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On a generalization of multiple zeta functions from the viewpoint of symmetric functions*

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1 Introduction

The multiple zeta and the multiple zeta-star function (MZF and MZSF for short) of Euler-Zagier type are respectively defined by the series

$$\zeta(\boldsymbol{s})=\sum_{1\le m_1<\cdots<m_n}\frac{1}{m_1^{s_1}\cdots m_n^{s_n}},\quad \zeta^{\star}(\boldsymbol{s})=\sum_{1\le m_1\le\cdots\le m_n}\frac{1}{m_1^{s_1}\cdots m_n^{s_n}},$$

where $\boldsymbol{s}=(s_1,\ldots,s_n)\in\mathbb{C}^n$. These series converge absolutely for $\Re(s_1),\ldots,\Re(s_{n-1})\geq 1$ and $\Re(s_n)>1$. One easily sees that a MZSF can be expressed as a linear combination of MZFs, and vice versa. For instance,

$$\begin{aligned}
\zeta^{\star}(s_1,s_2)&=\zeta(s_1,s_2)+\zeta(s_1+s_2),\\
\zeta(s_1,s_2)&=\zeta^{\star}(s_1,s_2)-\zeta^{\star}(s_1+s_2),\\
\zeta^{\star}(s_1,s_2,s_3)&=\zeta(s_1,s_2,s_3)+\zeta(s_1+s_2,s_3)+\zeta(s_1,s_2+s_3)+\zeta(s_1+s_2+s_3),\\
\zeta(s_1,s_2,s_3)&=\zeta^{\star}(s_1,s_2,s_3)-\zeta^{\star}(s_1+s_2,s_3)-\zeta^{\star}(s_1,s_2+s_3)+\zeta^{\star}(s_1+s_2+s_3),
\end{aligned}$$

where $\zeta(s)=\zeta^{\star}(s)$ is the Riemann zeta function. The special value of $\zeta(s_1,\ldots,s_n)$ and $\zeta^{\star}(s_1,\ldots,s_n)$ at positive integers were first introduced by Euler [E] for $n=2$, and by Hoffman [H] and Zagier [Z] for general n, independently. It is known that they appear in various branches of mathematics and mathematical physics, such as quantum field theory, knot theory, mixed Tate motive and quantum groups.

The aim of this article is to introduce a (skew) *Schur multiple zeta function* $\zeta_{\lambda/\mu}(\boldsymbol{s})$ for each (skew) Young diagram λ/μ from the view point of symmetric functions (as an analogue of the (skew) Schur function $s_{\lambda/\mu}$) and study its combinatorial and arithmetic properties. For instance, we will show a Jacobi-Trudi formula, Giambelli formula and dual Cauchy formula for Schur multiple zeta functions as analogues of those for Schur functions. Moreover, we will also give so-called $1,3$ formulas for them as analogues of those for MZFs and MZSFs.

2 Definition of SMZFs

2.1 Combinatorial settings

We first set up some notions of partitions.

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A partition of $n \in \mathbb{N}$ is a tuple $\lambda = (\lambda_1, \dots, \lambda_p)$ of positive integers $\lambda_1 \geq \cdots \geq \lambda_p \geq 1$ with $n = |\lambda| = \lambda_1 + \cdots + \lambda_p$. In this case, we write $\lambda \vdash n$. For another partition $\mu = (\mu_1, \dots, \mu_q)$, we write $\mu \subset \lambda$ if $q \leq p$ and $\mu_i \leq \lambda_i$ for all $1 \leq i \leq q$. For partitions λ, μ with $\mu \subset \lambda$, we identify the pair $\lambda/\mu = (\lambda, \mu)$ with its (skew) Young diagram $D(\lambda/\mu) = \{(i,j) \in \mathbb{Z}^2 \,|\, 1 \leq i \leq p,\ \mu_i < j \leq \lambda_i\}$ where we set $\mu_i = 0$ for $i > q$. In the case where $\mu = \emptyset$ is the empty partition, we just write $\lambda/\mu = \lambda$. We put $|\lambda/\mu| = |\lambda| - |\mu|$. An entry $(i,j) \in D(\lambda/\mu)$ is called a corner of λ/μ if $(i, j+1) \notin D(\lambda/\mu)$ and $(i+1, j) \notin D(\lambda/\mu)$. We denote the set of all corners of λ/μ by $\mathrm{Cor}(\lambda/\mu)$. The conjugate of λ/μ is the pair λ'/μ' with $\lambda' = (\lambda'_1, \dots, \lambda'_{p'})$ and $\mu' = (\mu'_1, \dots, \mu'_{q'})$ where $p' = \lambda_1$ and $\mu' = \mu_1$ whose Young diagram is the transpose of that of λ/μ. A (skew) Young tableau $s = (s_{i,j})_{(i,j)\in D(\lambda/\mu)}$ of shape λ/μ is a filling of $D(\lambda/\mu)$ obtained by putting $s_{i,j} \in \mathbb{C}$ into the (i,j)-entry of $D(\lambda/\mu)$. We will also just write $(s_{i,j})$ if the shape λ/μ is clear from the context. A Young tableau $(m_{i,j})$ is called semi-standard if $m_{i,j} \in \mathbb{N}$, $m_{i,j} < m_{i+1,j}$ and $m_{i,j} \leq m_{i,j+1}$ for all i and j. The set of all Young tableaux and all semi-standard Young tableaux of shape λ/μ are denoted by $T(\lambda/\mu)$ and $\mathrm{SSYT}(\lambda/\mu)$, respectively.

2.2 Definition

We call a Young tableau $s = (s_{i,j}) \in T(\lambda/\mu)$ admissible if $\Re(s_{i,j}) > 1$ for $(i,j) \in \mathrm{Cor}(\lambda/\mu)$ and $\Re(s_{i,j}) \geq 1$ otherwise. Let $W_{\lambda/\mu} \subset T(\lambda/\mu)$ be the set of all admissible Young tableaux of shape λ/μ. For $s \in W_{\lambda/\mu}$, the Schur multiple zeta function (SMZF for short) associated with λ/μ is defined by

$$\zeta(s) = \zeta_{\lambda/\mu}(s) = \sum_{(m_{i,j})\in \mathrm{SSYT}(\lambda/\mu)} \prod_{(i,j)\in D(\lambda/\mu)} \frac{1}{m_{i,j}^{s_{i,j}}}$$

and $\zeta_\emptyset = 1$ for convenience. It is essentially shown in [NPY, Lemma 2.1] that the series converges absolutely for $s \in W_{\lambda/\mu}$. Clearly, this is a generalization of both MZFs and MZSFs since

$$\zeta(s_1, \dots, s_n) = \zeta_{(1^n)}\left(\begin{array}{|c|}\hline s_1 \\ \hline \vdots \\ \hline s_n \\ \hline \end{array}\right), \quad \zeta^\star(s_1, \dots, s_n) = \zeta_{(n)}\left(\begin{array}{|c|c|c|}\hline s_1 & \cdots & s_n \\ \hline \end{array}\right). \tag{2.1}$$

Remark 2.1. Our Schur multiple zeta functions is not new in the sense that it can be written as a linear combination of MZFs or MZSFs. For example, when $\lambda/\mu = (2,1)/(1)$, we have

$$\begin{aligned}
\zeta\left(\begin{array}{|c|c|}\hline s_{1,1} & s_{1,2} \\ \hline s_{2,1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\right) &= \sum_{\substack{m_{1,1} \leq m_{1,2} \\ \wedge \\ m_{2,1}}} \frac{1}{m_{1,1}^{s_{1,1}} m_{1,2}^{s_{1,2}} m_{2,1}^{s_{2,1}}} \\
&= \zeta(s_{1,1}, s_{1,2}, s_{2,1}) + \zeta(s_{1,1}, s_{2,1}, s_{1,2}) + \zeta(s_{1,1}, s_{1,2} + s_{2,1}) + \zeta(s_{1,1} + s_{1,2}, s_{2,1}) \\
&= \zeta^\star(s_{1,1}, s_{1,2}, s_{2,1}) + \zeta^\star(s_{1,1}, s_{2,1}, s_{1,2}) - \zeta^\star(s_{1,1}, s_{1,2} + s_{2,1}) - \zeta^\star(s_{1,1} + s_{2,1}, s_{1,2}).
\end{aligned}$$

In the following discussion, we often study such multiple zeta functions by regarding them as analogues of symmetric functions. Actually, let $x = (x_1, x_2, \dots)$ be variables and

$$s_{\lambda/\mu} = s_{\lambda/\mu}(x) = \sum_{(m_{i,j})\in \mathrm{SSYT}(\lambda/\mu)} \prod_{(i,j)\in D(\lambda/\mu)} x_{m_{i,j}}$$

the Schur function associated with λ/μ. One easily sees that $\zeta_{\lambda/\mu}$ is an analogue of $s_{\lambda/\mu}$. Moreover, for $n \in \mathbb{Z}_{\geq 0}$, let

$$e_n = e_n(x) = \sum_{1\leq m_1 < \cdots < m_n} x_{m_1} \cdots x_{m_n}, \quad h_n = h_n(x) = \sum_{1\leq m_1 \leq \cdots \leq m_n} x_{m_1} \cdots x_{m_n}$$

be the ementary and complete symmetric functions, respectively. Then, since $s_{(1^n)} = e_n$ and $s_{(n)} = h_n$, from (2.1), we can say that ζ and $\zeta^\star$ are respectively analogues of e_n and h_n. Notice that the power-sum symmetric function

$$p_n = p_n(x) = \sum_{i=1}^{\infty} x_i^n,$$

which is another important class of symmetric functions, corresponds to $\zeta(ns)$.

2.3 A special case

For $s \in \mathbb{C}$, let $\{s\}^{\lambda/\mu} = (s_{i,j}) \in T(\lambda/\mu)$ be the Young tableau of shape λ/μ defined by $s_{i,j} = s$ for all $(i,j) \in D(\lambda/\mu)$. We here notice that, though it is not true for general $s \in W_{\lambda/\mu}$, $\zeta(\{s\}^{\lambda/\mu})$ is realized as a specialization of the Schur function, namely,

$$\zeta(\{s\}^{\lambda/\mu}) = s_{\lambda/\mu}(1^{-s}, 2^{-s}, \ldots).$$

This leads the following result.

Proposition 2.2. Let $\Re(s) > 1$. Then, $\zeta(\{s\}^{\lambda/\mu})$ can be written as a polynomial in $\zeta(s), \zeta(2s), \zeta(3s), \ldots$ with rational number coefficients. More precisely, if we write each monomial modulo coefficient as $\prod_i^r \zeta(\nu_i s)$ satisfying $\nu_1 \geq \cdots \geq \nu_r$, then $(\nu_1, \ldots, \nu_r) \vdash |\lambda/\mu|$.

Proof. Since $s_{\lambda/\mu}(x)$ is symmetric, it can be written as a $\mathbb{Q}$-linear combination of $p_\nu(x) = \prod_{i=1}^r p_{\nu_i}(x)$ for $\nu = (\nu_1, \ldots, \nu_r) \vdash |\lambda/\mu|$ (see [Ma]). Therefore, $\zeta(\{s\}^{\lambda/\mu}) = s_{\lambda/\mu}(1^{-s}, 2^{-s}, \ldots)$ as a $\mathbb{Q}$-linear combination of $p_\nu(1^{-s}, 2^{-s}, \ldots) = \prod_{i=1}^r p_{\nu_i}(1^{-s}, 2^{-s}, \ldots) = \prod_{i=1}^r \zeta(\nu_i s)$. $\square$

Corollary 2.3. For $k \in \mathbb{N}$, $\zeta(\{2k\}^{\lambda/\mu}) \in \mathbb{Q}\pi^{2k|\lambda/\mu|}$.

Proof. This immediately follows from Proposition 2.2 with the well-known formula $\zeta(2k) = (-1)^{k-1}\frac{2^{2k}B_{2k}}{2(2k)!}\pi^{2k} \in \mathbb{Q}\pi^{2k}$ where $B_{2k} \in \mathbb{Q}$ is the Bernoulli number. $\square$

Example 2.4. When $\lambda/\mu = (3,2)/(1)$, since

$$s_{(3,2)/(1)} = -\frac{1}{4}p_4 - \frac{1}{3}p_3p_1 + \frac{1}{8}p_2^2 + \frac{1}{4}p_2p_1^2 + \frac{5}{24}p_1^4,$$

we have

$$\zeta\left(\begin{array}{ccc} & \boxed{s} & \boxed{s} \\ \boxed{s} & \boxed{s} & \end{array}\right) = -\frac{1}{4}\zeta(4s) - \frac{1}{3}\zeta(3s)\zeta(s) + \frac{1}{8}\zeta(2s)^2 + \frac{1}{4}\zeta(2s)\zeta(s)^2 + \frac{5}{24}\zeta(s)^4$$

and hence

$$\zeta\left(\begin{array}{ccc} & \boxed{2} & \boxed{2} \\ \boxed{2} & \boxed{2} & \end{array}\right) = \frac{61}{362880}\pi^8,$$

$$\zeta\left(\begin{array}{ccc} & \boxed{4} & \boxed{4} \\ \boxed{4} & \boxed{4} & \end{array}\right) = \frac{667}{631547280000}\pi^{16},$$

$$\zeta\left(\begin{array}{ccc} & \boxed{6} & \boxed{6} \\ \boxed{6} & \boxed{6} & \end{array}\right) = \frac{9077644}{432684797065192546875}\pi^{24}.$$

Remark 2.5. Let $f^{\lambda/\mu}$ be the number of standard Young tableaux of shape λ/μ (i.e., semi-standard tableaux $(m_{i,j}) \in \mathrm{SSYT}(\lambda/\mu)$ such that $\{m_{i,j} \mid (i,j) \in D(\lambda/\mu)\} = \{1,2,\ldots,|\lambda/\mu|\}$). It is shown in [S] that, only for $k=1,2,3$, there exists Young tableaux σ/τ such that $\zeta(\{2k\}^{\lambda/\mu}) = C_{\lambda/\mu}(2k)\pi^{2k|\lambda/\mu|}$ where $C_{\lambda/\mu}(2k) \in \mathbb{Q}$ is involved in $f^{\sigma/\tau}$. More precisely, for any λ/μ satisfying $\lambda_1 \leq m$ and $|\lambda/\mu| = n$, it can be written as

$$\zeta(\{2\}^{\lambda/\mu}) = \frac{f^{(2\lambda'+\gamma_m+\delta_{m-1})/(2\mu'+\delta_{m-1})}}{(2n+m)!}\pi^{2n},$$

$$\zeta(\{4\}^{\lambda/\mu}) = \frac{2^{m+2n} f^{(4\lambda'+2\gamma_m+3\delta_{m-1})/(4\mu'+3\delta_{m-1})}}{(4n+2m)!}\pi^{4n},$$

$$\zeta(\{6\}^{\lambda/\mu}) = \frac{6^m 2^{6n} f^{(6\lambda'+3\gamma_m+5\delta_{m-1})/(6\mu'+5\delta_{m-1})}}{(6n+3m)!}\pi^{6n},$$

where $\gamma_m = (1^m)$ and $\delta_m = (m, m-1, \ldots, 2, 1)$. For example, when $\lambda/\mu = (3,2)/(1)$, $m = \lambda_1 = 3$ (and $n=4$), we have

$$\zeta\left(\begin{array}{ccc} & \boxed{2} & \boxed{2} \\ \boxed{2} & \boxed{2} & \end{array}\right) = \frac{f^{(7,6,3)/(4,1)}}{11!}\pi^8,$$

$$\zeta\left(\begin{array}{ccc} & \boxed{4} & \boxed{4} \\ \boxed{4} & \boxed{4} & \end{array}\right) = \frac{2^{11} f^{(16,13,6)/(10,3)}}{22!}\pi^{16},$$

$$\zeta\left(\begin{array}{ccc} & \boxed{6} & \boxed{6} \\ \boxed{6} & \boxed{6} & \end{array}\right) = \frac{6^3 2^{24} f^{(25,20,9)/(16,5)}}{33!}\pi^{24}.$$

Together with the expressions in Example 2.4, we have respectively

$$f^{(7,6,3)/(4,1)} = 6710, \quad f^{(16,13,6)/(10,3)} = 579637674, \quad f^{(25,20,9)/(16,5)} = 50270540048960.$$

3 Relations among SMZFs

This section is devoted to give relations among Schur multiple zeta functions which are anlogues of those for Schur functions. To describe our results, we need the set

$$W^{\mathrm{diag}}_{\lambda/\mu} = \left\{ (s_{i,j}) \in W_{\lambda/\mu} \mid s_{i,j} = s_{k,l} \text{ if } j-i = l-k \right\}.$$

For a tableau $s = (s_{i,j}) \in W^{\mathrm{diag}}_{\lambda/\mu}$, we always write $s_u = s_{c(u)}$ where $c(u) = j-i$ is the content of $u = (i,j) \in D(\lambda/\mu)$. For example, $s = (s_{i,j}) \in W^{\mathrm{diag}}_{(4,3,2)}$ implies that s is of the form

$$s = \begin{array}{|c|c|c|c|} \hline s_{1,1} & s_{1,2} & s_{1,3} & s_{1,4} \\ \hline s_{2,1} & s_{2,2} & s_{2,3} & \\ \cline{1-3} s_{3,1} & s_{3,2} & & \\ \cline{1-2} \end{array} = \begin{array}{|c|c|c|c|} \hline s_0 & s_1 & s_2 & s_3 \\ \hline s_{-1} & s_0 & s_1 & \\ \cline{1-3} s_{-2} & s_{-1} & & \\ \cline{1-2} \end{array}.$$

3.1 Jacobi-Trudi formula

Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ and $\mu = (\mu_1, \ldots, \mu_q)$ be partitions satisfying $\mu \subset \lambda$. Write $\lambda' = (\lambda'_1, \ldots, \lambda'_{p'})$ and $\mu' = (\mu'_1, \ldots, \mu'_{q'})$ with $p' = \lambda_1$ and $q' = \mu_1$. Recall that the Schur function $s_{\lambda/\mu}$ satisfies the following Jacobi-Trudi formula

$$s_{\lambda/\mu} = \det\left[h_{\lambda_i-\mu_j-i+j}\right]_{1\leq i,j\leq p},$$

$$s_{\lambda/\mu} = \det\left[e_{\lambda'_i-\mu'_j-i+j}\right]_{1\leq i,j\leq p'},$$

where we understand that $h_0 = e_0 = 1$ and $h_n = e_n = 0$ if $n < 0$. For example, when $\lambda/\mu = (4,3,2)/(2,1)$, we have

$$s_{(4,3,2)/(2,1)} = \begin{vmatrix} h_2 & h_4 & h_6 \\ 1 & h_2 & h_4 \\ 0 & 1 & h_2 \end{vmatrix}, \quad s_{(4,3,2)/(2,1)} = \begin{vmatrix} e_1 & e_3 & e_5 & e_6 \\ 1 & e_2 & e_4 & e_5 \\ 0 & 1 & e_2 & e_3 \\ 0 & 0 & 1 & e_1 \end{vmatrix}.$$

As an analogue of these formulas, SMZFs satisfy the following formula.

Theorem 3.1 ([NPY, Theorem 1.1 for $\mu = \emptyset$ and Theorem 4.3 for general μ]). Retain the above notations. Assume that $\boldsymbol{s} = (s_{i,j})_{(i,j)\in D(\lambda/\mu)} = (s_{c(u)})_{u\in D(\lambda/\mu)} \in W^{\mathrm{diag}}_{\lambda/\mu}$.

(1) Assume further that $\Re(s_{i,\lambda_i}) > 1$ for all $1 \le i \le p$. Then, we have

$$\zeta_{\lambda/\mu}(\boldsymbol{s}) = \det\left[\zeta^{\star}(s_{\mu_j-j+1}, s_{\mu_j-j+2}, \ldots, s_{\mu_j-j+(\lambda_i-\mu_j-i+j)})\right]_{1\le i,j\le p}.$$

Here, we understand that $\zeta^{\star}(\cdots) = 1$ if $\lambda_i - \mu_j - i + j = 0$ and 0 if $\lambda_i - \mu_j - i + j < 0$.

(2) Assume further that $\Re(s_{\lambda'_i,i}) > 1$ for all $1 \le i \le p'$. Then, we have

$$\zeta_{\lambda/\mu}(\boldsymbol{s}) = \det\left[\zeta(s_{-\mu'_j+j-1}, s_{-\mu'_j+j-2}, \ldots, s_{-\mu'_j+j-(\lambda'_i-\mu'_j-i+j)})\right]_{1\le i,j\le p'}.$$

Here, we understand that $\zeta(\cdots) = 1$ if $\lambda'_i - \mu'_j - i + j = 0$ and 0 if $\lambda'_i - \mu'_j - i + j < 0$.

We notice that just combining these two expressions for $\zeta_{\lambda/\mu}(\boldsymbol{s})$, we obtain a family of algebraic relations among MZFs and MZSFs.

Example 3.2. When $\lambda/\mu = (4,3,2)/(2,1)$, we have

$$\zeta\left(\begin{array}{cccc} & & \boxed{s_2} & \boxed{s_3} \\ & \boxed{s_0} & \boxed{s_1} & \\ \boxed{s_{-2}} & \boxed{s_{-1}} & & \end{array}\right) = \begin{vmatrix} \zeta^{\star}(s_2,s_3) & \zeta^{\star}(s_0,s_1,s_2,s_3) & \zeta^{\star}(s_{-2},s_{-1},s_0,s_1,s_2,s_3) \\ 1 & \zeta^{\star}(s_0,s_1) & \zeta^{\star}(s_{-2},s_{-1},s_0,s_1) \\ 0 & 1 & \zeta^{\star}(s_{-2},s_{-1}) \end{vmatrix},$$

$$\zeta\left(\begin{array}{cccc} & & \boxed{s_2} & \boxed{s_3} \\ & \boxed{s_0} & \boxed{s_1} & \\ \boxed{s_{-2}} & \boxed{s_{-1}} & & \end{array}\right) = \begin{vmatrix} \zeta(s_{-2}) & \zeta(s_0,s_{-1},s_{-2}) & \zeta(s_2,s_1,s_0,s_{-1},s_{-2}) & \zeta(s_3,s_2,s_1,s_0,s_{-1},s_{-2}) \\ 1 & \zeta(s_0,s_{-1}) & \zeta(s_2,s_1,s_0,s_{-1}) & \zeta(s_3,s_2,s_1,s_0,s_{-1}) \\ 0 & 1 & \zeta(s_2,s_1) & \zeta(s_3,s_2,s_1) \\ 0 & 0 & 1 & \zeta(s_3) \end{vmatrix}.$$

In [NPY], the proof of Theorem 3.1 is given in two ways: One is obtained by establishing an analogue of Lindström-Gessel-Viennot Lemma. Namely, we can regard $\zeta_{\lambda/\mu}(\boldsymbol{s})$ for $\boldsymbol{s} \in W^{\mathrm{diag}}_{\lambda/\mu}$ as a sum of weights, defined by the variable $\boldsymbol{s}$, of certain lattice paths in $\mathbb{Z}^2$ determined by λ/μ (remark that, in [NPY], this proof is given only for the case of $\mu = \emptyset$, however, one can easily generalize it for general μ). Another is obtained by regarding $\zeta_{\lambda/\mu}(\boldsymbol{s})$ as a specialization of Macdonald's ninth variation of Schur functions studied by Nakagawa, Noumi, Shirakawa and Yamada [NNSY], which satisfy the Jacobi-Trudi formulas.

Remark 3.3. In general, for $\boldsymbol{s} \in W_{\lambda/\mu}$, we can also find a kind of Jacobi-Trudi formulas for $\zeta_{\lambda/\mu}(\boldsymbol{s})$, however, in this case, we encounter some "error terms" which disappear when $\boldsymbol{s} \in W^{\mathrm{diag}}_{\lambda/\mu}$, in the formulas. For example, when $\lambda = (2,2)$, we have

$$\zeta\left(\begin{ytableau} a & b \\ c & d \end{ytableau}\right) = \begin{vmatrix} \zeta^\star(a,b) & \zeta^\star(c,d,b) \\ \zeta^\star(a) & \zeta^\star(c,d) \end{vmatrix}$$
$$+\,\zeta^\star(c,d,b,a) - \zeta^\star(c,a,b,d) + \zeta^\star(c,a,b+d) - \zeta^\star(c,d,b+a),$$
$$\zeta\left(\begin{ytableau} a & b \\ c & d \end{ytableau}\right) = \begin{vmatrix} \zeta(a,c) & \zeta(b,d,c) \\ \zeta(a) & \zeta(b,d) \end{vmatrix}$$
$$+\,\zeta(b,d,c,a) - \zeta(b,a,c,d) + \zeta(b,d,c+a) - \zeta(b,a,c+d).$$

Notice that the error terms $\zeta^\star(c,d,b,a) - \zeta^\star(c,a,b,d) + \zeta^\star(c,a,b+d) - \zeta^\star(c,d,b+a)$ and $\zeta(b,d,c,a) - \zeta(b,a,c,d) + \zeta(b,d,c+a) - \zeta(b,a,c+d)$ disappear when $a = d$. It seems to be interesting to find explicit expressions of the error terms for the Jacobi-Trudi formulas of $\zeta_{\lambda/\mu}(\boldsymbol{s})$ for $\boldsymbol{s} \in W_{\lambda/\mu}$.

3.2 Giambelli formula

Let $\lambda = (p_1 - 1, \dots, p_t - 1 \,|\, q_1, \dots, q_t)$ be the Frobenius notation of the partition λ, that is, t is the number of diagonal entries of λ and $p_1, \dots, p_t$ and $q_1, \dots, q_t$ are respectively defined by $p_i = \lambda_i - i + 1$ and $q_i = \lambda'_i - i$ for $1 \le i \le t$. Notice that $p_1 > p_2 > \cdots > p_t > 0$ and $q_1 > q_2 > \cdots > q_t \ge 0$. The Giambelli formula for Schur function s_λ is given by

$$s_\lambda = \det\left[s_{(p_i,1^{q_j})}\right]_{1\le i,j\le t}.$$

For example, when $\lambda = (4,3,3,2) = (3,1,0\,|\,3,2,0)$, we have

$$s_\lambda = \begin{vmatrix} s_{(4,1^3)} & s_{(4,1^2)} & s_{(4,1^0)} \\ s_{(2,1^3)} & s_{(2,1^2)} & s_{(2,1^0)} \\ s_{(1,1^3)} & s_{(1,1^2)} & s_{(1,1^0)} \end{vmatrix} = \begin{vmatrix} s_{\ydiagram{4,1,1,1}} & s_{\ydiagram{4,1,1}} & s_{\ydiagram{4}} \\ s_{\ydiagram{2,1,1,1}} & s_{\ydiagram{2,1,1}} & s_{\ydiagram{2}} \\ s_{\ydiagram{1,1,1,1}} & s_{\ydiagram{1,1,1}} & s_{\ydiagram{1}} \end{vmatrix}$$

As an analogue of these formulas, SMZFs satisfy the following formula.

Theorem 3.4 ([NPY, Theorem 4.5]). Retain the above notations. Assume that $\boldsymbol{s} = (s_{i,j})_{(i,j)\in D(\lambda)} = (s_{c(u)})_{u\in D(\lambda)} \in W_\lambda^{\mathrm{diag}}$. Moreover, assume further that $\Re(s_{i,\lambda_i}) = \Re(s_{p_i-1}) > 1$ and $\Re(s_{\lambda'_i,i}) = \Re(s_{-q_i}) > 1$ for $1 \le i \le t$. Then, we have

$$\zeta_\lambda(\boldsymbol{s}) = \det\left[\zeta_{(p_i,1^{q_j})}(\boldsymbol{s}_{i,j})\right]_{1\le i,j\le t},$$

where

$$\boldsymbol{s}_{i,j} = \begin{ytableau} s_0 & s_1 & s_2 & \cdots & s_{p_i-1} \\ s_{-1} \\ \vdots \\ s_{-q_j} \end{ytableau} \in W_{(p_i,1^{q_j})}.$$

This is also obtained by regarding $\zeta_\lambda(\boldsymbol{s})$ as a specialization of Macdonald's ninth variation of Schur functions, which satisfy the Giambelli formula.

Example 3.5. When $\lambda = (4,3,3,2) = (3,1,0\,|\,3,2,0)$, we have

$$\zeta\left(\begin{ytableau} s_0 & s_1 & s_2 & s_3 \\ s_{-1} & s_0 & s_1 \\ s_{-2} & s_{-1} & s_0 \\ s_{-3} & s_{-2} \end{ytableau}\right) = \begin{vmatrix} \zeta\left(\begin{ytableau} s_0 & s_1 & s_2 & s_3 \\ s_{-1} \\ s_{-2} \\ s_{-3} \end{ytableau}\right) & \zeta\left(\begin{ytableau} s_0 & s_1 & s_2 & s_3 \\ s_{-1} \\ s_{-2} \end{ytableau}\right) & \zeta\left(\begin{ytableau} s_0 & s_1 & s_2 & s_3 \end{ytableau}\right) \\ \zeta\left(\begin{ytableau} s_0 & s_1 \\ s_{-1} \\ s_{-2} \\ s_{-3} \end{ytableau}\right) & \zeta\left(\begin{ytableau} s_0 & s_1 \\ s_{-1} \\ s_{-2} \end{ytableau}\right) & \zeta\left(\begin{ytableau} s_0 & s_1 \end{ytableau}\right) \\ \zeta\left(\begin{ytableau} s_0 \\ s_{-1} \\ s_{-2} \\ s_{-3} \end{ytableau}\right) & \zeta\left(\begin{ytableau} s_0 \\ s_{-1} \\ s_{-2} \end{ytableau}\right) & \zeta\left(\begin{ytableau} s_0 \end{ytableau}\right) \end{vmatrix}.$$

3.3 Dual Cauchy formula

Let $p, q \in \mathbb{N}$ and put $r = p+q$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_p) \subset (q^p)$, define the complementary partition $\lambda^* \subset (q^p)$ of λ by $\lambda^* = (p - \lambda'_q, \ldots, p - \lambda'_1) \subset (p^q)$. For example, when $p = 5$, $q = 7$ and $\lambda = (6,4,4,3,1)$, we have $\lambda^* = (5,4,4,2,1,1,0)$.

For variables $\boldsymbol{x} = (x_1, x_2, \ldots, x_p)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_q)$, the dual Cauchy formula for Schur function is given by

$$\sum_{\lambda \subset (q^p)} s_\lambda(\boldsymbol{x}) s_{\lambda^*}(\boldsymbol{y}) = \prod_{i=1}^{p} \prod_{j=1}^{q} (x_i + y_j)$$

As an analogue of these formulas, the SMZFs satisfy the following formula.

Theorem 3.6 ([NPY, Theorem 4.8]). Retain the above notations. Let $\boldsymbol{s} = (s_{i,j})_{(i,j)\in D((q^p))} = (s_{c(u)})_{u \in D((q^p))} \in W^{\mathrm{diag}}_{(q^p)}$ and $\boldsymbol{t} = (t_{i,j})_{(i,j)\in D((p^q))} = (t_{c(u)})_{u\in D((p^q))} \in W^{\mathrm{diag}}_{(p^q)}$. Assume that $\Re(s_i) \geq 2$ and $\Re(t_i) \geq 2$ for all $i \in \mathbb{Z}$. Then, we have

$$\sum_{\lambda \subset (q^p)} (-1)^{|\lambda|} \zeta_\lambda(\boldsymbol{s}|_\lambda)\, \zeta_{\lambda^*}(\boldsymbol{t}|_{\lambda^*})$$

$$= \det \left[\begin{array}{cccccccc} 1 & \zeta^\star(s_{1-p}) & \zeta^\star(s_{1-p}, s_{2-p}) & \cdots & \zeta^\star(s_{1-p}, \ldots, s_0) & \cdots & \zeta^\star(s_{1-p}, \ldots, s_{r-1-p}) \\ 0 & 1 & \zeta^\star(s_{2-p}) & \cdots & \zeta^\star(s_{2-p}, \ldots, s_0) & \cdots & \zeta^\star(s_{2-p}, \ldots, s_{r-1-p}) \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & \zeta^\star(s_0) & \cdots & \zeta^\star(s_0, \ldots, s_{r-1-p}) \\ \hline 1 & \zeta^\star(t_{1-q}) & \zeta^\star(t_{1-q}, t_{2-q}) & \cdots & \zeta^\star(t_{1-q}, \ldots, t_0) & \cdots & \zeta^\star(t_{1-q}, \ldots, t_{r-1-q}) \\ 0 & 1 & \zeta^\star(t_{2-q}) & \cdots & \zeta^\star(t_{2-q}, \ldots, t_0) & \cdots & \zeta^\star(t_{2-q}, \ldots, t_{r-1-q}) \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & \zeta^\star(t_0) & \cdots & \zeta^\star(t_0, \ldots, t_{r-1-q}) \end{array}\right].$$

Here, $\boldsymbol{s}|_\lambda \in W^{\mathrm{diag}}_\lambda$ and $\boldsymbol{t}|_{\lambda^*} \in W^{\mathrm{diag}}_{\lambda^*}$ are the shape restrictions of $\boldsymbol{s}$ and $\boldsymbol{t}$ to λ and λ^*, respectively.

This is again proved by regarding $\zeta_\lambda(\boldsymbol{s})$ as a specialization of Macdonald's ninth variation of Schur functions, which satisfy the dual Cauchy formula.

Example 3.7. When $p = 2$ and $q = 3$, the lefthand side of the dual Cauchy formula is given by

$$
\begin{aligned}
&\zeta\left(\begin{ytableau} s_0 & s_1 & s_2 \\ s_{-1} & s_0 & s_1 \end{ytableau}\right) - \zeta\left(\begin{ytableau} s_0 & s_1 & s_2 \\ s_{-1} & s_0 \end{ytableau}\right)\zeta\left(\begin{ytableau} t_0 \end{ytableau}\right) + \zeta\left(\begin{ytableau} s_0 & s_1 & s_2 \\ s_{-1} \end{ytableau}\right)\zeta\left(\begin{ytableau} t_0 \\ t_{-1} \end{ytableau}\right) \\
&- \zeta\left(\begin{ytableau} s_0 & s_1 & s_2 \end{ytableau}\right)\zeta\left(\begin{ytableau} t_0 \\ t_{-1} \\ t_{-2} \end{ytableau}\right) + \zeta\left(\begin{ytableau} s_0 & s_1 \\ s_{-1} & s_0 \end{ytableau}\right)\zeta\left(\begin{ytableau} t_0 & t_1 \end{ytableau}\right) - \zeta\left(\begin{ytableau} s_0 & s_1 \\ s_{-1} \end{ytableau}\right)\zeta\left(\begin{ytableau} t_0 & t_1 \\ t_{-1} \end{ytableau}\right) \\
&+ \zeta\left(\begin{ytableau} s_0 & s_1 \end{ytableau}\right)\zeta\left(\begin{ytableau} t_0 & t_1 \\ t_{-1} \\ t_{-2} \end{ytableau}\right) + \zeta\left(\begin{ytableau} s_0 \\ s_{-1} \end{ytableau}\right)\zeta\left(\begin{ytableau} t_0 & t_1 \\ t_{-1} & t_0 \end{ytableau}\right) - \zeta\left(\begin{ytableau} s_0 \end{ytableau}\right)\zeta\left(\begin{ytableau} t_0 & t_1 \\ t_{-1} & t_0 \\ t_{-2} \end{ytableau}\right) + \zeta\left(\begin{ytableau} t_0 & t_1 \\ t_{-1} & t_0 \\ t_{-2} & t_{-1} \end{ytableau}\right).
\end{aligned}
$$

On the other hand, the righthand side is

$$
\det\left[\begin{array}{ccccc}
1 & \zeta^\star(s_{-1}) & \zeta^\star(s_{-1},s_0) & \zeta^\star(s_{-1},s_0,s_1) & \zeta^\star(s_{-1},s_0,s_1,s_2) \\
0 & 1 & \zeta^\star(s_0) & \zeta^\star(s_0,s_1) & \zeta^\star(s_0,s_1,s_2) \\
\hline
1 & \zeta^\star(t_{-2}) & \zeta^\star(t_{-2},t_{-1}) & \zeta^\star(t_{-2},t_{-1},t_0) & \zeta^\star(t_{-2},t_{-1},t_0,t_1) \\
0 & 1 & \zeta^\star(t_{-1}) & \zeta^\star(t_{-1},t_0) & \zeta^\star(t_{-1},t_0,t_1) \\
0 & 0 & 1 & \zeta^\star(t_0) & \zeta^\star(t_0,t_1)
\end{array}\right].
$$

4 1,3 formulas for SMZVs

To find explicit expressions for special values at positive integers of Schur multiple zeta functions (SMZVs for short) are another interesting problem. Here, we finally show so-called $1,3$ formulas for SMZVs, which can be regarded as analogues of formulas

$$\zeta(\{1,3\}^n) = \frac{2\pi^{4n}}{(4n+2)!} = \frac{1}{4^n}\zeta(\{4\}^n), \tag{4.1}$$

$$\zeta(3,\{1,3\}^n) = \sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k \zeta(4k+3)\zeta(\{1,3\}^{n-k}), \tag{4.2}$$

respectively obtained in [BBB, BB] for MZVs. Here, for $k_1,\dots,k_r \in \mathbb{N}$, $\{k_1,\dots,k_r\}^n$ is the n times repetition of $k_1,\dots,k_r$. Remark that the corresponding formulas for MZSVs are obtained in [Mu].

4.1 Stairs

In this section, we show $1,3$ formula for SMZVs of stair type, that is, of shape $\delta_N = (N, N-1,\dots,2,1)$. In the following, the coloring is just for optical reasons.

Theorem 4.1 ([BY, Corollary 4.6]). (1) For odd $N \geq 1$, we have

$$\zeta_{\delta_N}\left(\begin{ytableau} 3 & 1 & \iddots & 1 & 3 \\ 1 & \iddots & 1 & 3 \\ \iddots & 1 & 3 \\ 1 & 3 \\ 3 \end{ytableau}\right) = 4^{-\frac{1}{4}(N+1)(N-1)}\det\Big[\zeta\left(4(i+j)-5\right)\Big]_{1\leq i,j\leq \frac{N+1}{2}}.$$

(2) For even $N \geq 2$, we have

$$\zeta_{\delta_N}\left(\begin{array}{cccccc} 1 & 3 & \cdots & \cdot & 1 & 3 \\ 3 & \ddots & \cdot & 1 & 3 & \\ \vdots & \cdot & 1 & 3 & & \\ \cdot & 1 & 3 & & & \\ 1 & 3 & & & & \\ 3 & & & & & \end{array}\right) = 4^{-\frac{1}{4}N^2} \det\Big[\zeta\left(4(i+j)-1\right)\Big]_{1\leq i,j\leq \frac{N}{2}}.$$

These formulas are obtained by using the Jacobi-Trudi formulas obtained in Theorem 3.1 together with the help of (4.1) and (4.2). Notice that, in [BY, Corollary 4.6], more generally, 1, 3 formulas for SMZVs of shape δ_N/μ are obtained.

Remark 4.2. The righthand side of the formulas in Theorem 4.1 are so-called the Hankel determinant. See [Mo] and [HZ] for the similar topics.

Example 4.3.

$$\zeta\left(\begin{array}{c} 3 \end{array}\right) = \left|\zeta(3)\right|, \qquad \zeta\left(\begin{array}{cc} 1 & 3 \\ 3 & \end{array}\right) = \frac{1}{4}\left|\zeta(7)\right|,$$

$$\zeta\left(\begin{array}{ccc} 3 & 1 & 3 \\ 1 & 3 & \\ 3 & & \end{array}\right) = \frac{1}{4^2}\begin{vmatrix} \zeta(3) & \zeta(7) \\ \zeta(7) & \zeta(11) \end{vmatrix}, \qquad \zeta\left(\begin{array}{cccc} 1 & 3 & 1 & 3 \\ 3 & 1 & 3 & \\ 1 & 3 & & \\ 3 & & & \end{array}\right) = \frac{1}{4^4}\begin{vmatrix} \zeta(7) & \zeta(11) \\ \zeta(11) & \zeta(15) \end{vmatrix},$$

$$\zeta\left(\begin{array}{ccccc} 3 & 1 & 3 & 1 & 3 \\ 1 & 3 & 1 & 3 & \\ 3 & 1 & 3 & & \\ 1 & 3 & & & \\ 3 & & & & \end{array}\right) = \frac{1}{4^6}\begin{vmatrix} \zeta(3) & \zeta(7) & \zeta(11) \\ \zeta(7) & \zeta(11) & \zeta(15) \\ \zeta(11) & \zeta(15) & \zeta(19) \end{vmatrix}, \quad \zeta\left(\begin{array}{cccccc} 1 & 3 & 1 & 3 & 1 & 3 \\ 3 & 1 & 3 & 1 & 3 & \\ 1 & 3 & 1 & 3 & & \\ 3 & 1 & 3 & & & \\ 1 & 3 & & & & \\ 3 & & & & & \end{array}\right) = \frac{1}{4^9}\begin{vmatrix} \zeta(7) & \zeta(11) & \zeta(15) \\ \zeta(11) & \zeta(15) & \zeta(19) \\ \zeta(15) & \zeta(19) & \zeta(23) \end{vmatrix}.$$

4.2 Ribbons

A Young diagram λ/μ is called a ribbon if it is connected and does not contain any 2×2 blocks. In this section, we show 1, 3 formulas for SMZVs of ribbon type.

Theorem 4.4 ([BY, Theorem 3.4]). For $n \geq 1$, let $\sigma_n = (n, n, n-1, \ldots, 2, 1)/\delta_{n-1}$ and $\sigma'_n = (n+1, n, \ldots, 3, 2)/\delta_{n-1}$. We have

$$\text{(4.3)} \quad \zeta_{\sigma_n}\left(\begin{array}{ccc} & & 1 \\ & \cdot^{\cdot^{\cdot}} & 3 \\ 1 & \cdot^{\cdot^{\cdot}} & \\ 3 & & \end{array}\right) = \frac{1}{4^n}\zeta^{\star}(\{4\}^n), \quad \zeta_{\sigma'_n}\left(\begin{array}{cccc} & & 1 & 3 \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 1 & 3 & & \end{array}\right) = \sum_{k=0}^{n} \frac{1}{4^k}\zeta^{\star}(\{4\}^k)\zeta(\{4\}^{n-k}).$$

In particular, these values are in $\mathbb{Q}\pi^{4n}$.

The next theorem asserts that all odd Riemann zeta values are realized as a SMZV by adding a 1 on the bottom left or a 3 on the top right of the former tableau in Theorem 4.4.

Theorem 4.5 ([BY, Theorem 3.5]). For $n \geq 1$, let $\alpha_n = (n+1, n+1, n, n-1, \ldots, 3, 2)/\delta_n$ and $\beta_n = \delta_{n+1}/\delta_{n-1}$. we have

$$\text{(4.4)} \quad \zeta_{\alpha_n}\left(\begin{array}{cccc} & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & 3 \\ & 1 & \cdot^{\cdot^{\cdot}} & \\ 1 & 3 & & \end{array}\right) = \frac{2}{4^n}\zeta(4n+1), \quad \zeta_{\beta_n}\left(\begin{array}{cccc} & & 1 & 3 \\ & \cdot^{\cdot^{\cdot}} & 3 & \\ 1 & \cdot^{\cdot^{\cdot}} & & \\ 3 & & & \end{array}\right) = \frac{1}{4^n}\zeta(4n+3).$$

These formulas (4.3) and (4.4) are obtained by considering the corresponding generating functions, which can be written in terms of the Gauss hypergeometric function. We notice that the second one in (4.4) is also derived from the Jacobi-Trudi formula studied in the previous section together with (4.1) and (4.2).

With Theorem 4.4 and Theorem 4.5, one reaches the following result.

Theorem 4.6 ([BY, Theorem 3.1]). All Schur multiple zeta values of ribbons type whose entries are 1, 3 and are arranged as in a Checkerboard style are in $\mathbb{Q}[\pi^4, \zeta(3), \zeta(5), \zeta(7), \ldots]$.

Let us check this by the following example (then one can understand the general case). We notice that a harmonic product formula (see [BY, Lemma 2.2]), which gives an expression of a product of SMZVs as a sum of SMZVs, plays a crucial role in the calculation:

$$\begin{aligned}
\zeta\left(\begin{ytableau} \none & \none & 3 \\ \none & \none & 1 \\ 3 & 1 & 3 \end{ytableau}\right) &= \zeta\left(\begin{ytableau} \none & \none & 1 \\ 3 & 1 & 3 \end{ytableau}\right)\zeta\left(\begin{ytableau} 3 \end{ytableau}\right) - \zeta\left(\begin{ytableau} \none & \none & 1 & 3 \\ 3 & 1 & 3 \end{ytableau}\right) \\
&= \left[\zeta\left(\begin{ytableau} 3 \end{ytableau}\right)\zeta\left(\begin{ytableau} \none & 1 \\ 1 & 3 \end{ytableau}\right) - \zeta\left(\begin{ytableau} \none & 1 \\ 1 & 3 \\ 3 \end{ytableau}\right)\right]\zeta\left(\begin{ytableau} 3 \end{ytableau}\right) \\
&\quad - \left[\zeta\left(\begin{ytableau} 3 \end{ytableau}\right)\zeta\left(\begin{ytableau} \none & 1 & 3 \\ 1 & 3 \end{ytableau}\right) - \zeta\left(\begin{ytableau} \none & 1 & 3 \\ 1 & 3 \\ 3 \end{ytableau}\right)\right] \\
&= \frac{1}{2}\zeta(3)^2\zeta(5) - \frac{7\pi^8}{129600}\zeta(3) + \frac{1}{16}\zeta(11).
\end{aligned}$$

5 Concluding remark

For $k \geq 0$, let $\mathcal{Z}_k$ be the $\mathbb{Q}$-vector space spanned by all multiple zeta values of weight k. It is conjectured by Zagier that $\dim \mathcal{Z}_k = d_k$ where $\{d_k\}_{k\geq 0}$ is defined by the recurrence formula $d_0 = 1$, $d_1 = 0$, $d_2 = 1$ and $d_k = d_{k-2} + d_{k-3}$ for $k \geq 3$. To solve this conjecture, one needs to find all linear relations among MZVs of fixed weight.

It is worth mentioning that Kaneko and Yamamoto [KY, Conjecture 4.3] conjectured that any linear dependency of MZVs over $\mathbb{Q}$ can be deduced from the iterated integral representations of Schur multiple zeta values associated with some anti-hooks. Here, we mean that an anti-hook is anti-diagonal transpose of a hook. Let us consider the simplest case, that is, $\lambda/\mu = (2,2)/(1)$ with $\boldsymbol{k} = \begin{ytableau} \none & 1 \\ 1 & 2 \end{ytableau}$. From the definition (series expression), we have

$$\begin{aligned}
\zeta\left(\begin{ytableau} \none & 1 \\ 1 & 2 \end{ytableau}\right) = \sum_{\substack{m \\ \wedge \\ k \leq n}} \frac{1}{kmn^2} &= \sum_{k<m<n} + \sum_{k=m<n} + \sum_{m<k<n} + \sum_{m<k=n} \\
&= 2\zeta(1,1,2) + \zeta(2,2) + \zeta(1,3).
\end{aligned}$$

On the other hand, as is proved in [KY] (see also § 6 in [NPY] for more general cases), it has an iterated integral representation;

$$\begin{aligned}
\zeta\left(\begin{ytableau} \none & 1 \\ 1 & 2 \end{ytableau}\right) = \int_{\substack{x<y<z \gt w \\ 0<x,y,z,w<1}} \frac{dx}{1-x}\frac{dy}{1-y}\frac{dz}{z}\frac{dw}{1-w} &= \int_{\substack{w<x<y<z \\ 0<x,y,z,w<1}} + \int_{\substack{x<w<y<z \\ 0<x,y,z,w<1}} + \int_{\substack{x<y<w<z \\ 0<x,y,z,w<1}} \\
&= 3\zeta(1,1,2).
\end{aligned}$$

Combining these equations, we have a linear relation

$$\zeta(2,2)+\zeta(1,3)=\zeta(1,1,2).$$

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