; in this case

$$L(k\Lambda_0) = Vac^k/I(k), \quad \text{where } I(k) \text{ is generated by } f_0^{k+1}|0\rangle.$$

Note that the vector $f_0^{k+1}|0\rangle$ is singular if $\delta - \theta \in \Sigma$.

2.1.2. *Remark.* Recall that $L(k\Lambda_0)$ does not depend on the choice of $\dot{\Sigma}$. Combining § 1.2 and § 5, we see that computing $DS_x(L(k\Lambda_0))$ we can always assume that $supp(x)$ is an isotropic set which lies in S satisfying (P1), (P2), (P3) in § 5.

2.2. Theorem. *Let $x \in \mathfrak{g}_\Gamma^-$ be such that $[x, x] = 0$ and $supp(x)$ has a maximal rank.*

(i) *If $\mathfrak{g}_x = 0$ and $k \neq -h^\vee$, then $DS_x(L(k\Lambda_0))$ is one-dimensional.*

(ii) *Assume that $\dot{\Sigma}$ contains $S := supp(x)$ and the following inclusion holds*

$$(3) \quad (\mathbb{Q}_{\geq 0}\Sigma \cap S^\perp) \subset (\mathbb{Q}S + \mathbb{Q}_{\geq 0}\Sigma_x).$$

If $L(\lambda)$ is integrable and $(\lambda, S) = 0$, then

$$DS_x(L(\lambda)) \cong L_{\mathfrak{g}_x}(\lambda|_{\mathfrak{h}_x}).$$

(iii) *If $\mathfrak{g}_x \neq \mathbb{C}$ and $k \in \mathbb{Z}_{\geq 0}$, one has*

$$DS_x(L(k\Lambda_0)) \cong L_{\mathfrak{g}_x}(k\Lambda_0).$$

2.3. Proof of Theorem 2.2. For (i), (iii) we set $\lambda := k\Lambda_0 \in \mathfrak{h}^*$ (so $L(\lambda) = L(k\Lambda_0)$) and $S := supp(x)$. Using Remark 2.1.2, we assume for (i), (iii) that $S, \dot{\Sigma}$ satisfies (P1), (P3) of § 5, i.e. $S \subset \dot{\Sigma}$ and (3) holds except for $\mathfrak{g} = D(n+2|n), D(n+1|n)$.

We introduce

$$\lambda' := \lambda|_{\mathfrak{h}_x} \in \mathfrak{h}_x^*.$$

Since $(\lambda, S) = 0$ one has $\dim L(\lambda)_{\lambda-\nu} = \delta_{0,\nu}$ for $\nu \in \mathbb{Z}S$. Thus the singular vector in $L(\lambda)$ has a non-trivial image in $DS_x(L(k\lambda))$ which is singular; moreover,

$$(4) \quad [DS_x(L(\lambda)) : L_{\mathfrak{g}_x}(\lambda')] = 1.$$

For (i) $\mathfrak{g}_x = \mathbb{C}K \times \mathbb{C}d$. By § 1.3, the Casimir $\Omega_x = 2(K + h^\vee)d$ acts on $\mathrm{DS}_x(L(k\Lambda_0))$ by a scalar, so d acts on $\mathrm{DS}_x(L(k\Lambda_0))$ by a scalar. Now (i) follows from (4).

For (ii), (iii) assume that $[\mathrm{DS}_x(L(\lambda)) : L_{\mathfrak{g}_x}(\lambda' - \nu')] \neq 0$ for some $\nu' \in \mathfrak{h}_x^*$ with $\nu' \neq 0$. Since $\lambda' - \nu'$ is a weight of $\mathrm{DS}_x(L(\lambda))$, there exists $\nu \in \mathfrak{h}^*$ such that $\nu|_{\mathfrak{h}_x} = \nu'$, $(\nu, S) = 0$ and $L(\lambda)_{\lambda - \nu} \neq 0$. In particular,

$$(5) \quad \nu \in \mathbb{Z}_{\geq 0}\Sigma \cap S^\perp.$$

Let us prove (ii). Combining (5) and (3), $\nu \in \mathbb{Q}S + \mathbb{Q}_{\geq 0}\Sigma_x$, that is $\nu' \in \mathbb{Q}_{\geq 0}\Sigma_x$. If Δ_x is empty, we obtain $\nu' = 0$, a contradiction. Now we assume that Δ_x is not empty. By § 1.3, $\mathrm{DS}_x(L(\lambda))$ is \mathfrak{g}_x -integrable, so $L_{\mathfrak{g}_x}(\lambda')$, $L_{\mathfrak{g}_x}(\lambda' - \nu')$ are integrable modules and $\|\lambda' - \nu' + \rho_x\|^2 = \|\lambda' + \rho_x\|^2$, that is

$$(\lambda' - \nu' + \rho_x, \nu') + (\lambda' + \rho_x, \nu') = 0.$$

Since \mathfrak{g}_x has zero defect, the integrability of $L_{\mathfrak{g}_x}(\lambda')$ and $L_{\mathfrak{g}_x}(\lambda' - \nu')$ gives $(\lambda', \nu'), (\lambda' - \nu', \nu') \geq 0$ and $(\nu', \rho_x) > 0$ (for $\nu' \neq 0$), a contradiction. This establishes (ii). Recall that for our choice of $(S, \dot{\Sigma})$, (3) holds except for $\mathfrak{g} = D(n+2|n), D(n+1|n)$. Thus (ii) implies (iii).

It remains to verify (iii) for $\mathfrak{g} = D(n+2|n)$. Consider the short exact sequence

$$0 \rightarrow I(k) \rightarrow \mathrm{Vac}^k \rightarrow L(k\Lambda_0) \rightarrow 0,$$

where $I(k)$ is the maximal proper submodule of Vac^k . It is easy to see that $\mathrm{DS}_x(\mathrm{Vac}^k) = \mathrm{Vac}^k(\mathfrak{g}_x)$. Thus the corresponding long exact sequence is

$$0 \rightarrow E \rightarrow \mathrm{DS}_x(I(k)) \xrightarrow{\phi} \mathrm{Vac}^k(\mathfrak{g}_x) \xrightarrow{\psi} \mathrm{DS}_x(L(k\Lambda_0)) \rightarrow \Pi(E) \rightarrow 0.$$

Since $\mathrm{DS}_x(L(k\Lambda_0))$ is \mathfrak{g}_x -integrable, the image of ψ is an integrable quotient of Vac^k , that is $L_{\mathfrak{g}_x}(k\Lambda_0)$. Since $L_{\mathfrak{g}_x}(k\Lambda_0 - \nu')$ is a subquotient of $\mathrm{DS}_x(L(k\Lambda_0))$ and $\nu' \neq 0$, it is a subquotient of $\Pi(E)$. Therefore $\Pi(L_{\mathfrak{g}_x}(k\Lambda_0 - \nu'))$ is a subquotient of $\mathrm{DS}_x(I(k))$. Take Σ such that $\|\alpha_0\|^2 > 0$. By § 2.1, $I(k)$ is generated by a singular vector of the weight $k\Lambda_0 - (k+1)\alpha_0$. Therefore

$$\lambda' - \nu' = k\Lambda_0 - (k+1)\alpha_0 - \mu',$$

where $\mu' = \mu|_{\mathfrak{g}_x}$ for some $\mu \in \mathfrak{h}^*$ such that $\mu \in \mathbb{Z}_{\geq 0}\Sigma$. Then

$$(6) \quad (\nu', \Lambda_0) \geq k+1.$$

Take $S := \{\varepsilon_{i+1} - \delta_i\}_{i=1}^n$ and $\dot{\Sigma} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1, \dots, \varepsilon_{n+1} - \delta_n, \delta_n \pm \varepsilon_{n+2}\}$; then $\alpha_0 = \delta - \varepsilon_1 - \varepsilon_2$, so $\|\alpha_0\|^2 = 2$. One has

$$\Sigma_x = \{\varepsilon_1 \pm \varepsilon_{n+2}; \delta - (\varepsilon_1 \pm \varepsilon_{n+2})\}, \quad \rho_x = 2\Lambda_0 + \varepsilon_1.$$

One readily sees that $(\mathbb{C}\Sigma \cap S^\perp) \subset (\mathbb{C}S + \mathbb{C}\Sigma_x)$, so $\nu' \in \mathbb{C}\Sigma_x$, so

$$\nu' = j\delta - s_+(\varepsilon_1 + \varepsilon_{n+2}) - s_-(\varepsilon_1 - \varepsilon_{n+2}).$$

The integrability of $L_{\mathfrak{g}_x}(k\Lambda_0 - \nu')$ implies $0 \leq s_{\pm} \leq k/2$. In addition, (2) gives

$$(k + 2)j - s_+ - s_- = s_+^2 + s_-^2,$$

so $j \leq k/2$. However, $j = (\nu', \Lambda_0) \geq k + 1$ by (6). This contradiction completes the proof.

2.4. Example: $\mathfrak{g} = \mathfrak{sl}(1|2)^{(1)}$, $k = -1$. Take $\mathfrak{g} = \mathfrak{sl}(1|2)^{(1)}$ with $\Sigma = \{\delta - \varepsilon_1 + \delta_2, \varepsilon_1 - \delta_1, \delta_1 - \delta_2\}$ and $S = \{\varepsilon_1 - \delta_1\}$. Using the character formula (3.20) in [KW4] it is not hard to show that $DS_x((L(-\Lambda_0))$ is not one-dimensional.

3. DS FUNCTOR FOR VERTEX SUPERALGEBRAS

3.1. Vertex algebras. Recall that a vertex (super)algebra $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a vector superspace endowed with a vacuum vector $|0\rangle$, an even linear endomorphism T and a parity preserving linear map

$$Y : V \rightarrow (\text{End } V)[[z, z^{-1}]], \quad a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$$

subject to the following axioms ($a, b \in V, m, n \in \mathbb{Z}$)

(translation covariance) $[T, Y(a, z)] = \partial_z Y(a, z)$;

(vacuum) $T|0\rangle = 0$; $Y(|0\rangle, z) = Id_V$; $a_{(-1)}|0\rangle = a$, $a_{(n)}|0\rangle = 0$ for $n \geq 0$;

and the locality axiom which we use in the Borcherds form

$$(a_{(m)}b)_{(n)} = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} (a_{(m-i)}b_{(n+i)} - (-1)^{m+p(a)p(b)} b_{(m+n-i)}a_{(i)}).$$

For $m = 0$ this gives

$$(7) \quad (a_{(0)}b)_{(n)} = [a_{(0)}, b_{(n)}].$$

Note that T is “determined” by Y , i.e.

$$(8) \quad Ta = a_{(-2)}|0\rangle.$$

3.1.1. Modules. A weak module over a vertex superalgebra V in a vector superspace M with a parity preserving linear map

$$Y^M : V \rightarrow (\text{End } M)[[z, z^{-1}]] \quad a \mapsto Y_M(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)}^M z^{-n-1},$$

such that for each $v \in M$ one has $a_{(n)}^M v = 0$ for $n \gg 0$, $Y_M(|0\rangle, z) = Id_M$ and $a_{(m)}^M, b_{(n)}^M$ satisfy the locality axiom. As above, the locality axiom gives

$$(9) \quad (a_{(0)}b)_{(n)}^M = [a_{(0)}^M, b_{(n)}^M].$$

An ideal of a vertex algebra is a subspace $I \subset V$ such that $a_{(m)}b, b_{(m)}a \in I$ for each $a \in I, b \in V, m \in \mathbb{Z}$. If I is an ideal of V , then the quotient V/I inherits the structure of a vertex algebra.

If I is an ideal of V , the V/I -modules are the V -modules annihilated by I , that is $a_{(m)}^M = 0$ for each $a \in I, m \in \mathbb{Z}$. We say that an ideal I is generated by a set E if I is a minimal ideal containing E . The locality axiom implies that in this case M is V/I -module if and only if $a_{(m)}^M = 0$ for each $a \in E, m \in \mathbb{Z}$.

3.2. Definition of $DS_x(V)$. Let V be a vertex superalgebra and $x \in V$ be such that

$$(10) \quad x \in V_{\bar{1}}, \quad x_{(0)}x = 0 \quad \text{and} \quad |0\rangle \notin Imx_{(0)}.$$

By (7) one has $x_{(0)}^2 = 0$. We define the vector space $DS_x(V)$ as follows:

$$DS_x(V) := \text{Ker}_V x_{(0)} / Im_V x_{(0)}.$$

By the vacuum axiom, $|0\rangle \in \text{Ker } x_{(0)}$, so $|0\rangle$ has a non-zero image $|0\rangle' \in DS_x(V)$. From the translation axiom $[T, x_{(0)}] = 0$, so T induces an even map $T' \in \text{End } DS_x(V)$.

Take $b \in \text{Ker}_V x_{(0)}$. From (7) it follows that $[b_{(n)}, x_{(0)}] = 0$ for each n , so $b_{(n)}$ induces $b'_{(n)} \in \text{End}(DS_x(V))$ and $b'_{(n)} = 0$ if $b \in Im_V x_{(0)}$. This gives a parity preserving linear map

$$DS_x(V) \rightarrow (\text{End } DS_x(V))[[z, z^{-1}]] \quad b \mapsto Y'(b, z) = \sum_{n \in \mathbb{Z}} b'_{(n)} z^{-n-1}.$$

The space $DS_x(V)$ equipped with $|0\rangle', T'$ and the fields $Y'(b, z)$ form a vertex algebra (the axioms for V' follow from the corresponding axioms for the vertex algebra V). We denote this vertex algebra by $DS_x(V)$.

3.2.1. Remark. A vertex algebra V is $\mathbb{Z}_{\geq 0}$ -graded if $V = \bigoplus_{s=0}^{\infty} V_s$ with

$$deg(a_{(j)}b) = deg(a) + deg(b) - j - 1,$$

where deg stands for the degree of a homogeneous vector in V . We claim that the condition $|0\rangle \notin Im a_{(0)}$ holds for each $a \in V$ if V is a $\mathbb{Z}_{\geq 0}$ -graded vertex algebra with $V_0 = \mathbb{C}|0\rangle$.

Indeed, assume that $deg(a_{(0)}b) = 0$ for homogeneous a, b . Then $deg(a) + deg(b) = 1$, so a or b lie in $V_0 = \mathbb{C}|0\rangle$. However, $|0\rangle_{(0)} = 0$ and $a_{(0)}|0\rangle = 0$ for each a , that is $a_{(0)}b = 0$. Hence $|0\rangle \notin Im a_{(0)}$ for each $a \in V$.

3.2.2. Modules. Let M be a V -module. The condition $x_{(0)}x = 0$ gives $[x_{(0)}^M, x_{(0)}^M] = 0$. We introduce

$$DS_x(M) = \text{Ker}_M x_{(0)}^M / Im_M x_{(0)}^M.$$

Using (9) it is easy to check that $DS_x(M)$ inherits a structure of $DS_x(V)$ -module (i.e., Y^M induces a map $DS_x(V) \rightarrow (\text{End } DS_x(M))[[z, z^{-1}]]$ which satisfy the corresponding axioms).

3.3. Affine vertex superalgebras. Let $\hat{\mathfrak{g}}$ be a finite-dimensional Kac-Moody superalgebra and let $\mathfrak{g} = \hat{\mathfrak{g}}^{(1)}$.

By [FZ], the vacuum module $Vac^k(\mathfrak{g})$ has a structure of a vertex superalgebra with

$$(11) \quad Y(at^{-1}|0\rangle, z) = \sum_{n \in \mathbb{Z}} (at^n) z^{-n-1} \quad \text{for } a \in \hat{\mathfrak{g}}.$$

We denote this vertex superalgebra by $V^k(\mathfrak{g})$.

The weak $V^k(\mathfrak{g})$ -modules are the restricted $[\mathfrak{g}, \mathfrak{g}]$ -module of level k (M is "restricted" if for each $v \in M$ one has $(\hat{\mathfrak{g}}t^s)v = 0$ for $s \gg 0$).

The above correspondence between $V^k(\mathfrak{g})$ -modules and $[\mathfrak{g}, \mathfrak{g}]$ -modules implies that the maximal proper submodule $I(k)$ of $Vac^k(\mathfrak{g})$ is the maximal ideal in the vertex algebra $V^k(\mathfrak{g})$. Moreover, if $I(k)$ is generated by E as a \mathfrak{g} -module, then $I(k)$ is generated by E as an ideal in $V^k(\mathfrak{g})$. In particular, $L(k\Lambda_0)$ inherits a structure of a vertex superalgebra, which is simple; it is denoted by $V_k(\mathfrak{g})$.

For $\hat{\mathfrak{g}} = 0$ or $\hat{\mathfrak{g}} = \mathbb{C}$, the vacuum module $Vac^k(\mathfrak{g})$ is one-dimensional and $V^k(\mathfrak{g}) = V_k(\mathfrak{g})$ is a one-dimensional vertex algebra.

If \mathfrak{g} is a Lie algebra, then for $k \in \mathbb{Z}_{\geq 0}$ the $V_k(\mathfrak{g})$ -modules correspond to the restricted integrable \mathfrak{g} -modules of level k , see [DLM], Thm. 3.7; these modules are completely reducible and the irreducible modules are the integrable highest weight modules of level k (there are infinitely many such modules and $V_k(\mathfrak{g})$ is a *rational* vertex algebra). The following result was proven in [GS], Thm. 6.3.1 for $\hat{\mathfrak{g}}^\# \neq D_2$.

3.3.1. Theorem. *If $L(k\Lambda_0)$ are integrable, then $V_k(\mathfrak{g})$ -modules are the restricted $[\mathfrak{g}, \mathfrak{g}]$ -module of level k which are $[\mathfrak{g}^\#, \mathfrak{g}^\#]$ -integrable.*

Proof. Let $I(k)$ be the maximal proper submodule of Vac^k , so $L(k\Lambda_0) = Vac^k/I(k)$.

For $\hat{\mathfrak{g}}^\# \neq D_2$ consider the natural embedding $Vac^k(\mathfrak{g}^\#) \subset Vac^k$ and denote by $I^\#$ the maximal proper submodule of $Vac^k(\mathfrak{g}^\#)$.

If $\hat{\mathfrak{g}}^\# = D_2 = A_1 \times A_1$ consider the natural embeddings $Vac^k(A_1)', Vac^k(A_1)'' \subset Vac^k$ which correspond to two copies of A_1 in D_2 ; let I', I'' be the maximal proper submodules in $Vac^k(A_1)', Vac^k(A_1)''$ respectively. Set $I^\# := I' + I''$.

By § 2.1, the submodule $I(k)$ is generated by $I^\#$. By above, a restricted $[\mathfrak{g}, \mathfrak{g}]$ -module N of level k is $V_k(\mathfrak{g})$ -module if and only if it is annihilated by $a_{(m)}$ for each $a \in I^\#, m \in \mathbb{Z}$. Since $\hat{\mathfrak{g}}^\#$ is a Lie algebra, N is annihilated by $a_{(m)}$ for each $a \in I^\#, m \in \mathbb{Z}$ if and only if N is $\mathfrak{g}^\#$ -integrable ([DLM], Thm. 3.7). \square

3.4. DS_x for affine vertex algebras. Fix $x \in \hat{\mathfrak{g}}_{\bar{1}}$ satisfying $[x, x] = 0$. View

$$x' := xt^{-1}|0\rangle$$

as a vector in $V^k(\mathfrak{g})$ and $V_k(\mathfrak{g})$ respectively. Note that $x'_{(0)} = x$.

One has $x'_{(0)}x' = x(xt^{-1}|0\rangle) = 0$. The vertex algebras $V^k(\mathfrak{g})$, $V_k(\mathfrak{g})$ are $\mathbb{Z}_{\geq 0}$ -graded (the grading is given by the action of $-d \in \mathfrak{g}$) and the zero component is spanned by $|0\rangle$. Hence x' satisfies (10).

Consider the vertex algebras $DS_{x'}(V^k(\mathfrak{g}))$, $DS_{x'}(V_k(\mathfrak{g}))$.

It is easy to see that $DS_x(Vac^k(\mathfrak{g}))$ is canonically isomorphic to $Vac^k(\mathfrak{g}_x)$ as a \mathfrak{g}_x -module. Choose a vacuum vector $|0\rangle$ in $Vac^k(\mathfrak{g})$ and let the vacuum vector $|0'\rangle$ in $Vac^k(\mathfrak{g}_x)$ be the image of $|0\rangle$.

3.4.1. Theorem. *Let \mathfrak{g} be an affine (non-twisted) Lie superalgebra and let $x \in \mathfrak{g}_{\bar{1}}$ be such that $[x, x] = 0$; set $x' := xt^{-1}|0\rangle$.*

(i) *The canonical isomorphism $DS_x(Vac^k(\mathfrak{g})) \xrightarrow{\sim} Vac^k(\mathfrak{g}_x)$ induces a vertex algebra isomorphism $DS_{x'}(V^k(\mathfrak{g})) \xrightarrow{\sim} V^k(\mathfrak{g}_x)$.*

(ii) *If $DS_x(L_{\mathfrak{g}}(k\Lambda_0)) \cong L_{\mathfrak{g}_x}(k\Lambda_0)$, then ι induces the vertex algebra isomorphism*

$$DS_{x'}(V_k(\mathfrak{g})) \xrightarrow{\sim} V_k(\mathfrak{g}_x).$$

Proof. By above, ι is an isomorphism of \mathfrak{g}_x -modules and $\iota(|0\rangle) = |0'\rangle$. If $V^k(\mathfrak{g}_x)$ is one-dimensional, this implies (i) and (ii). Assume that $V^k(\mathfrak{g}_x)$ is not one-dimensional. Then $\Delta(\mathfrak{g}_x) \neq \emptyset$. Since ι is an isomorphism of \mathfrak{g}_x -modules,

$$\iota(at^{-1}|0\rangle) = (DS_x(a)t^{-1})|0'\rangle$$

for each $a \in \mathfrak{g}$ such that $[x, a] = 0$. By (11) we obtain

$$(12) \quad Y(\iota(v), z) = \iota(Y(v, z))$$

for each $v = bt^{-1}|0\rangle$ with $b \in \mathfrak{g}_x$.

Let V be a vertex algebra and E be a subspace of V . Denote by $\langle E \rangle$ the smallest subspace V' of V which contains E and such that $b_{(j)}v \in \langle E \rangle$ for each $b, v \in V'$ and $j \in \mathbb{Z}$. The locality axiom and (8) imply that if V admits two vertex algebra structures $(|0\rangle, T, Y)$ and $(|0\rangle, T', Y')$ such that $Y(v, z) = Y'(v, z)$ for each $v \in E$, then these structures coincide on $\langle E \rangle$ (i.e., $TV = T'v$ and $Y(v, z) = Y'(v, z)$ for each $v \in \langle E \rangle$).

Now let $E \subset V^k(\mathfrak{g}_x)$ (resp., $E \subset V_k(\mathfrak{g}_x)$) be the span of $|0\rangle$ and $bt^{-1}|0\rangle$ with $b \in \mathfrak{g}_x$. Since $V^k(\mathfrak{g}_x)$ and $V_k(\mathfrak{g}_x)$ are generated by $|0\rangle$ as a $[\mathfrak{g}_x, \mathfrak{g}_x]$ -modules, $V^k(\mathfrak{g}_x) = \langle E \rangle$ (resp., $V_k(\mathfrak{g}_x) = \langle E \rangle$). Thus ι is an isomorphism of the vertex algebras. \square

Using Theorem 2.2 we obtain the

3.4.2. Corollary. *If k is a non-negative integer, x has a maximal rank and \mathfrak{g} differ from $D(n+1|n), D(2|1, a)$, then $DS_{x'}(V_k(\mathfrak{g}))$ and $V_k(\mathfrak{g}_x)$ are isomorphic as vertex algebras.*

3.4.3. Take x as above. Let M be a weak $V^k(\mathfrak{g}_x)$ -module which we view as a restricted \mathfrak{g}_x -modules of level k . One readily sees that, as a \mathfrak{g}_x -module, the $DS_{x'}(V^k(\mathfrak{g}))$ -module $DS_{x'}(M)$ is $DS_x(M)$, so $DS_{x'}$ for $V^k(\mathfrak{g})$ -modules correspond to DS_x for $[\mathfrak{g}, \mathfrak{g}]$ -modules. We will denote the functor $DS_{x'}$ by DS_x .

3.4.4. Corollary. *Let $L(k\Lambda_0)$ be integrable. For any $V_k(\mathfrak{g})$ -module M the $V^k(\mathfrak{g}_x)$ -module $DS_x(M)$ is a $V_k(\mathfrak{g}_x)$ -module.*

Proof. By Theorem 3.3.1, M is $[\mathfrak{g}^\#, \mathfrak{g}^\#]$ -integrable. Note that $\mathfrak{g}_x^\#$ is the image of $\mathfrak{g}^\# \cap Ker_{\mathfrak{g}} x$ in $\mathfrak{g}_x = Ker_x \mathfrak{g} / Im_x \mathfrak{g}$. Therefore $DS_x(M)$ is $[\mathfrak{g}_x^\#, \mathfrak{g}_x^\#]$ -integrable, so $DS_x(M)$ is a $V_k(\mathfrak{g}_x)$ -module by Theorem 3.3.1. \square

4. PRINCIPAL ADMISSIBLE VACUUM MODULES

In this section we define admissible weights for affine Lie superalgebras and prove Theorem 4.4.2.

4.1. Affine Lie algebra case. Let \mathfrak{t} be a finite-dimensional simple Lie algebra; let $\mathfrak{t} = \mathfrak{t}^{(1)}$ be the corresponding affine Lie algebra with a Cartan subalgebra \mathfrak{h} . We denote by Δ_{re} the set of real roots of \mathfrak{t} .

4.1.1. For a non-critical weight $\lambda \in \mathfrak{h}^*$ the set of λ -integral real roots is defined as

$$\Delta_{re}(\lambda) = \{ \alpha \in \Delta_{re} \mid \frac{2(\lambda + \rho, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \}.$$

For our purposes we consider only λ s where $\mathbb{C}\Delta_{re}(\lambda) = \mathbb{C}\Delta_{re}$. In this case $\Delta_{re}(\lambda)$ is the set of real roots of an affine Lie algebra algebra $\bar{\mathfrak{t}}$ with the same Cartan algebra \mathfrak{h} and the triangular decomposition induced by the triangular decomposition of \mathfrak{t} , i.e.

$$\Delta_{re}(\lambda)^+ := \Delta_{re}(\lambda) \cap \Delta^+.$$

We denote by $\rho, \bar{\rho}$ the Weyl vectors of $\mathfrak{t}, \bar{\mathfrak{t}}$ respectively. The character of $L_{\mathfrak{t}}(\lambda)$ and the character of the highest weight $\bar{\mathfrak{t}}$ -module $L_{\bar{\mathfrak{t}}}(\lambda + \rho - \bar{\rho})$ are related by the following formula:

$$(13) \quad Re^\rho \text{ch } L_{\mathfrak{t}}(\lambda) = \bar{R}e^{\bar{\rho}} \text{ch } L_{\bar{\mathfrak{t}}}(\lambda + \rho - \bar{\rho}),$$

where R, \bar{R} stand for the respective Weyl denominators (see [KT1],[KT2] and references there).

4.1.2. Admissible weights. A non-critical weight $\lambda \in \mathfrak{h}^*$ is called *admissible* if $\mathbb{C}\Delta_{re}(\lambda) = \mathbb{C}\Delta_{re}$ and $L_{\bar{\mathfrak{t}}}(\lambda + \rho - \bar{\rho})$ is an integrable $\bar{\mathfrak{t}}$ -module.

If λ is admissible, then $chL_{\bar{\mathfrak{t}}}(\lambda + \rho - \bar{\rho})$ is given by the Weyl-Kac character formula and $chL(\lambda)$, suitably normalized, is a ratio of theta functions, which is a modular function, see [KW1],[KW2]. The admissible weights were classified in [KW2]. An admissible weight λ (and a module $L_{\mathfrak{t}}(\lambda)$) is called *principal admissible* if $\Delta_{re}(\lambda) \cong \Delta_{re}$, that is $\bar{\mathfrak{t}} \cong \mathfrak{t}$; the principal admissible weights were classified in [KW5].

4.1.3. Principal admissible levels. A level k is called *principal admissible* if $k\Lambda_0$ is principal admissible.

It is easy to see that k is principal admissible if and only if

$$k + h^\vee = \frac{p + h^\vee}{u}, \text{ where } p, u \in \mathbb{Z}_{\geq 0}, u > 0 \quad (p + h^\vee, u) = (u, r^\vee) = 1,$$

where r^\vee is the lacity of \mathfrak{t} (see the definition below in 4.2.1).

4.1.4. The following Adamović-Milas conjecture [AM] was proven by T. Arakawa in [A].

Theorem, Arakawa, 2014.

Let k be an admissible level for an affine Lie algebra \mathfrak{t} . The $V_k(\mathfrak{t})$ -modules in the category \mathcal{O} are completely reducible and the irreducible modules are $L_{\mathfrak{t}}(\lambda)$, where λ are the principal admissible weights of level k .

4.2. Admissibility for affine Lie superalgebras. Let \mathfrak{g} be a (non-twisted) affine Lie superalgebra.

4.2.1. Lacity. Let $\mathfrak{g} \neq D(2|1, a)$. We call $\alpha \in \Delta$ (resp., $\alpha \in \dot{\Delta}$) a short root if $|(\alpha, \alpha)|$ takes the smallest non-zero value. We define the lacity for \mathfrak{g} and for $\dot{\mathfrak{g}}$ as

$$r^\vee = \frac{2}{|(\alpha, \alpha)|},$$

where α is a short root. Observe that the lacities for Δ and for $\dot{\Delta}$ are equal. Moreover, this lacity is equal to the lacity of $\mathfrak{g}^\#$ if $\mathfrak{g} \neq B(0|n)$; for $\mathfrak{g} = B(0|n)$ one has $r^\vee = 4$.

The set $\Delta_{re}(\lambda)$ was introduced in [GK]. As for Lie algebra case, we define the admissible weights as follows.

4.2.2. Definitions. A non-critical weight $\lambda \in \mathfrak{h}^*$ is *admissible* if $\mathbb{C}\Delta_{re}(\lambda) = \mathbb{C}\Delta_{re}$ and $L_{\bar{\mathfrak{g}}}(\lambda + \rho - \bar{\rho})$ is an integrable $\bar{\mathfrak{g}}$ -module.

An admissible weight λ is called *principal admissible* if $\Delta_{re}(\lambda) \cong \Delta_{re}$.

We say that k is an *admissible* (resp., *principal admissible*) level if $k\Lambda_0$ is admissible (resp., principal admissible).

By [GK], Thm. 11.2.3, $\text{ch } L(k\Lambda_0)$ is given by (13) if k is admissible.

4.3. Principal admissible levels. It is not hard to show that for $\dot{\mathfrak{g}} \neq D(2|1, a)$ the level k is principal admissible if and only if

$$k + h^\vee = \frac{p + h^\vee}{u}, \text{ where } p, u \in \mathbb{Z}_{\geq 0}, u > 0 \quad (r^\vee(p + h^\vee), u) = 1,$$

where r^\vee is the lacity of $\dot{\mathfrak{g}}$. Note that $r^\vee(p + h^\vee)$ is integral: h^\vee is integral for $\dot{\mathfrak{g}} \neq B(m|n), m \leq n$, and $h^\vee = n - m + \frac{1}{2}$ for $\dot{\mathfrak{g}} = B(m|n), m \leq n$.

Let k be a principal admissible level. Then $\Delta_{re}(\bar{\mathfrak{g}}) = \dot{\Delta} + \mathbb{Z}u\delta$, where u is as above and the formula (13) takes the form

$$(14) \quad Re^p \text{ch } L_{\mathfrak{g}}(k\Lambda_0) = \overline{R}e^{\bar{p}} \text{ch } L_{\bar{\mathfrak{g}}}(p\bar{\Lambda}_0).$$

Note that $\Delta_{re}(\bar{\mathfrak{g}}) \cap \Delta^+$ vhas the base $\dot{\Sigma} \cup \{\alpha'_0\}$, where

$$\alpha'_0 = (u - 1)\delta + \alpha_0,$$

where $\Sigma = \dot{\Sigma} \cup \{\alpha_0\}$.

Recall that for $x \in \dot{\mathfrak{g}}$ such that $\dot{\Delta}_x$ is non-empty, $\dot{\mathfrak{g}}$ and $\text{DS}_x(\dot{\mathfrak{g}})$ have the same dual Coxeter numbers. If, in addition, $\dot{\Delta}_x$ has rank more than one, then $\dot{\mathfrak{g}}$ and $\text{DS}_x(\dot{\mathfrak{g}})$ have the same lacity r^\vee , so the principal admissible levels for \mathfrak{g} and $\text{DS}_x(\mathfrak{g})$ coincide. If $\dot{\Delta}_x$ has rank one, then the lacity of $\dot{\mathfrak{g}}$ is 1 for $A(n \pm 1|n)$ and 2 for other cases, whereas the lacity of $\text{DS}_x(\dot{\mathfrak{g}})$ is 1; hence each principal admissible levels for \mathfrak{g} is principle admissible for $\text{DS}_x(\mathfrak{g})$.

4.4. Vacuum modules for principal admissible levels. Retain notation of § 4.3.

4.4.1. Take $x \in \dot{\mathfrak{g}}$ satisfying (1) such that x has a maximal rank, i.e.

$$\dot{\mathfrak{t}} := \text{DS}_x(\dot{\mathfrak{g}})$$

has zero defect. We denote by $I_{\dot{\mathfrak{t}}}(k)$ the maximal proper submodule of $Vac_{\dot{\mathfrak{t}}}^k$. Let k be an admissible level for $\dot{\mathfrak{t}}$. The vacuum module $Vac_{\dot{\mathfrak{t}}}^k$ has a singular vector of weight $r'_0.k\Lambda_0$, where

$$r'_0 := r_{\alpha'_0} \in W.$$

From [F] it follows that in the case when $\dot{\mathfrak{t}}$ is a simple Lie algebra, this singular vector generates $I_{\dot{\mathfrak{t}}}(k)$.

4.4.2. Theorem. *Let $\dot{\mathfrak{g}} \neq B(n+1|n)$ be a finite-dimensional Kac-Moody algebra and let k be a principal admissible level. Let $x \in \dot{\mathfrak{g}}_{\bar{1}}$ be of the maximal rank.*

Assume that $\mathfrak{t} := \mathfrak{g}_x$ satisfies the following: \mathfrak{t} is simple,

(A1) $I_{\mathfrak{t}}(k)$ is generated by a singular vector of weight $r'_0 k \Lambda_0$;

(A2) any irreducible $V_k(\mathfrak{t})$ -module in the category \mathcal{O} is principal admissible.

Then

(i) $DS_x(L(k\Lambda_0)) \cong L_{\mathfrak{g}_x}(k\Lambda_0)$ as \mathfrak{g}_x -modules;

(ii) $DS_x(V_k(\mathfrak{g})) \cong V_k(\mathfrak{g}_x)$ as vertex algebras;

(iii) for any $V_k(\mathfrak{g})$ -module N , $DS_x(N)$ is a $V_k(\mathfrak{g}_x)$ -module;

(iv) if N is a $V_k(\mathfrak{g})$ -module in \mathcal{O} , then $DS_x(N)$ is either zero or the direct sum of principal admissible modules of level k .

4.5. Proof of Theorem 4.4.2. Note that $\dot{\mathfrak{g}} \neq A(m|n), B(m|n), D(m|n)$ with $m = n, n+1$ and $D(n+2|n)$. Using Remark 2.1.2, we assume for that $S, \dot{\Sigma}$ satisfies (P1), (P2), (P3) of § 5, i.e.

$$S \subset \dot{\Sigma}, \quad (S, \theta) = 0, \quad \|\theta\|^2 = 2,$$

where θ is the maximal root in $\Delta^+(\dot{\Sigma})$, and (20) holds.

In particular, $\alpha_0 := \delta - \theta$ is the affine root for \mathfrak{g} and for \mathfrak{t} .

Since $L(k\Lambda_0)$ is $\dot{\mathfrak{g}}$ -integrable, we can (and will) assume that $\text{supp}(x) = S$.

We fix the \mathbb{Z}_2 -grading on Vac^k and all its subquotients by letting the highest weight vector to be even. For a \mathbb{Z}_2 -graded space E we write $\dim E = (a|b)$ if $\dim E_{\bar{0}} = a, \dim E_{\bar{1}} = b$. Retain notation of § 4.3.

4.5.1. Denote by $I(k)$ the maximal submodule of Vac^k and by $I_{\bar{\mathfrak{g}}}(p)$ the maximal submodule of the vacuum $\bar{\mathfrak{g}}$ -module $Vac_{\bar{\mathfrak{g}}}^p$. One has

$$Re^{-k\Lambda_0} ch I(k) = Re^{-k\Lambda_0} (ch Vac^k - ch L(k\Lambda_0)) = \dot{R} - Re^{-k\Lambda_0} ch L(k\Lambda_0),$$

where R, \dot{R}, \bar{R} are the Weyl denominators for $\Delta^+, \dot{\Delta}^+, \bar{\Delta}^+$ respectively; recall that $\dot{\Delta} \subset \bar{\Delta}$, so $\dot{\bar{R}} = \dot{R}$.

From (14) we have $Re^{-k\Lambda_0} ch L(k\Lambda_0) = \bar{R}e^{-p\bar{\Lambda}_0} ch L_{\bar{\mathfrak{g}}}(p\bar{\Lambda}_0)$. This gives

$$(15) \quad Re^{-k\Lambda_0} ch I(k) = \dot{R} - \bar{R}e^{-p\bar{\Lambda}_0} ch L_{\bar{\mathfrak{g}}}(p\bar{\Lambda}_0) = \bar{R}e^{-p\bar{\Lambda}_0} ch I_{\bar{\mathfrak{g}}}(p).$$

By § 2.1, $I_{\bar{\mathfrak{g}}}(p)$ is generated by a singular vector v' of the weight

$$\mu := p\bar{\Lambda}_0 - (p+1)\alpha'_0.$$

Now the formula (15) can be rewritten as

$$(16) \quad Re^{-r'_0.k\Lambda_0}chI(k) = \overline{R}e^{-\mu}chI_{\overline{\mathfrak{g}}}(p)$$

since $k\Lambda_0 + \mu - p\overline{\Lambda}_0 = k\Lambda_0 - (p+1)\alpha'_0 = r'_0.(k\Lambda_0)$.

Recall that $v' = f_0^{p+1}|0\rangle$, where $f_0 \in \mathfrak{g}_{-\alpha'_0}$. For any $\beta \in S$ we have $(\alpha'_0, \beta) = 0$, so $[\mathfrak{g}_{-\alpha'_0}, \mathfrak{g}_{\pm\beta}] = 0$. Therefore $\mathfrak{g}_{\pm\beta}v' = 0$ and so

$$(17) \quad \dim I_{\overline{\mathfrak{g}}}(p\Lambda_0)_\mu = (1|0), \quad I_{\overline{\mathfrak{g}}}(p\Lambda_0)_{\mu-\gamma} = 0 \quad \text{for } \gamma \in \mathbb{Z}S \setminus \{0\}.$$

4.5.2. Set

$$\mathcal{R} := \left\{ \sum_{\nu \in \mathbb{Z}_{\geq 0}S} a_\nu e^{-\nu} \mid a_\nu \in \mathbb{C} \right\}, \quad P_S \left(\sum_{\nu \in \mathbb{Z}_{\geq 0}S} a_\nu e^{-\nu} \right) := \sum_{\nu \in \mathbb{Z}_{\geq 0}S} a_\nu e^{-\nu}.$$

Clearly, \mathcal{R} has a ring structure; this ring does not have zero divisors. Note that P_S is a ring homomorphism (since $S \subset \Sigma$) and $P_S^2 = P_S$.

The ring \mathcal{R} contains $R, \dot{R}, \overline{R}, R^{-1}, \overline{R}^{-1}$ and

$$P_S(R) = P_S(\dot{R}) = P_S(\overline{R}).$$

Since $I_{\overline{\mathfrak{g}}}(p)$ is generated by a singular vector of weight μ , one has $e^{-\mu}chI_{\overline{\mathfrak{g}}}(p) \in \mathcal{R}$. By (17), $P_S(e^{-\mu}ch_{\overline{\mathfrak{g}}}I(p)) = 1$. Using (16) we get

$$(18) \quad e^{-r'_0.k\Lambda_0}chI(k) \in \mathcal{R}, \quad P_S(e^{-r'_0.k\Lambda_0}chI(k)) = 1,$$

By (16), $r'_0.k\Lambda_0$ is the highest weight of $I(k)$ and $\dim I(k)_{r'_0.k\Lambda_0} = 1$. Thus $I(k)_{r'_0.k\Lambda_0}$ is spanned by an even singular vector. By (18), this vector has non-zero image in $\text{DS}_x(I(k))$. $\text{DS}_x(I(k))_{r'_0.k\Lambda_0}$ is spanned by this image.

We conclude that $\text{DS}_x(I(k))_{r'_0.k\Lambda_0}$ is spanned by an even singular vector, which we denote by v_0 .

4.5.3. Recall that $\text{DS}_x(\text{Vac}^k) = \text{Vac}_t^k$. Consider the short exact sequence

$$0 \rightarrow I(k) \rightarrow \text{Vac}^k \rightarrow \text{Vac}_k \rightarrow 0$$

and the corresponding long exact sequence

$$0 \rightarrow E \rightarrow \text{DS}_x(I(k)) \xrightarrow{\phi} \text{Vac}_t^k \xrightarrow{\psi} \text{DS}_x(\text{Vac}_k) \rightarrow \Pi(E) \rightarrow 0.$$

By (A1), $I_t(k)$ is generated by a singular vector v'_0 of weight $r'_0.k\Lambda_0$. Since v_0, v'_0 are singular, $\phi(v_0)$ is proportional to v'_0 . There are two possibilities: either $\phi(v_0) = v'_0$ (up to a non-zero scalar) or $\phi(v_0) = 0$.

Assume that $\phi(v_0) = 0$. Since v_0 spans $DS_x(I(k))_{r'_0, k\Lambda_0}$ one has $v'_0 \notin \text{Im}\phi = \text{Ker}\psi$. Since $v_0 \in \text{Ker}\phi$ one has

$$(19) \quad \dim E_{r'_0, k\Lambda_0} = \dim DS_x(I(k))_{r'_0, k\Lambda_0} = (1|0).$$

Since $v'_0 \notin \text{Ker}\psi$, the \mathfrak{t} -module $DS_x(\text{Vac}_k)$ has an even indecomposable subquotient of length two with the socle $L_{\mathfrak{t}}(r'_0, k\Lambda_0)$ and the cosocle $L_{\mathfrak{t}}(k\Lambda_0)$. Since $\text{Vac}_k \cong L(k\Lambda_0)$ is self-dual, $DS_x(\text{Vac}_k)$ is also self-dual (see § 1.4); thus $DS_x(\text{Vac}_k)$ has an even indecomposable subquotient of length two with the cosocle $L_{\mathfrak{t}}(r'_0, k\Lambda_0)$ and the socle $L_{\mathfrak{t}}(k\Lambda_0)$. Since $I_{\mathfrak{t}}(k)$ is generated by a singular vector of weight $r'_0, k\Lambda_0$, one has $[\text{Vac}_k^k : L_{\mathfrak{t}}(r'_0, k\Lambda_0)] = 1$, so $\text{Im}\psi$ does not have such subquotient. Then $\Pi(E)$ has an even subquotient $L_{\mathfrak{t}}(r'_0, k\Lambda_0)$, which contradicts to (19).

We conclude that $\phi(v_0) = v'_0$ up to a non-zero scalar. Denote by a a preimage of v_0 in $I(k) \subset \text{Vac}_k$. Let N be a $V_k(\mathfrak{g})$ -module and $DS_x(N) \neq 0$. Since $\text{Vac}_k = \text{Vac}_k/I(k)$, § 3.1.1 gives $Y(a, z)N = 0$, so $Y(v_0, z)DS_x(N) = 0$. Since $\text{Vac}_{\mathfrak{t}, k} = \text{Vac}_k^k/I_{\mathfrak{t}}(k)$ with $I_{\mathfrak{t}}(k)$ generated by v'_0 , § 3.1.1 implies that $DS_x(N)$ is a $V_k(\mathfrak{t})$ -module. This establishes (iii).

Let us prove that $DS_x(\text{Vac}_k) = L_{\mathfrak{t}}(k\Lambda_0)$. Clearly, $[DS_x(\text{Vac}_k) : L_{\mathfrak{t}}(k\Lambda_0)] = 1$. Let $L_{\mathfrak{t}}(\lambda'')$ be a subquotient of $DS_x(\text{Vac}_k)$ and $\lambda'' \neq k\Lambda_0$. By (iii) and (A2), λ'' is a \mathfrak{t} -admissible weight. One has $\lambda'' = k\Lambda_0 - (\nu|_{\mathfrak{h}_S})$ for some $\nu \in (\mathbb{Z}_{\geq 0}\Sigma \cap S^{\perp})$. Recall that $(S, \check{\Sigma})$ satisfies (20), so $\nu|_{\mathfrak{h}_S} \in \mathbb{Z}\Sigma_S$, which contradicts to Lemma 4.7. This gives (i); (ii) follows from Theorem 3.4.1 (ii). \square

4.6. Corollary. *Let \mathfrak{g} is one of the following algebras: $A(m|n), C(n); B(m|n), m \geq n + 2; D(m|n), m \neq n + 1, n + 2, B(n|n), F(4)$ or $G(2)$. Take $x \in \mathfrak{g}_{\overline{1}}$ such that $\text{supp}(x)$ is maximal. Let k be an admissible level. Then*

- (i) $DS_x(L(k\Lambda_0)) \cong L_{\mathfrak{g}_x}(k\Lambda_0)$ as \mathfrak{g}_x -modules;
- (ii) $DS_x(V_k(\mathfrak{g})) \cong V_k(\mathfrak{g}_x)$ as vertex algebras;
- (iii) for any $V_k(\mathfrak{g})$ -module N , $DS_x(N)$ is a $V_k(\mathfrak{g}_x)$ -module;
- (iv) if N is a $V_k(\mathfrak{g})$ -module in \mathcal{O} , then $DS_x(N)$ is either zero or the direct sum of principal modules of level k .

Proof. For $\mathfrak{g}_x \neq 0$, the assumption (A1) of Theorem 4.4.2 follows from [F] and the assumption (A2) follows from Theorem 4.1.4. This gives (i)–(iii) for $\mathfrak{g}_x \neq 0$; (iv) follows from (iii) and Theorem 4.1.4.

If $\mathfrak{g}_x = 0$, then (i) is a particular case of Theorem 2.2 (i); moreover, (ii)–(iv) follow from (i). \square

4.7. The following lemma was used in the proof.

Lemma. *Let \mathfrak{t} has zero defect and let k be a principal admissible level. If λ is an admissible weight such that $k\Lambda_0 - \lambda \in \mathbb{Z}\Sigma$, then $\lambda = k\Lambda_0$.*

Proof. Since $k\Lambda_0 - \lambda \in \mathbb{Z}\Sigma$ one has $\Delta_{re}(\lambda) = \Delta_{re}(k\Lambda_0)$. Set

$$\lambda' := \lambda + (p + h^\vee)\left(1 - \frac{1}{u}\right)\Lambda_0.$$

One readily sees that λ' is a dominant weight of level p . One has $p\Lambda_0 - \lambda' = k\Lambda_0 - \lambda \in \mathbb{Z}\Sigma$. Since λ' is dominant,

$$0 \leq (\Lambda_0, p\Lambda_0 - \lambda') = -(\lambda', \Lambda_0) \quad \text{and} \quad 0 \leq (p\Lambda_0 - \lambda', \lambda') = p(\Lambda_0, \lambda') - (\lambda', \lambda').$$

Therefore $(\lambda', \lambda') = (\Lambda_0, \lambda') = 0$. Since $p\Lambda_0 - \lambda' = k\Lambda_0 - \lambda \in \mathbb{Z}\Sigma$, we obtain $k\Lambda_0 - \lambda = 0$ as required. \square

5. APPENDIX

We fix the standard triangular decomposition in $\dot{\Delta}_{\bar{0}}$ and consider the bases $\dot{\Sigma}$ which are compatible with this decomposition. For each base $\dot{\Sigma}$ denote by $\theta_{\dot{\Sigma}}$ the maximal root of $\Delta^+(\dot{\Sigma})$.

Let \mathcal{S} be the set of maximal isotropic subsets of $\Delta_{\bar{1}}$. Consider the action of the Weyl group \dot{W} on \mathcal{S} . For each orbit it is not hard to give an example of a pair $(S, \dot{\Sigma})$ such that S is a representative of this orbit and

$$(P1) \quad S \subset \dot{\Sigma};$$

$$(P2) \quad \theta_{\dot{\Sigma}} \in \dot{\Delta}^\# \quad \text{and} \quad (\theta_{\dot{\Sigma}}, S) = 0 \quad \text{for} \quad \dot{\mathfrak{g}} \neq A(m|n), B(m|n), D(m|n) \quad \text{with} \quad m = n, n + 1;$$

$$(P3) \quad \text{if} \quad \dot{\mathfrak{g}} \neq D(n + 1, n), D(n + 2|n), \quad \text{then the following inclusion holds}$$

$$(20) \quad (\mathbb{Q}_{\geq 0}\Sigma \cap S^\perp) \subset (\mathbb{Q}S + \mathbb{Q}_{\geq 0}\Sigma_S),$$

where $\Sigma = \{\delta - \theta_{\dot{\Sigma}}\} \cup \dot{\Sigma}$ is a base for $\Delta = \dot{\Delta}^{(1)}$.

For instance, for $B(m, n), D(m, n), n > m$ one has $\dot{\Delta}^\# = C_n$. We take $S := \{\varepsilon_i - \delta_{i+1}\}_{i=1}^m$ and

$$\dot{\Sigma} := \{\delta_1 - \varepsilon_1, \varepsilon_1 - \delta_2, \dots, \varepsilon_m - \delta_{m+1}, \delta_{m+1} - \delta_{m+2}, \dots, \delta_{n-1} - \delta_n, a\delta_n\},$$

where $a = 1$ for $B(m|n)$ and $a = 2$ for $D(m|n)$. One has $\theta = 2\delta_1$, so (P1), (P2) are satisfied. One has

$$\dot{\Sigma}_S = \{\delta_1 - \delta_{m+2}, \dots, \delta_{n-1} - \delta_n, a\delta_n\},$$

of type $B(0|n - m)$ for $B(m|n)$ and C_{n-m} for $D(m|n)$; it is easy to see that (20) holds.

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