数理解析研究所講究録
第2077巻 2018年 10-21

Description of infinite orbits on multiple flag varieties of type A

Naoya SHIMAMOTO
Graduate School of Mathematical Sciences, The University of Tokyo

Abstract

Let G be a reductive group, P be its parabolic subgroup, and H be a closed subgroup of G. There are several studies on the orbit decomposition of the flag variety G/P by the H-action, and these studies are expected to play an important role in various problems such as branching problem of G with respect to H. In this note, we focus on explicit descriptions of the orbit decomposition of a multiple flag variety $(G \times G \times \cdots \times G)/(P_1 \times P_2 \times \cdots \times P_m)$ by the diagonal action of G.

Now, let G be a general linear group on an algebraically closed field with characteristic 0. Magyar-Weyman-Zelevinsky proved that there are only finitely many orbits only if $m \leq 3$. Furthermore, they also classified all tuples $(P_1, P_2, \ldots, P_m)$ of parabolic subgroups where the number of orbits are finite, and gave explicit orbit decompositions for these cases. The aim of this note is to give an explicit description of the orbit decomposition for $m \geq 4$, the case where infinitely many orbits exist.

1 Introduction

There have been many studies on the relations between orbit decompositions on flag varieties and representation theory. For example, for a real reductive algebraic group G, its minimal parabolic subgroup P_G (resp. a Borel subgroup B_G of the complexification G_c of G), and its algebraically defined closed subgroup H, Kobayashi-Oshima [2] proved the following theorem on the relationship between the H-orbits (resp. H_c-orbits) on the real flag variety G/P_G (resp. complex flag variety G_c/B_G) and the analysis on the homogeneous space G/H:

Theorem 1.1. *For an irreducible admissible representation π of G and a finite dimensional irreducible representation τ of H, let $c_{\mathfrak{g},K}(\pi, \mathrm{Ind}_H^G \tau)$ be the multiplicity of the underlying $(\mathfrak{g}, K)$-module π_K in the space of sections for the vector bundle $\mathrm{Ind}_H^G \tau$ on G/H associated with τ. Then the followings are equivalent:*

(1) H (resp. H_c) has an open orbit on G/P_G (resp. G_c/B_G);

(2) for all irreducible admissible representations π of G and finite dimensional irreducible representations τ of H, $c_{\mathfrak{g},K}(\pi, \mathrm{Ind}_H^G \tau) < \infty$. (resp. bounded proportionally to $\dim \tau$).

A homogeneous space G/H satisfying the first condition in the theorem is called a real spherical variety (resp. spherical variety).

A number of people, for instance, Brion, Vinberg, Kimelfeld, Bien, Matsuki, and Kobayashi, proved that G/H is real spherical (resp. spherical) if and only if the number of H-orbits (resp H_c-orbits) on the flag variety G/P_G (resp. G_c/B_G) is finite [3]. Remark that for a non-minimal parabolic subgroup P of G, it may occur that H has infinitely many orbits and some open orbits on G/P simultaneously.

In this note, we focus on the orbit decomposition of an m-tuple flag variety of $G = GL_n$ especially for the m-ary direct product of projective spaces under the diagonal action of G. This is because it is the simplest case among the cases where H has some open orbits and uncountably many orbits simultaneously on a flag variety G/P in some sense.

The main idea is to decompose the multiple flag variety into finitely many pieces by using the notion of direct sums of multiple flags, and to classify those pieces whether it becomes a single orbit or it decomposes into infinitely many orbits.

2 Main problem and settings

Let G be the general linear group of the degree $n \geq 2$ over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We also let P denote the parabolic subgroup of G with blocks of sizes $(1, n-1)$, and we consider the multiple flag variety G^m/P^m. Our main problem is to describe the diag(G)-orbit decomposition of this multiple flag variety.

We know that $G^m/P^m \simeq (\mathbb{P}^{n-1}\mathbb{K})^m$. Hence our main problem is equivalent to describing the orbit decomposition of $(\mathbb{P}^{n-1}\mathbb{K})^m$ under the diagonal action of $GL(n;\mathbb{K})$.

In this setting, the following proposition holds.

Proposition 2.1. *(1) $m \geq 4 \Leftrightarrow$ there are infinitely many orbits.*

(2) $m \leq n+1 \Leftrightarrow$ there exists an open orbit.

If there are only finitely many orbits, then there exists an open orbit. On the other hand, the converse does not hold in general. (If P is the minimal parabolic subgroup, then it holds.) In other words, there are some cases where open orbits and infinitely many orbits exist simultaneously. In our case, if $4 \leq m \leq n+1$, then open orbits and infinitely many orbits exist simultaneously. Hence, our setting includes the following 3 situations:

(1) the number of orbits is finite ($\Rightarrow$there exists an open orbit);

(2) the number of orbits is infinite, but there exists an open orbit;

(3) there is no open orbit ($\Rightarrow$ the number of orbits is infinite).

Example 2.2. If $(n, m) = (2, 3)$, we are considering the case where $GL(2;\mathbb{K})$ acts diagonally on the ternary product of $\mathbb{P}^1\mathbb{K}$ by the Möbius transformation. Any 3 distinct points can be transformed to $(0, 1, \infty)$ by this action, hence representatives of all orbits can be described as

$$(0,0,0),\ (0,0,1),\ (0,1,0),\ (1,0,0),\ (0,1,\infty).$$

In particular if $(n, m) = (2, 2)$, representatives of all orbits can be described as $(0, 0)$, $(0, 1)$.

Example 2.3. If $(n,m) = (2,4)$, we replace the ternary product of $\mathbb{P}^1\mathbb{K}$ with 4-ary product. For a 4-tuple (z_1, z_2, z_3, z_4) such that $z_1 \neq z_2 \neq z_3 \neq z_1$, by transforming (z_1, z_2, z_3) to $(0, 1, \infty)$, z_4 is transformed to

$$\frac{(z_2 - z_3)(z_4 - z_1)}{(z_1 - z_2)(z_3 - z_4)}$$

which is $GL(2;\mathbb{K})$-invariant. Hence there exist infinitely many orbits represented by

$$(0, 1, \infty, z) \quad (z \in \mathbb{P}^1\mathbb{K}).$$

3 Main result

To describe all orbits combinatorially, we introduce a set of some partitions of $\{1, 2, \ldots, m\}$ with some extra conditions and information. Let $\mathcal{P}$ denote the set of tuples

$$p = \left(\{I_a\}_{a=1}^{A}, \{J_b\}_{b=1}^{B}, \{(K_c, r_c)\}_{c=1}^{C}\right)$$

satisfying the conditions:

(1)
$$A, B, C, r_c \in \mathbb{N}, \quad \emptyset \neq I_a, J_b, K_c \subset \{1, 2, \ldots, m\};$$

(2)
$$\left(\coprod_{a=1}^{A} I_a\right) \sqcup \left(\coprod_{b=1}^{B} J_b\right) \sqcup \left(\coprod_{c=1}^{C} K_c\right) = \{1, 2, \ldots, m\};$$

(3)
$$3 \leq \# J_b \quad (1 \leq {}^\forall b \leq B);$$

(4)
$$4 \leq r_c + 2 \leq \# K_c \quad (1 \leq {}^\forall c \leq C);$$

(5)
$$0 \leq A + \sum_{b=1}^{B} (\# J_b - 1) + \sum_{c=1}^{C} r_c \leq n.$$

The number $r(p)$ denotes the summation in (5).

With this combinatorial material, we can claim the main result as follows:

Theorem 3.1 ([5]). *There exists a surjection*

$$\pi \colon \operatorname{diag}(G)\backslash G^m/P^m \twoheadrightarrow \mathcal{P}$$

onto the finite set $\mathcal{P}$ with the property that, for $p \in \mathcal{P}$, there exists an open dense embedding

$$\prod_{c=1}^{C} (\mathbb{P}^{r_c - 1}\mathbb{K})^{\# K_c - r_c - 1} \hookrightarrow \pi^{-1}(p)$$

in the classical topology.

To give explicit representatives of orbits, we introduce the following notations in general settings.

Let $F^{(1)}, F^{(2)}, \ldots,$ and $F^{(m)}$ be flags with the common whole space. More precisely, for a vector space V, each $F^{(i)}$ is a sequence of subspaces of V such as

$$F^{(i)} = \left(0 = V_0 \subset V_1 \subset \cdots \subset V_{n_i - 1} \subset V_{n_i} = V\right).$$

Then we call $(F^{(1)}, F^{(2)}, \ldots, F^{(m)}; V)$ an m-tuple flag.

We define the notion of the direct sum of multiple flags as the collection of direct sums of each subspaces. More precisely, given two flags $F = \left(0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V\right)$ and $G = \left(0 = W_0 \subset W_1 \subset \cdots \subset W_{n-1} \subset W_n = W\right)$, we define the direct sum of the flags F and G by

$$F \oplus G = \left(0 = V_0 \oplus W_0 \subset V_1 \oplus W_1 \subset \cdots \subset V_{n-1} \oplus W_{n-1} \subset V_n \oplus W_n = V \oplus W\right).$$

For two multiple flags $F = (F^{(1)}, F^{(2)}, \ldots, F^{(m)}; V)$ and $G = (G^{(1)}, G^{(2)}, \ldots, G^{(m)}; W)$, we define the direct sum of two multiple flags by

$$F \oplus G = (F^{(1)} \oplus G^{(1)}, F^{(2)} \oplus G^{(2)}, \ldots, F^{(m)} \oplus G^{(m)}; V \oplus W).$$

Example 3.2. Let F and G be double flags defined by

$$F = \left((0 \subset \mathbb{K} \subset \mathbb{K}), (0 \subset 0 \subset \mathbb{K}); \mathbb{K}\right),$$
$$G = \left((0 \subset 0 \subset \mathbb{K}), (0 \subset \mathbb{K} \subset \mathbb{K}); \mathbb{K}\right).$$

The whole space of $F \oplus G$ is $\mathbb{K}^{\oplus 2}$, and let $\{e_1, e_2\}$ be the standard basis. Then we have

$$F \oplus G = \left((0 \subset \mathbb{K}e_1 \subset \mathbb{K}^{\oplus 2}), (0 \subset \mathbb{K}e_2 \subset \mathbb{K}^{\oplus 2}); \mathbb{K}^{\oplus 2}\right).$$

On the other hand, if

$$F' = \left((0 \subset \mathbb{K} \subset \mathbb{K}), (0 \subset \mathbb{K} \subset \mathbb{K}); \mathbb{K}\right),$$
$$G' = \left((0 \subset 0 \subset \mathbb{K}), (0 \subset 0 \subset \mathbb{K}); \mathbb{K}\right),$$

then we have

$$F' \oplus G' = \left((0 \subset \mathbb{K}e_1 \subset \mathbb{K}^{\oplus 2}), (0 \subset \mathbb{K}e_1 \subset \mathbb{K}^{\oplus 2}); \mathbb{K}^{\oplus 2}\right).$$

We say that a multiple flag is *indecomposable* if it cannot be realised as the direct sum of two non-trivial (the whole space is non-zero) multiple flags.

Two multiple flags $F = (F^{(1)}, F^{(2)}, \ldots, F^{(m)}; V)$ and $G = (G^{(1)}, G^{(2)}, \ldots, G^{(m)}; W)$ are said to be *isomorphic* to each other if there exists a linear isomorphism $f \colon V \to W$, and f sends all subspaces in F to those of G. More precisely, if $F^{(i)}$ and $G^{(i)}$ are expressed as

$$F^{(i)} = \left(0 = V_0^{(i)} \subset V_1^{(i)} \subset \cdots \subset V_{n_i - 1}^{(i)} \subset V_{n_i}^{(i)} = V\right),$$
$$G^{(i)} = \left(0 = W_0^{(i)} \subset W_1^{(i)} \subset \cdots \subset W_{m_i - 1}^{(i)} \subset W_{m_i}^{(i)} = W\right),$$

then $n_i = m_i$ and $f(V_j^{(i)}) = W_j^{(i)}$ for all i's and j's.

Example 3.3. For two double flags $F \oplus G$ and $F' \oplus G'$ in Example 3.2, they are *not* isomorphic to each other.

For a multiple flag, we can define the notion of dimension vector by taking the difference of dimensions between the subspaces. Let $F = \big(0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V\big)$ be a flag, then we call $\big(\dim V_j - \dim V_{j-1}\big)_{j=1}^{n}$ the *dimension vector* of F and denote it $\dim F$. If we have a multiple flag $(F^{(1)}, F^{(2)}, \ldots, F^{(m)}; V)$, then we define the dimension vector of this multiple flag just by lining up the dimension vectors of $F^{(i)}$'s to be an m-tuple of vectors with non negative integral entries. Remark that for a dimension vector of a multiple flag, the sum of entries in each row is independent of i's, since it is equal to the dimension of the whole space V.

It is clear that $\dim(F \oplus G) = \dim F + \dim G$ for multiple flags F and G. Hence, if we have a decomposition of a multiple flag into indecomposables, then the summation of dimension vectors of each summand is equal to the original dimension vector.

Now, we define some indecomposable multiple flags to give explicit representatives of orbits in Theorem 3.1. From now on, we write the multiple flag $\big((0 \subset V^{(1)} \subset V), (0 \subset V^{(2)} \subset V), \ldots, (0 \subset V^{(m)} \subset V); V\big)$ just as $(V^{(1)}, V^{(2)}, \ldots, V^{(m)}; V)$.

(1) The cases for I_a's.

At first, we consider the case where $I_a = \{1, 2, \ldots, k\}$. Let $F(\{1, 2, \ldots, k\})$ be the m-tuple flag defined by

$$F(\{1,2,\ldots,k\}) = \Big(\overbrace{\mathbb{K}, \mathbb{K}, \ldots, \mathbb{K}}^{k}, \overbrace{0, 0, \ldots, 0}^{m-k}; \mathbb{K}\Big). \tag{3.1}$$

To cover permuted ones, we define $F(I)$ for general $I \subset \{1, 2, \ldots, m\}$ by

$$F(I)^{(i)} = \begin{cases} \mathbb{K} & i \in I, \\ 0 & i \notin I. \end{cases} \tag{3.2}$$

(2) The cases for J_b's.

At first, we consider the case where $J_b = \{1, 2, \ldots, r+1\}$. Let $G(\{1, 2, \ldots, r+1\})$ be the m-tuple flag such that

$$G(\{1,2,\ldots,r+1\}) = \Big(\overbrace{\mathbb{K}e_1, \mathbb{K}e_2, \ldots, \mathbb{K}e_r, \mathbb{K}\Big(\sum_{l=1}^{r} e_l\Big)}^{r+1}, \overbrace{0, 0, \ldots, 0}^{m-r-1}; \mathbb{K}^{\oplus r}\Big) \tag{3.3}$$

where $\{e_1, e_2, \ldots, e_r\}$ is the standard basis of $\mathbb{K}^{\oplus r}$.

To cover permuted ones, we define $G(J)$ for general $J = \{i_1 < i_2 < \cdots < i_r < i_{r+1}\}$ by

$$G(J)^{(i)} = \begin{cases} \mathbb{K}e_l & i = i_l \in J, \quad 1 \le l \le r, \\ \mathbb{K}\left(\sum_{l=1}^{r} e_l\right) & i = i_{r+1} \in J, \\ 0 & i \notin J. \end{cases} \tag{3.4}$$

(3) The cases for (K_c, r_c)'s.

Similarly to the previous cases, we consider the case where $K_c = \{1, 2, \ldots, k\}$. Note that $2 \leq r_c + 2 \leq k$ by (4) in the definition of $\mathcal{P}$. Now r denotes r_c for simplicity. For $q = (q_{r+2}, q_{r+3}, \ldots, q_k) \in (\mathbb{P}^{r-1}\mathbb{K})^{k-r-1}$, we define the m-tuple flag $G(K, r, q)$ by

$$G(K,r,q) = \Big(\overbrace{\mathbb{K}e_1, \mathbb{K}e_2, \ldots, \mathbb{K}e_r, \mathbb{K}\left(\sum_{l=1}^{r} e_l\right)}^{r+1}, \overbrace{q_{r+2}, q_{r+3}, \ldots, q_k}^{k-r-1}, \overbrace{0, 0, \ldots, 0}^{m-k}; \mathbb{K}^{\oplus r}\Big) \tag{3.5}$$

where $\{e_1, e_2, \ldots, e_r\}$ is the standard basis of $\mathbb{K}^{\oplus r}$.

To cover permuted ones, we define $G(K, r, q)$ for general $K = \{i_1 < i_2 < \cdots < i_k\} \subset \{1, 2, \ldots, m\}$ by

$$G(K,r,q)^{(i)} = \begin{cases} \mathbb{K}e_l & i = i_l \in K, \quad 1 \leq l \leq r, \\ \mathbb{K}\left(\sum_{l=1}^{r} e_l\right) & i = i_{r+1} \in K, \\ q_l & i = i_l \in K, \quad r+2 \leq l \leq k, \\ 0 & i \notin K. \end{cases} \tag{3.6}$$

Now, let

$$p = \left(\{I_a\}_{a=1}^{A}, \{J_b\}_{b=1}^{B}, \{(K_c, r_c)\}_{c=1}^{C}\right) \in \mathcal{P}$$

and F be a multiple flag expressed as

$$F = F(\emptyset)^{\oplus(n-r(p))} \oplus \left(\bigoplus_{a=1}^{A} F(I_a)\right) \oplus \left(\bigoplus_{b=1}^{B} G(J_b)\right) \oplus \left(\bigoplus_{c=1}^{C} G(K_c, r_c, q^{(c)})\right),$$

where

$$(q^{(1)}, q^{(2)}, \ldots, q^{(C)}) \in \prod_{c=1}^{C} (\mathbb{P}^{r_c-1}\mathbb{K})^{\#K_c - r_c - 1}.$$

Then F can be identified with an element of $G^m/P^m \simeq (\mathbb{P}^{n-1}\mathbb{K})^m$ and

$$\pi(\mathrm{diag}(G) \cdot F) = p.$$

Furthermore, this correspondence from $\prod_{c=1}^{C}(\mathbb{P}^{r_c-1}\mathbb{K})^{\#K_c-r_c-1}$ to $\pi^{-1}(\{p\})$ is one of the embeddings in the Theorem 3.1.

Example 3.4. For the case $(n, m) = (2, 2)$, we have

$$\mathcal{P} = \{p_c = (\{\{1,2\}\}, \emptyset, \emptyset), p_o = (\{\{1\}, \{2\}\}, \emptyset, \emptyset)\}.$$

The orbit corresponding to p_c is the orbit through

$$F(\emptyset) \oplus F(\{1,2\}) = \left(0, 0; \mathbb{K}\right) \oplus \left(\mathbb{K}, \mathbb{K}; \mathbb{K}\right) = \left(\mathbb{K}e_2, \mathbb{K}e_2; \mathbb{K}^{\oplus 2}\right)$$

where $\{e_1, e_2\}$ is the standard basis of $\mathbb{K}^{\oplus 2}$. This is the closed orbit in $\mathbb{P}^1\mathbb{K} \times \mathbb{P}^1\mathbb{K}$.

On the other hand, the orbit corresponding to p_o is through

$$F(\{1\}) \oplus F(\{2\}) = \left(\mathbb{K}, 0; \mathbb{K}\right) \oplus \left(0, \mathbb{K}; \mathbb{K}\right) = \left(\mathbb{K}e_1, \mathbb{K}e_2; \mathbb{K}^{\oplus 2}\right),$$

and this is the open orbit in $\mathbb{P}^1\mathbb{K} \times \mathbb{P}^1\mathbb{K}$. Recall Example 3.2.

Example 3.5. For the case $(n,m)=(2,3)$, we have

$$\mathcal{P}=\{p_c=(\{\{1,2,3\}\},\emptyset,\emptyset),p_1=(\{\{2,3\},\{1\}\},\emptyset,\emptyset),p_2=(\{\{1,3\},\{2\}\},\emptyset,\emptyset),\\ p_3=(\{\{1,2\},\{3\}\},\emptyset,\emptyset),p_o=(\emptyset,\{\{1,2,3\}\},\emptyset)\}$$

The orbit corresponding to p_c is the orbit through

$$F(\emptyset)\oplus F(\{1,2,3\})=\left(0,0,0;\mathbb{K}\right)\oplus\left(\mathbb{K},\mathbb{K},\mathbb{K};\mathbb{K}\right)=\left(\mathbb{K}e_2,\mathbb{K}e_2,\mathbb{K}e_2;\mathbb{K}^{\oplus 2}\right)$$

where $\{e_1,e_2\}$ is the standard basis of $\mathbb{K}^{\oplus 2}$, and this is the closed orbit in $\mathbb{P}^1\mathbb{K}\times\mathbb{P}^1\mathbb{K}\times\mathbb{P}^1\mathbb{K}$.

The orbit corresponding to p_1 is the orbit through

$$F(\{1\})\oplus F(\{2,3\})=\left(\mathbb{K},0,0;\mathbb{K}\right)\oplus\left(0,\mathbb{K},\mathbb{K};\mathbb{K}\right)=\left(\mathbb{K}e_1,\mathbb{K}e_2,\mathbb{K}e_2;\mathbb{K}^{\oplus 2}\right).$$

The representatives of the orbits corresponding to p_2 and p_3 are obtained similarly.

On the other hand, the orbit corresponding to p_o is through

$$G(\{1,2,3\})=\left(\mathbb{K}e_1,\mathbb{K}e_2,\mathbb{K}(e_1+e_2);\mathbb{K}^{\oplus 2}\right).$$

This is the open orbit in $\mathbb{P}^1\mathbb{K}\times\mathbb{P}^1\mathbb{K}\times\mathbb{P}^1\mathbb{K}$.

Example 3.6. For the case $(n,m)=(5,9)$, we set

$$p=(\{\{1,2\}\},\{\{3,4,5\}\},\{(\{6,7,8,9\},2)\})\in\mathcal{P}.$$

We have $r(p)=5$. For the tuple $(K_1,r_1)=(\{6,7,8,9\},2)$, we have $r_1=2$ and $k=\#K_1=4$ under the notation in the definition of $\mathcal{P}$. Thus, we now deal with the multiple flag $G(K_1,r_1,q)$ where $q\in\mathbb{P}^1\mathbb{K}$ with $q=\overline{\begin{pmatrix}q_1\\q_2\end{pmatrix}}\in\mathbb{P}^1\mathbb{K}$. The orbit corresponding to (p,q) is through

$$\begin{aligned}&F(\{1,2\})\oplus G(\{3,4,5\})\oplus G(\{6,7,8,9\},2,q)\\&\quad=(\mathbb{K},\mathbb{K},0,0,0,0,0,0,0;\mathbb{K})\oplus(0,0,\mathbb{K}e_1,\mathbb{K}e_2,\mathbb{K}(e_1+e_2),0,0,0,0;\mathbb{K}^{\oplus 2})\\&\qquad\oplus(0,0,0,0,0,\mathbb{K}e_1,\mathbb{K}e_2,\mathbb{K}(e_1+e_2),\mathbb{K}(q_1e_1+q_2e_2);\mathbb{K}^{\oplus 2})\\&\quad=(\mathbb{K}e_1,\mathbb{K}e_1,\mathbb{K}e_2,\mathbb{K}e_3,\mathbb{K}(e_2+e_3),\mathbb{K}e_4,\mathbb{K}e_5,\mathbb{K}(e_4+e_5),\mathbb{K}(q_1e_4+q_2e_5);\mathbb{K}^{\oplus 5})\end{aligned}$$

where $\{e_1,\ e_2,\dots,e_5\}$ is the standard basis of $\mathbb{K}^{\oplus 5}$. Remark that in this case there is no open orbit.

4 Splitting of flags into indecomposables

Let $\boldsymbol{a}=\left((a_j^{(i)})_{j=1}^{n_i}\right)_{i=1}^{m}$ be an m-tuple of vectors with non-negative integral entries such that the number $n=\sum_{j=1}^{n_i}a_j^{(i)}$ does not depend on i. We call such $\boldsymbol{a}$ an abstract dimension vector, and $|\boldsymbol{a}|$ denotes the number n. For an n-dimensional vector space V, the multiple flag variety $Fl_{\boldsymbol{a}}(V)$ is defined as the set of all multiple flags whose whole spaces are V and whose dimension vectors are $\boldsymbol{a}$.

Let $G = GL(n;\mathbb{K})$ with $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, and P be the the parabolic subgroup of G with blocks of sizes $(1, n-1)$. We consider the homogeneous space G^m/P^m. Clearly G^m/P^m is identified with $Fl_{\boldsymbol{a}}(\mathbb{K}^{\oplus n})$ where

$$\boldsymbol{a} = \Big(\overbrace{(1, n-1), (1, n-1), \ldots, (1, n-1)}^{m}\Big).$$

Under this identification we have

$$\operatorname{diag}(G)\backslash G^m/P^m \cong Fl_{\boldsymbol{a}}(\mathbb{K}^{\oplus n})/\simeq .$$

Now, our main problem, describing the double coset $\operatorname{diag}(G)\backslash G^m/P^m$, is interpreted as describing isomorphic classes of multiple flags with the dimension vectors $\boldsymbol{a}$. To determine all isomorphic classes, it suffices to consider the decompositions of multiple flags with the dimension vectors $\boldsymbol{a}$ into indecomposable ones. More precisely,

$$\begin{array}{ccc} Fl_{\boldsymbol{a}}(\mathbb{K}^{\oplus n})/\simeq & \cong & \Big\{\{m_I\}_{I\in\tilde{\Lambda}} \,|\, \sum_{I\in\tilde{\Lambda}} m_I \dim I = \boldsymbol{a},\ m_I \in \mathbb{Z}_{\geq 0}\Big\} \\ \cup\!\!\!\!\!\!\;\shortmid & & \cup\!\!\!\!\!\!\;\shortmid \\ \bigoplus_{I\in\tilde{\Lambda}} I^{\oplus m_I} & \mapsto & \{m_I\}_{I\in\tilde{\Lambda}} \end{array}$$

where $\tilde{\Lambda}$ is a system of representatives of all indecomposable multiple flags. Furthermore, we have a surjection

$$\begin{array}{ccc} \Big\{\{m_I\}_{I\in\tilde{\Lambda}} \,|\, \sum_{I\in\tilde{\Lambda}} m_I \dim I = \boldsymbol{a},\ m_I \in \mathbb{Z}_{\geq 0}\Big\} & \twoheadrightarrow & \{\{m_{\boldsymbol{d}}\}_{\boldsymbol{d}\in\Lambda} \,|\, \sum_{\boldsymbol{d}\in\Lambda} m_{\boldsymbol{d}}\boldsymbol{d} = \boldsymbol{a},\ m_{\boldsymbol{d}} \in \mathbb{Z}_{\geq 0}\} \\ \cup\!\!\!\!\!\!\;\shortmid & & \cup\!\!\!\!\!\!\;\shortmid \\ \{m_I\}_{I\in\tilde{\Lambda}} & \mapsto & \{\sum_{\dim I = \boldsymbol{d}} m_I\}_{\boldsymbol{d}\in\Lambda} \end{array} \tag{4.1}$$

where Λ is the set of all abstract dimension vectors $\boldsymbol{d}$ such that there exists an indecomposable multiple flag with the dimension vector $\boldsymbol{d}$, and $\boldsymbol{d}$ is an summand of $\boldsymbol{a}$.

Considering these maps, our problem is separated into the following 2 problems:

- determining all combinations of dimension vectors whose sum is equal to the original dimension vector $\boldsymbol{a}$, which is purely combinatorial problem (determining the set in the right-hand side of the surjection above);
- determining all isomorphic classes of indecomposable multiple flags with the given dimension vectors, which is a slightly complicated problem (determining the preimages of the surjection).

The surjection in Theorem 3.1 is the composition of three maps above. More precisely, for an orbit $\mathcal{O}$ in G^m/P^m, I_a, J_b, and (K_c, r_c, q_c) occuring in $\pi(\mathcal{O})$ express that the multiplicities of $F(I_a), G(J_b)$, and $G(K_c, r_c, q_c)$ defined in (3.1)–(3.6) are all 1, and multiplicities of other indecomposable multiple flags are 0 but $F(\emptyset)$. The multiplicity of $F(\emptyset)$ is $n - r(p)$.

First of all, we determine all possibilities of abstract dimension vectors which are summands of $\boldsymbol{a}$. All summands of $\boldsymbol{a}$ are of the forms

$$\boldsymbol{d}(r,k) = \Big(\overbrace{(1, r-1), \ldots, (1, r-1)}^{k}, \overbrace{(0, r), \ldots, (0, r)}^{m-k}\Big),$$

or its permutations of entries. We see that $|\boldsymbol{d}(r,k)| = r$. The numbers r and k should satisfy $1 \leq r \leq n$ and $0 \leq k \leq m$. Remark that if $r = n$ then $k = m$.

Now to determine all indecomposable multiple flags for each dimension vector $\boldsymbol{d}(r,k)$, we introduce the Tits form.

For a dimension vector $\boldsymbol{d}$, the Tits form $Q(\boldsymbol{d})$ is defined as

$$Q(\boldsymbol{d}) = \dim GL(W) - \dim FL_{\boldsymbol{d}}(W),$$

where $|\boldsymbol{d}| = r$, and W is an r-dimensional vector space.

For this quadratic form, there is an important property.

Theorem 4.1 ([1], [4]). *If there exists an indecomposable multiple flag with the dimension vector* $\mathbf{d}$*, then* $Q(\mathbf{d}) \leq 1$*. Furthermore, if there exists an indecomposable multiple flag with the dimension vector* $\mathbf{d}$ *and* $Q(\mathbf{d}) = 1$*, then all indecomposable multiple flags with the dimension vectors* $\mathbf{d}$ *are isomorphic to each others.*

Permuting the entries preserves the Tits form, hence it suffices to consider the dimension vectors of the form $\boldsymbol{d}(r,k)$ now. One can easily compute that

$$Q(\boldsymbol{d}(r,k)) = (r-1)(r-k+1)+1.$$

(1) The case where $r = 1$.

Clearly $Fl_{\boldsymbol{d}(1,k)}(\mathbb{K})$ is a singleton, and the only multiple flag with this dimension vector is of course indecomposable. The only isomorphic class of multiple flags with the dimension vectors $\boldsymbol{d}(1,k)$ is represented by

$$\Big(\overbrace{\mathbb{K},\mathbb{K},\ldots,\mathbb{K}}^{k}, \overbrace{0,0,\ldots,0}^{m-k};\ \mathbb{K}\Big) \in Fl_{\boldsymbol{d}(1,k)}(\mathbb{K}). \tag{4.2}$$

This is just $F(\{1,2,\ldots,k\})$ which we defined in (3.1).

To cover all permuted ones, we redefine the dimension vector $\mathbf{1}(I)$ as

$$\mathbf{1}(I)^{(i)} = \begin{cases} (1,0) & i \in I, \\ (0,1) & i \notin I \end{cases} \tag{4.3}$$

for $I \subset \{1,2,\ldots,m\}$ with $\#I = k$. It is clear that $\boldsymbol{d}(1,k) = \mathbf{1}(\{1,2,\ldots,k\})$, and the indecomposable multiple flags corresponding to $\mathbf{1}(I)$ are given by $F(I)$ in (3.2).

(2) The case where $r \geq 2$, and $0 \leq k \leq 2$.

Now $Q(\boldsymbol{d}(r,k)) \geq 2$ holds always. Hence there is no indecomposable multiple flag with the dimension vector $\boldsymbol{d}(r,k)$ by Theorem 4.1 in this case.

(3) The case where $r \geq 2$, and $k \geq 3$.

Now the value of $Q(\boldsymbol{d}(r,k))$ is

$$\begin{aligned} Q(\boldsymbol{d}(r,k)) \leq 0 \ &\Leftrightarrow\ 4 \leq r+2 \leq k, \\ Q(\boldsymbol{d}(r,k)) = 1 \ &\Leftrightarrow\ r+1 = k, \\ Q(\boldsymbol{d}(r,k)) \geq 2 \ &\Leftrightarrow\ k \leq r. \end{aligned}$$

Hence there is no indecomposable multiple flag with the dimension vector $\boldsymbol{d}(r,k)$ if $k \leq r$.

If $r+1=k$, the multiple flag

$$\Big(\overbrace{\mathbb{K}e_1, \mathbb{K}e_2, \ldots, \mathbb{K}e_r, \mathbb{K}\left(\sum_{k=1}^{r} e_k\right)}^{r+1}, \overbrace{0, 0, \ldots, 0}^{m-r-1} \ ; \mathbb{K}^{\oplus r}\Big) \in Fl_{\boldsymbol{d}(r,r+1)}(\mathbb{K}^{\oplus r}) \tag{4.4}$$

is just $G(\{1, 2, \ldots, r+1\})$ defined in (3.3), and this is indecomposable. By Theorem 4.1, $G(\{1, 2, \ldots, r+1\})$ is contained in the unique isomorphic class of indecomposable multiple flags with the dimension vectors $\boldsymbol{d}(r, r+1)$.

To cover permuted ones, we redefine the dimension vector $\boldsymbol{d}(J)$ for $J \subset \{1, 2, \ldots, m\}$ as

$$\boldsymbol{d}(J)^{(i)} = \begin{cases} (1, r-1) & i \in J, \\ (0, r) & i \notin J \end{cases} \tag{4.5}$$

where $r+1 = k = \#J$. It is clear that $\boldsymbol{d}(r, r+1) = \boldsymbol{d}(\{1, 2, \ldots, r, r+1\})$, and the indecomposable multiple falg corresponding to $\boldsymbol{d}(J)$ are given by $G(J)$ in (3.4).

On the other hand if $4 \leq r+2 \leq k$, then there may exist more than two isomorphic classes of indecomposable multiple flags with the dimension vectors $\boldsymbol{d}(r,k)$ by Theorem 4.1. $\Lambda_{\boldsymbol{d}(r,k)}$ denotes the set of all isomorphic classes of indecomposable multiple flags with the dimension vectors $\boldsymbol{d}(r,k)$, and consider the topology relatively defined from the classical topology of $\mathbb{P}^1\mathbb{K}$. In fact, there is an open dense embedding of $(\mathbb{P}^{r-1}\mathbb{K})^{k-r-1}$ into $\Lambda_{\boldsymbol{d}(r,k)}$ defined by

$$(p_{r+2}, p_{r+3}, \ldots, p_k) \mapsto$$

$$\Big(\overbrace{\mathbb{K}e_1, \mathbb{K}e_2, \ldots, \mathbb{K}e_r, \mathbb{K}\left(\sum_{k=1}^{r} e_k\right)}^{r+1}, \overbrace{p_{r+2}, p_{r+3}, \ldots, p_k}^{k-r-1}, \overbrace{0, 0, \ldots, 0}^{m-k}; \ \mathbb{K}^{\oplus r}\Big) \in Fl_{\boldsymbol{d}(r,k)}(\mathbb{K}^{\oplus r}).$$

Hence $\Lambda_{\boldsymbol{d}(r,k)}$ has uncountably many elements.

To cover permuted ones, we redefine the dimension vector $\boldsymbol{e}(K,r)$ as

$$\boldsymbol{e}(K,r)^{(i)} = \begin{cases} (1, r-1) & i \in K, \\ (0, r) & i \notin K \end{cases} \tag{4.6}$$

for $K \subset \{1, 2, \ldots, m\}$ where $r+2 \leq k = \#K$.

It is clear that $\boldsymbol{d}(r,k) = \boldsymbol{e}(\{1, 2, \ldots, k\}, r)$, and the indecomposable multiple flags corresponding to $\boldsymbol{e}(K,r)$ are given by $G(K, r, q)$ in (3.6).

If we consider other embeddings of $(\mathbb{P}^{r-1}\mathbb{K})^{\#K-r-1}$ into $\Lambda_{e(K,r)}$ by permuting the first $r+1$ vectors in general position and the remaining $k-r-1$ vectors, these are all isomorphic classes of indecomposable multiple flags occuring in the decomposition of multiple flags in $Fl_{\boldsymbol{a}}(\mathbb{K}^{\oplus n})$.

Next, we want to determine $(m_{\boldsymbol{d}})_{\boldsymbol{d}\in\Lambda}$ satisfying $\sum_{\boldsymbol{d}\in\Lambda} m_{\boldsymbol{d}}\boldsymbol{d} = \boldsymbol{a}$ in (4.1). As we saw before, all dimension vectors in $\boldsymbol{d}$ which are possibly summands of $\boldsymbol{a}$ and which have indecomposable multiple flags are expressed as $\mathbf{1}(I)$, $\boldsymbol{d}(J)$, or $\boldsymbol{e}(K,r)$. One can prove that all multiplicities of each dimension vectors are at most 1 but $\mathbf{1}(\emptyset)$, and I, J, K's form a partition of $\{1,2,\ldots,m\}$.

Hence the multiplicity problem is reduced to determining which dimension vectors occur or not. The equality

$$\sum_{\boldsymbol{d}\in\Lambda}\boldsymbol{d} = m_{\mathbf{1}(\emptyset)}\mathbf{1}(\emptyset) + \sum_{a=1}^{A}\mathbf{1}(I_a) + \sum_{b=1}^{B}\boldsymbol{d}(J_b) + \sum_{c=1}^{C}\boldsymbol{e}(K_c,r_c) = \boldsymbol{a}$$

leads us to the condition

$$m_{\mathbf{1}(\emptyset)} + A + \sum_{b=1}^{B}(\#J_b - 1) + \sum_{c=1}^{C} r_c = n$$

which is a crucial part of the definition of $\mathcal{P}$.

5 Infiniteness and dimension of orbits

By the expression in Theorem 3.1, we can calculate the dimension of each orbit explicitly.

Corollary 5.1. *For*

$$p = \left(\{I_a\}_{a=1}^{A}, \{J_b\}_{b=1}^{B}, \{(K_c,r_c)\}_{c=1}^{C}\right) \in \mathcal{P}$$

and $O \in \pi^{-1}(p)$*,*

$$\dim O = nr(p) - A - B - C.$$

Using this formula, the open orbit is explicitly obtained.

(1) If $n \geq m$, consider $p_0 = \left(\{\{j\}\}_{j=1}^{m}, \emptyset, \emptyset\right) \in \mathcal{P}$. Then for $\{O_0\} = \pi^{-1}(p_0)$,

$$\dim O_0 = nm - m - 0 - 0 = \dim(\mathbb{P}^{n-1}\mathbb{K})^m.$$

(2) If $n+1 = m$, consider $p_0 = (\emptyset, \{\{1,2,\ldots,m\}\}, \emptyset) \in \mathcal{P}$. Then for $\{O_0\} = \pi^{-1}(p_0)$,

$$\dim O_0 = n(m-1) - 0 - 1 - 0 = \dim(\mathbb{P}^{n-1}\mathbb{K})^m.$$

(3) If $n+2 \leq m$, then for any $p \in \mathcal{P}$ and any $O \in \pi^{-1}(p)$, $\dim O < \dim(\mathbb{P}^{n-1}\mathbb{K})^m$.

Also, the infiniteness of orbits is obtained from Theorem 3.1.

Corollary 5.2.

$$\#\,\mathrm{diag}(GL_n)\backslash GL_n^m/P^m = \infty \Leftrightarrow m \geq 4.$$

This equivalence is already proved by Magyar-Weyman-Zelevinsky in [1], but by using Theorem 3.1, we can easily see what kind of infinitely many orbits occur and their topology explicitly.

References

[1] P. Magyar, J. Weyman, and A. Zelevinsky, Multiple flag varieties of finite type, *Adv. Math.* **141** (1999), pp. 97–118.

[2] T. Kobayashi and T. Oshima, Finite multiplicity theorems for induction and restriction, *Adv. Math.* **248** (2013), pp. 921–944.

[3] M. Brion, Classification des espaces homogenes spheriques, *Compositio Math.* **63** (1987), pp. 189–208.

[4] V. Kac, Infinite root systems, representations of graphs and invariant theory, *Invent. Math.* **56** (1980), pp. 57–92.

[5] N. Shimamoto, Description of orbits on multiple flag varieties of type A with infinite orbits, in preparation.