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ラマン散乱項を持つ非線形シュレディンガー方程式の初期値問題の非適切性

Ill-posedness of the Cauchy problem for the nonlinear Schrödinger equation with Raman scattering term

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概　要

光ファイバー中のパルスの伝播を記述する，3階の分散項とラマン散乱項を持つ微分型非線形シュレディンガー方程式を考える．周期境界条件を課さない場合，初期値問題はソボレフ空間において時間局所的に適切であることが知られている．ここでは，周期境界条件下における初期値問題のソボレフ空間での非適切性（解の非存在，もしくは初期値連続依存性の破綻）に関する，著者による最近の結果を紹介する．

1. Introduction

We consider the ill-posedness of the Cauchy problem for the nonlinear Schrödinger equation with third order dispersion and intrapulse Raman scattering term:

$$\partial_t u = \alpha_1 \partial_x^3 u + i\alpha_2 \partial_x^2 u + i\gamma_1 |u|^2 u + \gamma_2 \partial_x\big(|u|^2 u\big) - i\Gamma u \partial_x\big(|u|^2\big), \tag{1.1}$$
$$t \in [-T,T], \quad x \in \mathbf{T} = \mathbf{R}/2\pi\mathbf{Z},$$

$$u(0,x) = u_0(x), \qquad x \in \mathbf{T}, \tag{1.2}$$

where α_j, γ_j ($j = 1,2$) and Γ are real constants and T is a positive constant. Later we will assume that α_1 and Γ are non-zero, in particular that $\Gamma > 0$, though the corresponding results for $\Gamma < 0$ can be shown by the same argument.

The equation (1.1) arises in nonlinear optics as a model of propagation of pulses in an optical fiber ([4], [1, Eq.(2.3.43)]). The last and the last but one terms on the right-hand side of (1.1) represent the effect of the intrapulse Raman scattering, which is not negligible for ultrashort optical pulses (see [1, §2.3.2]). Our aim is to show that

In this note, we summarize the authors' article [5]. More information and detailed proofs can be found in [5].

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the last term $-i\Gamma u\partial_x(|u|^2)$ causes the ill-posedness of the Cauchy problem with the periodic boundary condition (1.1)–(1.2).

In the case of $x \in \mathbf{R}$ (i.e., spatially decaying initial data), the well-posedness in the Sobolev spaces $H^s(\mathbf{R})$, $s \geq \frac{1}{4}$ of the Cauchy problem (1.1)–(1.2) (with any $\gamma_1, \gamma_2, \Gamma$) was shown by Staffilani [9]. For our problem with the periodic boundary condition, the equation without Raman scattering terms ($\gamma_2 = \Gamma = 0$) has been studied by Miyaji and the second author [6, 7], where the Cauchy problems of (1.1) and the reduced equation relevant to (1.1) were shown to be well-posed in $H^s(\mathbf{T})$, $s \geq 0$ and $s > -\frac{1}{6}$, respectively. In the case $\Gamma = 0$ (or if $\Gamma \in i\mathbf{R}$), Takaoka [10] showed the well-posedness in $H^s(\mathbf{T})$ for $s \geq \frac{1}{2}$. Nevertheless, the Cauchy problem becomes ill-posed in $H^s(\mathbf{T})$ for any large s if $\Gamma \neq 0$, as we see below. The difference between the cases of $\mathbf{R}$ and $\mathbf{T}$ is that the spectrum of the Laplacian is continuous in the former case, while it is discrete in the latter case. We also note that the condition

$$\frac{2\alpha_2}{3\alpha_1} \notin \mathbf{Z} \tag{1.3}$$

was imposed in all the results concerning the periodic problem, and we also assume it.

The well/ill-posedness for general periodic derivative nonlinear Schrödinger equations was shown in the recent work of Tsugawa (see [12] for related results). His result implies in particular that the Cauchy problem (1.1)–(1.2) with $\alpha_1 = 0$ and $\Gamma \neq 0$ is ill-posed in $H^s(\mathbf{T})$ for all sufficiently large s. In [12], he introduced the notion of *parabolic resonance*, by which some nonlinear terms could yield the smoothing type effect either forward or backward in time, and hence lead to the ill-posedness. Our proof of the ill-posedness is indeed based on the same mechanism, but we focus on specific nonlinearities and capture the instability of the problem more precisely. Also, his proof is different from ours because it proceeds in the x variable space while our estimates are mainly done in the Fourier space.

We conclude this section with the notation. We employ the following definition of the Fourier coefficients of 2π-periodic functions:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{-ikx}\, dx \quad (k \in \mathbf{Z}); \qquad f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbf{Z}} \hat{f}(k) e^{ikx}.$$

The Sobolev space $H^s(\mathbf{T})$ is defined via the norm

$$\|f\|_{H^s} := \|\langle\cdot\rangle^s \hat{f}(\cdot)\|_{\ell^2(\mathbf{Z})}; \qquad s \in \mathbf{R}, \quad \langle\cdot\rangle := 1 + |\cdot|.$$

We define the operator $P_\pm$ on $L^2(\mathbf{T})$ by

$$P_\pm f(x) := \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbf{Z};\,\pm k>0} \hat{f}(k)e^{ikx}.$$

2. Main results

We obtain the following ill-posedness results in the sense of *nonexistence* of the solution:

Theorem 2.1. *We assume $\alpha_1 \neq 0$, $\Gamma > 0$, and (1.3). Let $1 \le s_1 \le s$ and $u_0 \in H^s(\mathbb{T})$. Then, the following holds.*

(i) If ($s < s_1 + \frac{1}{2}$ and) $P_+u_0 \notin H^{s_1+\frac{1}{2}-\varepsilon}$ for some $\varepsilon > 0$, then for any $T > 0$ there exists no solution $u \in C([0,T]; H^{s_1}(\mathbb{T}))$ to (1.1)–(1.2) on $[0,T]$.

(ii) If ($s < s_1 + \frac{1}{2}$ and) $P_-u_0 \notin H^{s_1+\frac{1}{2}-\varepsilon}$ for some $\varepsilon > 0$, then for any $T > 0$ there exists no solution $u \in C([-T,0]; H^{s_1}(\mathbb{T}))$ to (1.1)–(1.2) on $[-T,0]$.

(iii) If $u_0 \notin H^\infty$, then for any $T > 0$ there exists no solution $u \in C([-T,T]; H^1(\mathbb{T}))$ to (1.1)–(1.2) on $[-T,T]$.

(iv) If $u_0 \in H^\infty$ and the estimate

$$\sup_{s\ge 1} R^{-s^2}\|u_0\|_{H^s} < \infty$$

is false *for any $R > 1$, then for any $T > 0$ there exists no solution $u \in C([-T,T]; H^1(\mathbb{T}))$ to (1.1)–(1.2) on $[-T,T]$.*

Remark 2.2. (a) In (i) and (ii), the nonexistence of *one-sided* solutions is shown, but the regularities s, s_1 for initial data and solutions can differ at most by $\frac{1}{2} - \varepsilon$. A similar argument shows the existence of $u_0 \in H^s$ admitting no one-sided solution in $C_tH_x^{s_1}$, for $1 \le s_1 \le s < s_1 + 1$; see [5, §2].

On the other hand, (iii) and (iv) only show the nonexistence of *two-sided* solutions, but in the largest class $C_tH_x^1$ for initial data of any finite differentiability.

(b) The assertion (iv) suggests that a solution may not exist even for C^∞ initial data. For instance, the function $u_0 \in H^\infty$ defined by

$$\hat{u}_0(k) := e^{-[\log\langle k\rangle]^{4/3}}, \qquad k \in \mathbf{Z}$$

satisfies $\|u_0\|_{H^s} \ge \langle e^{s^2}-1\rangle^s |\hat{u}_0(e^{s^2}-1)| = e^{s^3-s^{8/3}}$ for any $s \ge 1$ such that $e^{s^2} \in \mathbf{Z}$, and thus it cannot be the initial data for a two-sided solution.

From Theorem 2.1, it seems impossible to solve the Cauchy problem (1.1)–(1.2) in Sobolev spaces. There is still a chance that it is solvable for a class of C^∞ initial data. In fact, we can show the unique solvability for real analytic initial data; see [5, §4]. However, even if a solution exists, the next theorem concerning the *norm inflation* phenomena shows that the solution map: $u_0 \mapsto u$ would be discontinuous everywhere in any Sobolev norm.

Theorem 2.3. *We assume* $\alpha_1 \neq 0$, $\Gamma > 0$, *and* (1.3). *Let* $s \geq 1$. *Then, for any* $\varepsilon, \tau > 0$ *there exists a real analytic function* $\phi_{\varepsilon,\tau}$ *satisfying* $\|\phi_{\varepsilon,\tau}\|_{H^s} \leq \varepsilon$ *such that the following holds: Let* $T > 0$ *and* $u^* \in C([-T,T]; H^1(\mathbf{T}))$ *be a solution to* (1.1) *on* $[-T,T]$. *Assume that* $\varepsilon, \tau > 0$ *satisfy* $0 < \tau \leq T$, $\sup_{t\in[-\tau,\tau]} \|u^*(t)\|_{H^1} \leq \varepsilon^{-1}$, *and* $2\varepsilon \leq \|u^*(0)\|_{L^2}$ *if* $u^*(0) \neq 0$. *Then, if there exists a solution* $u \in C([-\tau,\tau]; H^1(\mathbf{T}))$ *to* (1.1) *with the initial condition* $u(0) = u^*(0) + \phi_{\varepsilon,\tau}$, *it holds that*

$$\sup_{t\in[-\tau,\tau]} \|u(t) - u^*(t)\|_{H^1} \geq \varepsilon^{-1}.$$

Remark 2.4. Similarly to Theorem 2.1, we can show the corresponding result for one-sided solutions if the difference of the solutions is measured in H^{s_1} with $1 \leq s_1 \leq s < s_1 + 1$. If we further restrict to $u^* \equiv 0$, we can include the case $s_1 = s - 1$, which seems to be natural since the nonlinearity of (1.1) includes the first derivative of the unknown function. See [5, §3] for details.

A large number of numerical simulations for the Cauchy problem (1.1)–(1.2) have been made though it is ill-posed in Sobolev spaces as mentioned above. We note that our results do not necessarily suggest the failure of numerical attempts. Indeed, such analytic functions as Gaussian and super-Gaussian pulses have been chosen as initial data in those numerical computations, and we have the unique solvability as well as a weak form of continuous dependence on initial data in a class of analytic functions (see [5, §4] for the precise statement of this result). On the other hand, it is worth noticing that some sort of instability has been observed in numerical analysis of the Cauchy problem (1.1)–(1.2); see, e.g., [2, 8, 3, 11]. We remark again that the ill-posedness we present in this note is peculiar to the periodic boundary value problem and does not occur in the spatially decaying setting.

3. Idea for proof

The following smoothing type effect is the key ingredient of the proofs of Theorems 2.1, 2.3:

Lemma 3.1. *We assume $\alpha_1 \neq 0$, $\Gamma > 0$, and* (1.3)*. Let $s \geq 1$ and $u \in C([0,T]; H^s(\mathbf{T}))$ be a solution to the Cauchy problem* (1.1)–(1.2) *on $[0,T]$ for some $T > 0$. Then, there exist an absolute constant $R > 1$ and a positive constant C depending on $\|u\|_{L^\infty(0,T;H^1)}$, $\|u_0\|_{L^2}^{-1}$, T^{-1}, and the parameters in the equation such that*

$$\langle k \rangle^{s+1} |\hat{u}_0(k)| \leq C R^s \|u\|_{L^\infty(0,T;H^s)} \tag{3.1}$$

for any $k > 0$.

In fact, Theorem 2.1 (i) follows immediately from the above lemma and the Cauchy-Schwarz inequality, and (ii) is derived by a parallel argument or by time reflection $u(t,x) \mapsto \bar{u}(-t,x)$ of the equation. For two-sided solutions, such smoothing estimates from both positive and negative times can be iterated to yield H^∞ smoothing inside the interval $[-T,T]$, showing (iii). We can also derive (iv) from the precise dependence on s in the above estimate. Theorem 2.3 can be deduced by a similar idea.

In the following, we present the rough idea for the proof of Lemma 3.1. Let us consider the simple case that $\gamma_1 = \gamma_2 = 0$, $\Gamma = 2\pi$. The Fourier coefficients of a solution $u \in C([0,T]; H^s(\mathbf{T}))$ to (1.1) satisfies

$$\partial_t \hat{u}(t,k) = -i\phi(k)\hat{u}(t,k) + \sum_{\substack{k_1,k_2,k_3 \in \mathbf{Z} \\ k = k_1+k_2+k_3}} (k_1+k_2)\hat{u}(t,k_1)\hat{\bar{u}}(t,k_2)\hat{u}(t,k_3), \quad t \in [0,T],$$

where $\phi(k) := \alpha_1 k^3 + \alpha_2 k^2$. We next represent the equation in terms of $\hat{v}(t,k) := e^{it\phi(k)}\hat{u}(t,k)$ (this is called the interaction picture or the interaction representation):

$$\partial_t \hat{v}(t,k) = \sum_{k=k_1+k_2+k_3} e^{it\Phi}(k_1+k_2)\hat{v}(t,k_1)\hat{\bar{v}}(t,k_2)\hat{v}(t,k_3), \qquad t \in [0,T], \tag{3.2}$$

where we have $e^{it\Phi}$ as the product of all linear oscillations through the nonlinear interaction, that is,

$$\begin{aligned}\Phi &= \phi(k_1+k_2+k_3) - \phi(k_1) + \phi(-k_2) - \phi(k_3) \\ &= 3\alpha_1(k_1+k_2)(k_2+k_3)(k_3+k_1+\tfrac{2\alpha_2}{3\alpha_1}).\end{aligned}$$

For the terms with $\Phi \neq 0$ on the right-hand side (which we call the *non-resonant* part), we have a kind of smoothing effect by averaging (integrating) in time. However, this is not the case for the *resonant* part which does not oscillate.

The condition (1.3) plays an essential role in the analysis of the resonant part. Observing that $\Phi = 0$ if and only if $k_1 + k_2 = 0$ or $k_2 + k_3 = 0$ under (1.3), we have

$$\begin{aligned}
\sum_{\substack{k=k_1+k_2+k_3\\ \Phi=0}} (k_1+k_2)\hat{v}(k_1)\hat{\bar{v}}(k_2)\hat{v}(k_3) &= \sum_{\substack{k=k_1+k_2+k_3\\ k_1+k_2=0 \text{ or } k_2+k_3=0}} (k_1+k_2)\hat{v}(k_1)\bar{\hat{v}}(-k_2)\hat{v}(k_3)\\
&= \sum_{k=k_1,\, k_2+k_3=0} (k_1+k_2)\hat{v}(k_1)\bar{\hat{v}}(-k_2)\hat{v}(k_3) = \sum_{k_3\in\mathbf{Z}} (k-k_3)\hat{v}(k)|\hat{v}(k_3)|^2\\
&= \Big(\|v(t)\|_{L^2}^2 k - P(v(t))\Big)\hat{v}(k),
\end{aligned}$$

where $P(v)$ is the momentum defined by

$$P(v) := \sum_{k\in\mathbf{Z}} k|\hat{v}(k)|^2 = \Im \int_{\mathbf{T}} \bar{v}(x)\partial_x v(x)\,dx.$$

We notice that the L^2 norm of a solution to (1.1) is formally conserved (see [5, §2] for a rigorous proof). Hence, by (3.2) we have

$$\partial_t\hat{v}(k) - \|u_0\|_{L^2}^2 k\hat{v}(k) = \sum_{\substack{k=k_1+k_2+k_3\\ \Phi\neq 0}} e^{it\Phi}(k_1+k_2)\hat{v}(k_1)\hat{\bar{v}}(k_2)\hat{v}(k_3) - P(v)\hat{v}(k) \tag{3.3}$$

for $t \in [0,T]$. Observe that the Cauchy-Riemann type elliptic operator $\partial_t + i\|u_0\|_{L^2}^2\partial_x$ appears on the left-hand side of (3.3) due to the Raman scattering term. However, it seems difficult to derive the elliptic regularity theorem such as (3.1) simply by (3.3) without exploiting the dispersive nature, because the right-hand side of (3.3) also contains the first derivative of the unknown function.

We will recover the loss of one derivative on the right-hand side of (3.3) by use of the non-resonant property. Solving (3.3) from $t = T$ to $t = 0$ and applying (1.2), we have

$$\begin{aligned}
\hat{u}_0(k) = e^{-\|u_0\|_{L^2}^2 kT}\hat{v}(T,k) + \int_0^T e^{-\|u_0\|_{L^2}^2 kt}\Big[&P(v(t))\hat{v}(t,k)\\
&- \sum_{\substack{k=k_1+k_2+k_3\\ \Phi\neq 0}} e^{it\Phi}(k_1+k_2)\hat{v}(t,k_1)\hat{\bar{v}}(t,k_2)\hat{v}(t,k_3)\Big]\,dt.
\end{aligned} \tag{3.4}$$

To prove (3.1), we need to show that the right-hand side of (3.4) decays as $\langle k\rangle^{-s-1}$ for positive k if $v \in C([0,T];H^s)$ solves (3.2). The first term has an exponential decay. Moreover, since $|\hat{v}(t,k)| \le \|v\|_{L^\infty(0,T;H^s)}\langle k\rangle^{-s}$ for $t \in [0,T]$ and the integration of $e^{-\|u_0\|_{L^2}^2 kt}$ in t yields an additional decay $\langle k\rangle^{-1}$, the first term in the integral decays as $\langle k\rangle^{-s-1}$. Hence, we estimate the integral of the non-resonant part:

$$\int_0^T e^{-\|u_0\|_{L^2}^2 kt} \sum_{\substack{k=k_1+k_2+k_3\\ \Phi\neq 0}} e^{it\Phi}(k_1+k_2)\hat{v}(t,k_1)\hat{\bar{v}}(t,k_2)\hat{v}(t,k_3)\,dt.$$

First, we consider the contribution from interactions for which the size of $|k_1|$, $|k_2|$, and $|k_3|$ are all comparable. In this case we can shift 'one derivative' k_1+k_2 freely to any one of the three v's, so there is no loss of regularity, i.e., the summation has the decay of order $\langle k\rangle^{-s}$. Therefore, this part is treated by an additional decay from the integration in t. Secondly, we perform a 'differentiation by parts' for the remaining terms and reduce the problem to estimating

$$I := \int_0^T e^{-\|u_0\|_{L^2}^2 kt}\partial_t\Big(\sum_{k=k_1+k_2+k_3} \frac{e^{it\Phi}}{i\Phi}(k_1+k_2)\hat{v}(t,k_1)\hat{\bar{v}}(t,k_2)\hat{v}(t,k_3)\Big)\,dt,$$

$$II := \int_0^T e^{-\|u_0\|_{L^2}^2 kt} \sum_{k=k_1+k_2+k_3} \frac{e^{it\Phi}}{i\Phi}(k_1+k_2)\partial_t\big(\hat{v}(t,k_1)\hat{\bar{v}}(t,k_2)\hat{v}(t,k_3)\big)\,dt.$$

Now, we observe that $|\Phi| \ge C\max_{1\le j\le 3}\langle k_j\rangle^2$ for these terms. Then, for I, the t-derivative is cancelled with the integration, while the summation has an additional decay. For II, another loss of one derivative from $\partial_t\hat{v}$ (through the equation (3.2)) can be also recovered, and the t-integration gives an extra decay. Consequently, both I and II decay as $\langle k\rangle^{-s-1}$, which verifies (3.1) (except for an explicit dependence on s).

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