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# Fokas-Lenells 方程式の多成分系への拡張

## Multi-component generalization of the Fokas-Lenells equation

山口大学大学院創成科学研究科　松野　好雅 (Yoshimasa Matsuno)
Division of Applied Mathematical Science
Graduate School of Sciences and Technology for Innovation
Yamaguchi University
E-mail address: matsuno@yamaguchi-u.ac.jp

The Fokas-Lenells (FL) equation is an integrable model for the nonlinear propagation of short pulses in an optical fiber. We introduce an integrable multi-component FL system and provide its bright multisoliton solutions as well as an infinite number of conservation laws under the vanishing boundary conditions. We also give the dark multisoliton solutions of the system under the nonvanishing boundary conditions.

## 1. Introduction

### 1.1. Basic equation

The Fokas-Lenells (FL) equation is an integrable generalization of the nonlinear Schrödinger (NLS) equation. In the context of fiber optics, it describes the nonlinear propagation of short pulses in a monomode fiber. Starting from Maxwell's equation for an electric field, Lenells derived the following equation [1]

$$\begin{aligned} \mathrm{i}A_z + \frac{1}{\beta_0}A_{zz} - \frac{1}{\beta_0 v_g}A_{zT} + \gamma A_{TT} - \frac{\mathrm{i}\beta_3}{6}A_{TTT} \\ = -\rho A|A|^2 - \mathrm{i}s(A|A|^2)_T - \mathrm{i}\tau A(|A|^2)_T, \end{aligned} \tag{1.1}$$

where $A = A(z, T)$ is an envelope of an electric field, $z$ and $T = t - z/v_g$ denote the space and time variables, respectively, $\beta_0$ is a wave number, $v_g$ is a group velocity, and $\gamma, \beta_3, \rho, s, \tau$ are real constants.

The several completely integrable equations are obtained by the reductions of Eq. (1.1). Among them, the following four equations are well-known:

1) A modified NLS equation

$$\mathrm{i}A_z + \gamma A_{TT} = -\rho A|A|^2 - \mathrm{i}s(A|A|^2)_T. \tag{1.2}$$

2) Hirota equation (Hirota [2])

$$A_z + A_{TTT} = -6|A|^2 A_T. \tag{1.3}$$

3) Sasa-Satsuma equation (Sasa & Satsuma [3])

$$A_z + A_{TTT} = -6|A|^2 A_T - 3A(|A|^2)_T. \tag{1.4}$$

4) FL equation (Fokas [4], Lenells [1]

$$\mathrm{i}A_z - \frac{1}{\beta_0 v_g} A_{zT} + \gamma A_{TT} = -\rho|A|^2 \left( A + \mathrm{i}\frac{s}{\rho} A_T \right), \quad s + \tau = 0, \quad 1/\beta_0 v_g = s/\rho. \tag{1.5}$$

If we put $A = u, s/\rho = \nu$ in Eq. (1.5) and identify $z$ and $T$ with $t$ and $x$, respectively, the FL equation can be rewritten as

$$\mathrm{i}u_t - \nu u_{xt} + \gamma u_{xx} + \rho|u|^2(u + \mathrm{i}\nu u_x) = 0.$$

Replacing $u$ by $\sqrt{a/|\rho|}b\,\mathrm{e}^{\mathrm{i}(bx+2abt)}u$ $(a = \gamma/\nu > 0, b = 1/\nu)$, this equation becomes

$$u_{xt} - a u_{xx} = ab^2(-u + \mathrm{i}\sigma|u|^2 u_x), \quad (\sigma = \operatorname{sgn}\rho).$$

Last, by means of the transformations $x + at \to x, -ab^2 t \to t$, we arrive at the simplified form of the FL equation

$$u_{xt} = u - \mathrm{i}\sigma|u|^2 u_x, \quad \sigma = \pm 1. \tag{1.6}$$

### 1.2. Purpose

Here, we address the following issues:

- Generalization of the FL equation to an integrable multi-component system.
- Construction of the bright soliton solutions of the multi-component FL system by means of a direct method.
- Derivation of an infinite number of conservation laws of the multi-component FL system.
- Bilinearization under the nonvanishing boundary conditions and construction of the dark soliton solutions.

In this report, we outline the main results and the detail will be published in a separate paper.

## 2. Multi-component Fokas-Lenells system

### 2.1. Lax pair

The FL equation has an integrable multi-component generalization. Actually, it exhibits a Lax representation

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi, \quad (U, V : (n+1) \times (n+1) \text{ matrices}), \tag{2.1a}$$

$$U = \begin{pmatrix} \frac{\mathrm{i}}{2}\zeta^2 & -\mathrm{i}\zeta\mathbf{u}_x \\ \mathrm{i}\zeta\mathbf{v}_x^T & -\frac{\mathrm{i}}{2}\zeta^2 I \end{pmatrix} = (u_{jk})_{1\le j,k\le n+1}, \quad V = \begin{pmatrix} -\frac{\mathrm{i}}{2\zeta^2} - \mathrm{i}\mathbf{u}\mathbf{v}^T & \frac{1}{\zeta}\mathbf{u} \\ \frac{1}{\zeta}\mathbf{v}^T & \frac{\mathrm{i}}{2\zeta^2} I + \mathrm{i}\mathbf{v}^T\mathbf{u} \end{pmatrix} = (v_{jk})_{1\le j,k\le n+1}, \tag{2.1b}$$

$$\Psi = (\psi_1, \psi_2, ..., \psi_{n+1}), \quad \mathbf{u} = (u_1, u_2, ..., u_n), \quad \mathbf{v} = (v_1, v_2, ..., v_n), \quad \Psi \in \mathbb{C}^{n+1}, \quad \mathbf{u}, \mathbf{v} \in \mathbb{C}^n, \tag{2.1c}$$

where $\zeta$ is a spectral parameter. It follows from the compatibility condition of the Lax pair that $U_t - V_x + UV - VU = O$. This yields the system of equations for the vector variables $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u}_{xt} - \mathbf{u} + \mathrm{i}(\mathbf{u}_x\mathbf{v}^T\mathbf{u} + \mathbf{u}\mathbf{v}^T\mathbf{u}_x) = \mathbf{0}, \tag{2.2a}$$

$$\mathbf{v}_{xt} - \mathbf{v} - \mathrm{i}(\mathbf{v}_x\mathbf{u}^T\mathbf{v} + \mathbf{v}\mathbf{u}^T\mathbf{v}_x) = \mathbf{0}. \tag{2.2b}$$

Recall that the system of equations (2.2) can be reduced from the first negative flow of the matrix derivative NLS hierarchy. See, for example Fordy [5], Tsuchida & Wadati [6], Tsuchida [7], Guo & Ling [8].

### 2.2. Reduction

If we put $v_j = \sigma_j u_j^*$, $\sigma_j = \pm 1$ $(j = 1, 2, ..., n)$, then the system of equations (2.2) reduces to

$$u_{j,xt} = u_j - \mathrm{i}\left\{\left(\sum_{s=1}^{n}\sigma_s u_{s,x}u_s^*\right)u_j + \left(\sum_{s=1}^{n}\sigma_s u_s u_s^*\right)u_{j,x}\right\},\ (j = 1, 2, ..., n). \tag{2.3}$$

The following two special cases have been considered for the system (2.3):

1) $n = 1$: FL equation (Fokas [4], Lenells [1])

$$u_{xt} = u - 2\mathrm{i}\sigma|u|^2u_x,\ (u \equiv u_1, \sigma_1 = 1).$$

2) $n = 2$: Two-component FL system (Guo & Ling [8], Ling *et al* [9])

$$u_{1,xt} = u_1 - \mathrm{i}\left\{(2|u_1|^2 + \sigma|u_2|^2)u_{1,x} + \mathrm{i}\sigma u_1 u_2^* u_{2,x}\right\}, \tag{2.4a}$$

$$u_{2,xt} = u_2 - \mathrm{i}\left\{(|u_1|^2 + 2\sigma|u_2|^2)u_{2,x} + \mathrm{i}\sigma u_2 u_1^* u_{1,x}\right\},\quad (\sigma_1 = 1, \sigma_2 = \sigma). \tag{2.4b}$$

## 3. Soliton solutions

### 3.1. Bilinearization

There exist several exact methods of solution for solving integrable soliton equations. Among them, we employ a direct method [10] (or, bilinear transformation method [11]). Specifically, we construct the bright soliton solutions of the multi-component FL system (2.3) under the vanishing boundary conditions $u_j \to 0$ as $|x| \to \infty$ $(j = 1, 2, ..., n)$.

- **Proposition 1**

*Under the dependent variable transformations*

$$u_j = \frac{g_j}{f},\ (j = 1, 2, ..., n), \tag{3.1}$$

*the multi-component FL system (2.3) can be decoupled into the system of equations*

$$D_t f \cdot f^* = \mathrm{i}\sum_{k=1}^{n}\sigma_k g_k g_k^*, \tag{3.2}$$

$$D_xD_tf\cdot f^* = \mathrm{i}\sum_{k=1}^{n}\sigma_kD_xg_k\cdot g_k^*, \tag{3.3}$$

$$f^*(g_{j,xt}f - g_{j,t}f_x - g_jf) = f_t^*(g_{j,x}f - g_jf_x),\ (j=1,2,...,n), \tag{3.4}$$

*where* $f = f(x,t)$ *and* $g_j = g_j(x,t)$ *are the complex-valued functions of* $x$ *and* $t$ *and the bilinear operators* $D_x$ *and* $D_t$ *are defined by*

$$D_x^mD_t^nf\cdot g = \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n f(x,t)g(x',t')\Big|_{x'=x,\,t'=t}$$

*with* $m$ *and* $n$ *being nonnegative integers.*

- **Remarks**

1) We can decouple the trilinear equations (3.4) into a system of bilinear equations

$$g_{j,xt}f - g_{j,t}f_x - g_jf = h_jf_t^*,\ (j=1,2,...,n), \tag{3.5a}$$

$$g_{j,x}f - g_jf_x = h_jf^*,\ (j=1,2,...,n), \tag{3.5b}$$

where $h_j = h_j(x,t)$ are the complex-valued functions of $x$ and $t$. This system can be rewritten by using the bilinear operators

$$D_xD_tg_j\cdot f - 2g_jf = -D_th_j\cdot f^*,\ (j=1,2,...,n), \tag{3.6a}$$

$$D_xg_j\cdot f = h_jf^*,\ (j=1,2,...,n). \tag{3.6b}$$

2) If we introduce the variables $q_j = u_{j,x}$, then

$$q_j = \left(\frac{g_j}{f}\right)_x = \frac{h_jf^*}{f^2},\quad (j=1,2,...,n), \tag{3.7}$$

solve the $n$-component derivative NLS system

$$\mathrm{i}q_{j,t} + q_{j,xx} + 2\mathrm{i}\left[\left(\sum_{k=1}^{n}\sigma_k|q_k|^2\right)q_j\right]_x = 0,\ (j=1,2,...,n). \tag{3.8}$$

This comes from the fact that the $n$-component FL system (2.3) is the first negative flow of the $n$-component derivative NLS hierarchy.

### 3.2. The bright $N$-soliton solution

- **Proposition 2**

*The bright* $N$*-soliton solution of the system of equations (3.2)-(3.4) are given in terms of the following determinants*

$$f = |D|,\quad D = (d_{jk})_{1\le j,k\le N},\quad g_j = \begin{vmatrix} D & \mathbf{z}_t^T \\ \mathbf{a}_j^* & o \end{vmatrix},\ (j=1,2,...,n), \tag{3.9}$$

$$d_{jk} = \frac{z_jz_k^* - \mathrm{i}p_k^*C_{jk}}{p_j+p_k^*},\quad z_j = \mathrm{e}^{p_jx+\frac{1}{p_j}t},\quad C_{jk} = \sum_{s=1}^{n}\sigma_s\alpha_{sj}\alpha_{sk}^*, \tag{3.10}$$

$$\mathbf{z} = (z_1, z_2, ..., z_N), \quad \mathbf{z}_t = \left(\frac{z_1}{p_1}, \frac{z_2}{p_2}, ..., \frac{z_N}{p_N}\right), \quad \mathbf{a}_j = (\alpha_{j1}, \alpha_{j2}, ..., \alpha_{jN}), \quad (j = 1, 2, ..., n). \tag{3.11}$$

*Here,* $p_j$ $(j = 1, 2, ..., N)$ *and* $\alpha_{jk}$ $(j = 1, 2, ..., n; k = 1, 2, ..., N)$ *are arbitrary complex parameters.*

The proof of the Proposition 2 can be done by means of an elementary calculation using the basic formulas of determinants, i.e.,

$$\frac{\partial}{\partial x}|D| = \sum_{j,k=1}^{N} \frac{\partial d_{jk}}{\partial x} D_{jk}, \quad (D_{jk} : \text{cofactor of } d_{jk}),$$

$$\begin{vmatrix} D & \mathbf{a}^T \\ \mathbf{b} & z \end{vmatrix} = |D|z - \sum_{j,k=1}^{N} D_{jk} a_j b_k,$$

$$|D(\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d})||D| = |D(\mathbf{a}; \mathbf{c})||D(\mathbf{b}; \mathbf{d})| - |D(\mathbf{a}; \mathbf{d})||D(\mathbf{b}; \mathbf{c})| \quad : \text{Jacobi's identity}.$$

If one replaces $z_j$ by $z_j = e^{p_j x + i p_j^2 t}$, then Proposition 2 provides the bright $N$-soliton solution of the $n$-component derivative NLS system [12]

$$q_j = \frac{h_j f^*}{f^2}, \quad h_j = (-1)^N \prod_{j=1}^{N} \frac{p_j^*}{p_j} \begin{vmatrix} D & \mathbf{z}^T \\ \mathbf{a}_j^* & 0 \end{vmatrix}, \quad (j = 1, 2, ..., n).$$

## 4. Conservation laws

The several methods are available to derive an infinite number of conservation laws for integrable soliton equations. One of them is based on the inverse scattering method, which we apply to the system (2.3). First, we write the linear system (2.1$a$) in terms of its components

$$\psi_{j,x} = \sum_{k=1}^{n+1} u_{jk}\psi_k, \quad \psi_{j,t} = \sum_{k=1}^{n+1} v_{jk}\psi_k, \quad (j = 1, 2, ..., n+1). \tag{4.1}$$

The compatibility condition of this system gives

$$\left(\sum_{k=1}^{n+1} \frac{u_{jk}\psi_k}{\psi_j}\right)_t = \left(\sum_{k=1}^{n+1} \frac{v_{jk}\psi_k}{\psi_j}\right)_x, \quad (j = 1, 2, ..., n+1). \tag{4.2}$$

For $j = 1$, the relation (4.2) yields

$$\left(u_{11} + \sum_{k=2}^{n+1} \frac{u_{1k}\psi_k}{\psi_1}\right)_t = \left(v_{11} + \sum_{k=2}^{n+1} \frac{v_{1k}\psi_k}{\psi_1}\right)_x.$$

If we substitute the matrix elements of $U$ and $V$ from (2.1$b$) and introduce the new variables $\Gamma_j = \psi_{j+1}/\psi_1$ $(j = 1, 2, ..., n)$, this expression can be put into the form

$$\left(\sum_{j=1}^{n} q_j \Gamma_j\right)_t = \left(\frac{1}{\zeta}\sum_{k=1}^{n} \sigma_k u_k u_k^* + \frac{i}{\zeta^2}\sum_{j=1}^{n} u_j \Gamma_j\right)_x, \quad (q_j = u_{j,x}). \tag{4.3}$$

showing that the quantity $\int_{-\infty}^{\infty} \sum_{j=1}^{n} q_j \Gamma_j dx$ is conserved.

Similarly, it follows from the first equation in (4.1) that

$$q_j\Gamma_j = \frac{1}{\zeta}\sigma_j q_j q_j^* + \frac{\mathrm{i}}{\zeta^2} q_j\Gamma_{j,x} + \frac{1}{\zeta} q_j\Gamma_j \sum_{k=1}^{n} q_k\Gamma_k, \quad (j = 1, 2, ..., n). \tag{4.4}$$

We expand the quantity $q_j\Gamma_j$ in inverse powers of $\zeta$ as

$$q_j\Gamma_j = \sum_{k=1}^{\infty} \frac{f_j^{(k)}}{\zeta^{2k-1}}, \quad (j = 1, 2, ..., n), \tag{4.5}$$

subsitute it into (4.4) and compare the same power of $\zeta$. Then, we obtain the recursion relation that determines $f_j^{(k)}$:

$$f_j^{(1)} = \sigma_j q_j q_j^*, \quad (j = 1, 2, ..., n), \tag{4.6$a$}$$

$$f_j^{(k)} = \mathrm{i} q_j \left(\frac{f_j^{(k-1)}}{q_j}\right)_x + \sum_{l=1}^{k-1} f_j^{(k-l)} \sum_{s=1}^{n} f_s^{(l)}, \quad (j = 1, 2, ..., n,\ k \geq 2). \tag{4.6$b$}$$

Consequently, the quantity

$$I = \int_{-\infty}^{\infty} \sum_{j=1}^{n} q_j\Gamma_j dx = \sum_{k=1}^{\infty} \frac{1}{\zeta^{2k-1}} \int_{-\infty}^{\infty} \sum_{j=1}^{n} f_j^{(k)} dx \equiv \sum_{k=1}^{\infty} \frac{I_k}{\zeta^{2k-1}}, \tag{4.7}$$

is conserved. Thus, we obtain an infinite number of conservation laws

$$I_k = \int_{-\infty}^{\infty} \sum_{j=1}^{n} f_j^{(k)} dx, \quad (k = 1, 2, ...). \tag{4.8}$$

The first three of them read

$$I_1 = \int_{-\infty}^{\infty} \sum_{j=1}^{n} \sigma_j q_j q_j^* dx, \quad (q_j = u_{j,x}), \tag{4.9$a$}$$

$$I_2 = \int_{-\infty}^{\infty} \left[ \frac{\mathrm{i}}{2} \sum_{j=1}^{n} \sigma_j (q_j q_{j,x}^* - q_j^* q_{j,x}) + \left( \sum_{j=1}^{n} \sigma_j q_j q_j^* \right)^2 \right] dx, \tag{4.9$b$}$$

$$I_3 = \int_{-\infty}^{\infty} \left[ \sum_{j=1}^{n} \sigma_j q_{j,x} q_{j,x}^* + \frac{3}{2} \mathrm{i} \sum_{j=1}^{n} \sigma_j (q_j q_{j,x}^* - q_{j,x} q_j^*) \sum_{s=1}^{n} \sigma_s q_s q_s^* + 2 \left( \sum_{j=1}^{n} \sigma_j q_j q_j^* \right)^3 \right] dx. \tag{4.9$c$}$$

## 5. Discussion

We discuss solutions of the $n$-component FL system (2.3) under the nonvanishing boundary conditions

$$u_j \sim \rho_j \exp\left(\mathrm{i}k_j x - \mathrm{i}\omega_j t + \mathrm{i}\phi_j^{(\pm)}\right), \quad x \to \pm\infty, \quad (j = 1, 2, ..., n), \tag{5.1}$$

where $\rho_j \in \mathbb{C}, k_j, \omega_j \in \mathbb{R}$ represent the amplitude, wavenumber and angular frequency of the plane wave, respectively, and $\phi_j^{(\pm)}$ are phase constants. The linear dispersion relation of the system (2.3) then becomes

$$k_j\omega_j = 1 + \sum_{s=1}^{n} \sigma_s k_s |\rho_s|^2 + \sum_{s=1}^{n} \sigma_s |\rho_s|^2 k_j, \quad (j = 1, 2, ..., n). \tag{5.2}$$

Introducing the dependent variable transformations

$$u_j = \rho_j e^{\mathrm{i}(k_j x - \omega_j t)} \frac{g_j}{f}, \quad (j = 1, 2, ..., n), \tag{5.3}$$

and performing the bilinearization, we obtain

$$D_t f \cdot f^* = \mathrm{i} \sum_{k=1}^{n} \sigma_k |\rho_k|^2 (g_k g_k^* - f f^*), \tag{5.4}$$

$$D_x D_t f \cdot f^* - \mathrm{i} \sum_{k=1}^{n} \sigma_k |\rho_k|^2 D_x g_k \cdot g_k^* + \mathrm{i} \sum_{k=1}^{n} \sigma_k |\rho_k|^2 D_x f \cdot f^* + 2 \sum_{s=1}^{n} \sigma_s k_s |\rho_s|^2 (g_s g_s^* - f f^*) = 0, \tag{5.5}$$

$$\begin{aligned} f^* \left[ g_{j,xt} f - (f_x - \mathrm{i} k_j f) g_{j,t} - \frac{\mathrm{i}}{k_j} \left( 1 + \sum_{s=1}^{n} \sigma_s k_s |\rho_s|^2 \right) D_x g_j \cdot f \right] \\ = f_t^* (g_{j,x} f - g_j f_x + \mathrm{i} k_j g_j f), \ (j = 1, 2, ..., n). \end{aligned} \tag{5.6}$$

As in the case of Eqs. (3.4), the trilinear equations (5.6) can be decoupled to the bilinear equations.

In the special case of $n = 1$, the corresponding expressions are given by

$$u = \rho e^{\mathrm{i}(kx - \omega t + \phi^{(\pm)})} \frac{g}{f}, \tag{5.7}$$

$$D_t f \cdot f^* = \mathrm{i} \rho^2 (g g^* - f f^*), \tag{5.8}$$

$$D_x D_t f \cdot f^* = \mathrm{i} \rho^2 D_x g \cdot g^* + \mathrm{i} \rho^2 D_x f \cdot f^* + 2 \rho^2 k (g g^* - f f^*), \tag{5.9}$$

$$f^* \left[ g_{xt} f - (f_x - \mathrm{i} k f) g_t - \mathrm{i} \left( \frac{1}{k} + \rho^2 \right) D_x g \cdot g^* \right] = f_t^* (g_x f - g f_x + \mathrm{i} k f g), \tag{5.10}$$

where $g = g_1, \rho = \rho_1, k = k_1, \omega = \omega_1, \phi^{(\pm)} = \phi_1^{(\pm)}, \sigma_1 = 1$. This system of equations coincides with that given in Matsuno [13] for the FL equation under the boundary condition (5.1).

The construction of the dark $N$-soliton solution of the system of equations (5.4)-(5.6) can be done following the similar procedure as that developed for the vanishing boundary conditions. It is given compactly by the determinantal form

$$f = |D|, \quad D = \left( \delta_{jk} - \frac{\mathrm{i} p_j}{p_j + p_k^*} z_j z_k^* \right)_{1 \le j,k \le N}, \tag{5.11}$$

$$g_s = |G_s|, \quad G_s = \left( \delta_{jk} - \frac{\mathrm{i} p_k^*}{p_j + p_k^*} \frac{p_j - \mathrm{i} k_s}{p_k^* + \mathrm{i} k_s} z_j z_k^* \right)_{1 \le j,k \le N}, \quad (s = 1, 2, ..., n), \tag{5.12}$$

$$z_j = \exp \left[ p_j x + \frac{1 + \sum_{s=1}^{n} \sigma_s k_s |\rho_s|^2}{p_j} t + \zeta_{j0} \right], \quad (j = 1, 2, ..., N), \tag{5.13}$$

where $p_j$ and $\zeta_{j0}$ $(j = 1, 2, ..., N)$ are arbitrary complex parameters and the $N$ constraints are imposed on the former parameters

$$\sum_{s=1}^{n} \frac{p_j p_j^* \sigma_s k_s |a_s|^2}{(p_j - \mathrm{i} k_s)(p_j^* + \mathrm{i} k_s)} = 1 + \sum_{s=1}^{n} \sigma_s k_s |\rho_s|^2, \quad (j = 1, 2, ..., N). \tag{5.14}$$

We point out that the expressions (5.11)-(5.14) will provide the dark $N$-soliton solution of the $n$-component derivative NLS system (3.8) if one changes the time dependence of $z_j$ from (5.13) and the constraints (5.14) appropriately.

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