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INDEPENDENCE IN QUASIMINIMAL CLASSES

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1. Introduction

There are many interesting mathematical structures and classes of structures that cannot be formalised in a first order language, and sometimes the first order theory of a structure is not nice enough to be effectively studied. To give an easy example, the class of all groups where each element has finite order is not first order definable. Such structures and classes can often be studied using non-elementary approaches such as e.g. infinitary languages. The most general framework for studying non-elementary classes is that of *abstract elementary classes* (AECs). It was originally introduced to model theory by Saharon Shelah [9] in the 1980's. Shelah studied a class of structures $\mathcal{K}$ without specifying the language. Instead, he the defined the class in terms of a relation $\preccurlyeq$ between models, and gave axioms for $(\mathcal{K}, \preccurlyeq)$.

Model theoretic research, however, is still predominantly concerned with first order classes and structures. One of the reasons for this is that the powerful machinery of first order stability theory has made many applications possible. In the context of abstract elementary classes, we are only at the beginning of developing similar machinery. Here, we discuss work that is part of the wider project of developing stability theory in AEC contexts. We will develop a perfect independence calculus for one specific AEC setting. Our treatment is based on [4], and we will refer the reader there for omitted details.

One example of a structure that cannot be formalised in first order logic is the cover of an algebraically closed field (assuming that it has standard kernel, i.e. the kernel is isomorphic to the additive group of integers), see [3] for details.

Definition 1. *Let V be a vector space over $\mathbb{Q}$, and let F be an algebraically closed field of characteristic* 0. *A* cover *of the multiplicative group of F is a structure represented by an exact sequence*

$$0 \to K \to V \to F^* \to 1,$$

where the map $V \to F^*$ is a surjective group homomorphism from $(V, +)$ onto $(F^*, \cdot)$ with kernel K. We will call this map exp. If $K \cong (\mathbb{Z}, +)$, then we say that the cover has standard kernel. *We think of the cover as a structure V in the language $\mathcal{L} = \{0, +, f_q, R_+, R_0\}_{q \in \mathbb{Q}}$, where V consists of the elements in the vector space, 0 is a constant symbol denoting the zero element of the vector space V, $+$ is a binary function symbol denoting addition on V, and for each $q \in \mathbb{Q}$, f_q is a unary function symbol denoting scalar multiplication by the number q. The symbol R_+ is a ternary relation symbol interpreted so that $R_+(v_1, v_2, v_3)$ if and only if $exp(v_1) + exp(v_2) = exp(v_3)$, and R_0 is a binary relation symbol interpreted so that $R_0(v_1, v_2)$ if and only if $exp(v_1) + exp(v_2) = 0$.*

Note that for the cover, field multiplication is definable using vector space addition, and that we can express (for example) $exp(v_1) = exp(v_2)$ with the formula $\exists v_3(R_0(v_1, v_3) \wedge R_0(v_2, v_3))$.

In fact, the cover is an example of a *quasiminimal pregeometry structure* (we will give a precise definition in section 4). Quasiminimality is a non-elementary analogue for strong minimality. Simplifying a bit, we can say that it's what you get from a strongly minimal structure if you replace finiteness conditions with countability conditions. Quasiminimal pregeometry structures have been discussed in e.g. [1] and [7]. There is a natural way to construct an AEC from a quasiminimal pregeometry structure (see [1]), and an AEC that arises this way is called a *quasiminimal class* (again, we will give a definition in section 4).

We will present a theory of independence for quasiminimal classes. It turns out that there is an independence notion that has all the usual properties of non-forking (see Theorem 73 and Lemma 76). In [4], we used this independence calculus to prove a non-elementary analogue for Hrushovski's famous Group Configuration Theorem (see [8] for Hrushovski's theorem). This connects to a wider project of finding non-elementary analogues for Zariski geometries and showing that an analogue of Zilber's Trichotomy holds there (see [4], [6]).

We first introduce FUR-classes ("Finite U-Rank"), then show that they have a perfect theory of independence (Theorem 73) and that every quasiminimal class is a FUR-class (Theorem 76). The main reason for presenting FUR-classes is to show that quasiminimal classes have a perfect theory of independence. The class of all ω-saturated models of a first order ω-stable theory with finite Morley rank (with $\preccurlyeq$ the first order elementary submodel relation) provides one example of a FUR-class.

The basic idea is that we first define an auxiliary independence relation in terms of non-splitting and assume we are working in a class where this relation satisfies certain axioms. Our main independence notion, however, will be a different one and defined in terms of Lascar splitting (see Definition 42). We first prove some properties of non-splitting independence and then use them to show that the main independence notion actually has all the properties that we could hope to expect (Theorem 73).

However, since we will be working in the context of AECs, we will first give the basic notions related to them.

2. Abstract elementary classes

An abstract elementary class (AEC) is a generalisation of the class of models of a first order theory together with the first order elementary submodel relation.

Definition 2. *Let L be a countable language, let $\mathcal{K}$ be a class of L structures and let $\preccurlyeq$ be a binary relation on $\mathcal{K}$. We say $(\mathcal{K}, \preccurlyeq)$ is an* abstract elementary class *(AEC for short) if the following hold.*

(1) Both $\mathcal{K}$ and $\preccurlyeq$ are closed under isomorphisms.
(2) If $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ and $\mathcal{A} \preccurlyeq \mathcal{B}$, then $\mathcal{A}$ is a substructure of $\mathcal{B}$.
(3) The relation $\preccurlyeq$ is a partial order on $\mathcal{K}$.
(4) If δ is a cardinal and $\langle \mathcal{A}_i \,|\, i < \delta \rangle$ is an $\preccurlyeq$-increasing chain of structures, then
   a) $\bigcup_{i<\delta} \mathcal{A}_i \in \mathcal{K}$;
   b) for each $j < \delta$, $\mathcal{A}_j \preccurlyeq \bigcup_{i<\delta} \mathcal{A}_i$;
   c) if $\mathcal{B} \in \kappa$ and for each $i < \delta$, $\mathcal{A}_i \preccurlyeq \mathcal{B}$, then $\bigcup_{i<\delta} \mathcal{A}_i \preccurlyeq \mathcal{B}$.
(5) If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{K}$, $\mathcal{A} \preccurlyeq \mathcal{C}$, $\mathcal{B} \preccurlyeq \mathcal{C}$ and $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A} \preccurlyeq \mathcal{B}$.
(6) There is a Löwenheim-Skolem number $LS(\mathcal{K})$ such that if $\mathcal{A} \in \mathcal{K}$ and $B \subseteq \mathcal{A}$, then there is some structure $\mathcal{A}' \in \mathcal{K}$ such that $B \subseteq \mathcal{A}' \preccurlyeq \mathcal{A}$ and $|\mathcal{A}'| = |B| + LS(\mathcal{K})$.

If $\mathcal{A} \preccurlyeq \mathcal{B}$, we say that $\mathcal{A}$ is an elementary substructure *of $\mathcal{B}$.*

It is easy to see that the class $(\mathcal{K}, \preccurlyeq)$ of all models of some first order theory T, where $\preccurlyeq$ is interpreted as the elementary submodel relation, is an AEC. Moreover, the class of all groups where each element has finite order, equipped with the subgroup relation, is an AEC.

We also consider the following example, presented in [7].

Example 3. *Let $\mathcal{K}$ be the class of all models $M = (M, E)$ such that E is an equivalence relation on M with infinitely many classes, each of size $\aleph_0$. For any set X, we define the closure of X to be*

$$cl(X) = \bigcup \{x/E \mid x \in X\}.$$

We define $\preccurlyeq$ so that $\mathcal{A} \preccurlyeq \mathcal{B}$ if and only if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} = cl(\mathcal{A})$. Then, it is easy to see that $(\mathcal{K}, \preccurlyeq)$ is an AEC.

Next, we present the AEC counterparts for some common notions and techniques from first order model theory.

Definition 4. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{K}$. We say a function $f : \mathcal{A} \to \mathcal{B}$ is an* elementary embedding, *if there is some $\mathcal{C} \in \mathcal{K}$ such that $\mathcal{C} \preccurlyeq \mathcal{B}$ and f is an isomorphism from $\mathcal{A}$ to $\mathcal{C}$.*

In model theory, it is common to work in a very large, saturated and homogeneous model that is often called the *monster model.* In such a setting, we can assume that all the tuples we consider are from the monster model, all sets are subsets of the monster, and all models its submodels. In this framework, types can be seen as orbits of automorphisms of the monster. The usual construction of the monster model can be carried out in AEC context, assuming that the AEC has amalgamation property and joint embedding property (these properties are defined in a manner analoguous to the first order case, see definitions 7 and 8 below).

Definition 5. *Let $\mathbb{M} \in \mathcal{K}$, and let δ be a cardinal. We say $\mathbb{M}$ is δ-* model homogeneous *if whenever $\mathcal{A}, \mathcal{B} \preccurlyeq \mathbb{M}$ are such that $|\mathcal{A}|, |\mathcal{B}| < \delta$ and $f : \mathcal{A} \to \mathcal{B}$ is an isomorphism, there is some automorphism g of $\mathbb{M}$ such that $f \subseteq g$.*

Definition 6. *Let $\mathbb{M} \in \mathcal{K}$, and let δ be a cardinal. We say $\mathbb{M}$ is* δ-universal *if for every $\mathcal{A} \in \mathcal{K}$ such that $|\mathcal{A}| < \delta$ there is an elementary embedding $f : \mathcal{A} \to \mathbb{M}$.*

We note that if $\mathbb{M} \in \mathcal{K}$ is both δ- model homogeneous and δ-universal, then for any $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ such that $\mathcal{A} \preccurlyeq \mathcal{B}$ and $|\mathcal{B}| < \delta$, and any elementary embedding $f : \mathcal{A} \to \mathbb{M}$, there is an elementary embedding $g : \mathcal{B} \to \mathbb{M}$ such that $f \subseteq g$.

It follows that if all the structures we are considering are small compared to some cardinal δ and our class $\mathcal{K}$ contains a structure $\mathbb{M}$ of size δ that is both δ- model homogeneous and δ-universal, we can view all the other structures we are considering as elementary substructures of $\mathbb{M}$. If the class $\mathcal{K}$ has the amalgamation property and joint embedding property and contains arbitrarily large structures, then we can use the construction by Jónsson and Fraïssé [2] to build a δ- model homogeneous and δ-universal monster model $\mathbb{M} \in \mathcal{K}$ of size δ for arbitrarily large δ.

Definition 7. *We say a class of structures $\mathcal{K}$ has the* amalgamation property *(AP for short) if for all $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ and any map $f : \mathcal{A} \to \mathcal{B}$ such that $f : \mathcal{A}' \to \mathcal{B}$ is an elementary embedding for some $\mathcal{A}' \preccurlyeq \mathcal{A}$, there exists some $\mathcal{C} \in \mathcal{K}$ such that $\mathcal{B} \subseteq \mathcal{C}$ and an elementary embedding $g : \mathcal{A} \to \mathcal{C}$ such that $f \subseteq g$.*

Definition 8. *We say a class of structures $\mathcal{K}$ has the* joint embedding property *(JEP for short) if for all $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, there is some $\mathcal{C} \in \mathcal{K}$ such that $\mathcal{B} \preccurlyeq \mathcal{C}$ and an elementary embedding $f : \mathcal{A} \to \mathcal{C}$.*

We will assume that δ is a cardinal bigger than any structure we will be considering and call a δ- model homogeneous and δ-universal structure $\mathbb{M} \in \mathcal{K}$ of size δ a *monster model* for $\mathcal{K}$. Then, we may think we are always working inside the monster model $\mathbb{M}$. In practice, this means that every structure we will be considering will be an elementary substructure of $\mathbb{M}$ of cardinality less than δ, every set we will be considering will be a subset of $\mathbb{M}$ of cardinality less than δ, and every tuple we will be considering will be a tuple of elements of $\mathbb{M}$.

For an easy example of a monster model, consider the class of Example 3. There, all closed models of the same cardinality are isomorphic. It is easy to see that the class has AP, JEP and arbitrarily large models. For a monster model, one can just choose any closed structure that is large enough.

In the AEC setting, Galois types, defined as orbits of automorphisms, provide a natural analogue for first order types. However, in this presentation, we will use weak types as our main notion of type.

Definition 9. *Suppose $A \subset \mathbb{M}$. We denote by $Aut(\mathbb{M}/A)$ the subgroup of the automorphism group of $\mathbb{M}$ consisting of those automorphisms f that satisfy $f(a) = a$ for each $a \in A$.*

We say that a and b have the same Galois type *over A if there is some $f \in Aut(\mathbb{M}/A)$ such that $f(a) = b$. We write $t^g(a/A) = t^g(a/A;\mathbb{M})$ for the Galois-type of a over A.*

We say that a and b have the same weak type *over A if for all finite subsets $B \subseteq A$, it holds that $t^g(a/B) = t^g(b/B)$. We write $t(a/A)$ for the weak type of a over A.*

3. FUR-classes

In this section, we introduce FUR-classes and study the properties of non-splitting independence there. The main reason for introducing the notion is to show that quasiminimal classes (see section 4) have a perfect theory of independence. Indeed, we will use the properties of non-splitting independence to show that in FUR-classes, an independence notion obtained from Lascar splitting (see Definition 42) has all the usual properties of non-forking (Theorem 73). It will then turn out that every quasiminimal class is a FUR-class (Lemma 76).

We first define independence in terms of non-splitting.

Definition 10. *Let A and B be sets such that $A \subseteq B$ and A is finite. We say that $t(a/B)$* splits *over A if there are $b, c \in B$ such that $t(b/A) = t(c/A)$ but $t(ab/A) \neq t(ac/A)$.*

We write $a \downarrow^{ns}_B C$ if there is some finite $A \subseteq B$ such that $t(a/B \cup C)$ does not split over A. By $A \downarrow^{ns}_B C$ we mean that $a \downarrow^{ns}_B C$ for each $a \in A$.

We now proceed to defining a FUR-class using six axioms, AI-AVI. We will look at the properties that non-splitting independence has under these axioms. We will see that types over models have unique free extensions, that symmetry and transitivity hold over models, that the setting is ω-stable (in the sense of AECs), that weak types over models determine Galois types, and that there are no infinite splitting chains of models. We then define U-ranks over models and finite sets in terms of non-splitting (Definition 29), and show that U-rank is preserved in free extensions over models.

For the sake of readability, instead of first presenting all the definitions needed and then giving the axioms for a FUR-class in the form of a simple list, we will start listing the axioms and give the related definitions, lemmas and remarks in midst of them.

The reader can check that the class of ω-saturated models of an ω-stable first order theory with finite Morley rank satisfies the axioms. It is also easy to see that the class of Example 3 satisfies the axioms (details can be found in [5]).

The first axiom states that models are $\aleph_0$- Galois saturated.

AI: Every countable model $\mathcal{A} \in \mathcal{K}$ is s-saturated, i.e. for any $b \in \mathbb{M}$ and any finite $A \subseteq \mathcal{A}$, there is $a \in \mathcal{A}$ such that $t(a/A) = t(b/A)$.

We now apply AI to show that for non-splitting independence, free extensions of types over models are unique.

Lemma 11. *Let $\mathcal{B}$ be a model. If $a \downarrow^{ns}_{\mathcal{B}} A$, $b \downarrow^{ns}_{\mathcal{B}} A$ and $t(a/\mathcal{B}) = t(b/\mathcal{B})$, then $t(a/A) = t(b/A)$.*

Proof. Let $c \in A$ be arbitrary. We show that $t(ac/\emptyset) = t(bc/\emptyset)$. Let $B \subset \mathcal{B}$ be a finite set such that neither $t(a/\mathcal{B} \cup A)$ nor $t(b/\mathcal{B} \cup A)$ splits over B. By AI, there is some $d \in \mathcal{B}$ such that $t(d/B) = t(c/B)$. We have

$$t(ac/\emptyset) = t(ad/\emptyset) = t(bd/\emptyset) = t(bc/\emptyset).$$

□

Lemma 12. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are countable models, $t(a/\mathcal{A})$ does not split over some finite $A \subseteq \mathcal{A}$, and $\mathcal{A} \subseteq \mathcal{B}$. Then there is some b such that $t(b/\mathcal{A}) = t(a/\mathcal{A})$ and $b \downarrow^{ns}_{A} \mathcal{B}$.*

Proof. As both $\mathcal{A}$ and $\mathcal{B}$ are countable and contain A, we can, using AI and back-and-forth methods, construct an automorphism $f \in Aut(\mathbb{M}/A)$ such that $f(\mathcal{A}) = \mathcal{B}$. Choose $b = f(a)$. □

Axioms AII and AIII together will guarantee that unique prime models exist. To be able to state AII, we need the notion of s-primary models.

Definition 13. *We say that a model $\mathcal{B} = \mathcal{A}a \cup \bigcup_{i<\omega} a_i$, where a_i is a singleton for each i, is* s-primary *over $\mathcal{A}a$ if for all $n < \omega$, there is a finite $A_n \subset \mathcal{A}$ such that for all $(a', a'_0, \ldots, a'_n) \in \mathbb{M}$ such that $t(a'/\mathcal{A}) = t(a/\mathcal{A})$, $t(a', a'_0, \ldots, a'_n/A_n) = t(a, a_0, \ldots, a_n/A_n)$ implies $t(a', a'_0, \ldots, a'_n/\mathcal{A}) = t(a, a_0, \ldots, a_n/\mathcal{A})$*

AII: For all a and countable $\mathcal{A}$, there is an s-primary model $\mathcal{B} = \mathcal{A}a \cup \bigcup_{i<\omega} a_i$ ($\leq \mathbb{M}$) over $\mathcal{A}a$.

We denote a countable s-primary model $\mathcal{B} = \mathcal{A}a \cup \bigcup_{i<\omega} a_i$ over $\mathcal{A}a$ that is as above by $\mathcal{A}[a]$.

We will use AII to show that weak types over countable models determine Galois types. For this, we need the following lemma.

Lemma 14. *Let $\mathcal{A}$ be a countable model, and let $t(b/\mathcal{A}) = t(a/\mathcal{A})$. Then, there is an isomorphism $f : \mathcal{A}[a] \to \mathcal{A}[b]$ such that $f \restriction \mathcal{A} = id$ and $f(a) = b$.*

Proof. Let $\mathcal{A}[a] = \mathcal{A}a \cup \bigcup_{i<\omega} a_i$ and $\mathcal{A}[b] = \mathcal{A}b \cup \bigcup_{i<\omega} b_i$. Now there is some finite $A_0 \subset \mathcal{A}$ such that it holds for any a', a'_0 that if $t(a/\mathcal{A}) = t(a'/\mathcal{A})$ and $t(a', a'_0/A_0) = t(a, a_0/A_0)$, then $t(a, a_0/\mathcal{A}) = t(a', a'_0/\mathcal{A})$. As $t(b/\mathcal{A}) = t(a/\mathcal{A})$, there is an automorphism $F \in \mathrm{Aut}(\mathbb{M}/\mathcal{A})$ such that $F(a) = b$. Let $a'_0 = F(a_0)$. By AI, there is some i such that $t(b_i/A_0 b) = t(a'_0/A_0 b)$, and in particular $t(b_i, b/A_0) = t(a_0, a/A_0)$. Thus, $t(b_i, b/\mathcal{A}) = t(a_0, a/\mathcal{A})$. Now we can construct f using back and forth methods. □

Corollary 15. *If $\mathcal{A}$ is a countable model, then $t(a/\mathcal{A})$ determines $t^g(a/\mathcal{A})$.*

Definition 16. *We say a* dominates *B over A if the following holds for all C: If there is a finite $A_0 \subseteq A$ such that $t(a/AC)$ does not split over A_0, then $B \downarrow^{ns}_{A} C$.*

Lemma 17. *If $\mathcal{A}$ is a countable model, then the element a dominates $\mathcal{A}[a]$ over $\mathcal{A}$.*

Proof. See [4]. □

To be able to state AIII, we need the notion of a prime model. It is defined in terms of weakly elementary maps.

Definition 18. *Let α be a cardinal and $\mathcal{A}_i \preccurlyeq \mathbb{M}$ for $i < \alpha$, and let $A = \bigcup_{i<\alpha} \mathcal{A}_i$. We say that $f : A \to \mathbb{M}$ is* weakly elementary *with respect to the sequence $(\mathcal{A}_i)_{i<\alpha}$ if for all $a \in A$, $t(a/\emptyset) = t(f(a)/\emptyset)$ and for all $i < \alpha$, $f(\mathcal{A}_i) \preccurlyeq \mathbb{M}$.*

Definition 19. *We say a model $\mathcal{A}$ is s-*prime *over $A = \bigcup_{i<\alpha} \mathcal{A}_i$, where α is a cardinal and $\mathcal{A}_i$ is a model for each i, if for every model $\mathcal{B}$ and every map $f : A \to \mathcal{B}$ that is weakly elementary with respect to $(\mathcal{A}_i)_{i<\alpha}$, there is an elementary embedding $g : \mathcal{A} \to \mathcal{B}$ such that $f \subseteq g$.*

AIII: Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be models. If $\mathcal{A} \downarrow^{ns}_{\mathcal{B}} \mathcal{C}$ and $\mathcal{B} = \mathcal{A} \cap \mathcal{C}$, then there is a unique (not only up to isomorphism) s-prime model $\mathcal{D}$ over $\mathcal{A} \cup \mathcal{C}$. Furthermore, if $\mathcal{C}'$ is such that $\mathcal{C} \subseteq \mathcal{C}'$ and $\mathcal{A} \downarrow_{\mathcal{B}} \mathcal{C}'$, then $\mathcal{D} \downarrow_{\mathcal{C}} \mathcal{C}'$.

It follows that if also $\mathcal{A}', \mathcal{B}', \mathcal{C}'$ and $\mathcal{D}'$ are as in AIII, $f : \mathcal{A} \to \mathcal{A}'$ and $g : \mathcal{C} \to \mathcal{C}'$ are isomorphisms and $f \restriction \mathcal{B} = g \restriction \mathcal{B}$, then there is an isomorphism $h : \mathcal{D} \to \mathcal{D}'$ such that $f \cup g \subseteq h$.

The axiom AIV is given in terms of player II having a winning strategy in the game $GI(a, A, \mathcal{A})$, defined below. The motivation behind this is to be able to show that there are no infinite splitting chains of models (Lemma 28) so that a well behaving notion of rank can be developed. In the beginning of the game, $t(a/A)$ is considered. The idea is that player I has to show that $t(a/A)$ does not imply $t(a/\mathcal{A})$, and player II tries to isolate $t(a/\mathcal{A})$ by enlarging the set A. If the game is played in a setting with a well behaving rank, player II will be able to win in finitely many moves since there are no infinite descending chains of ordinals.

Definition 20. *Let $\mathcal{A}$ be a model, $A \subseteq \mathcal{A}$ finite and $a \in \mathbb{M}$. The game $GI(a, A, \mathcal{A})$ is played as follows: The game starts at the position $a_0 = a$ and $A_0 = A$. At each move n, player I first chooses $a_{n+1} \in \mathbb{M}$ and a finite subset $A'_{n+1} \subseteq \mathcal{A}$ such that $t(a_{n+1}/A_n) = t(a_n/A_n)$, $A_n \subseteq A'_{n+1}$ and $t(a_{n+1}/A'_{n+1}) \neq t(a_n/A'_{n+1})$. Then player II chooses a finite subset $A_{n+1} \subseteq \mathcal{A}$ such that $A'_{n+1} \subseteq A_{n+1}$. Player II wins if player I can no longer make a move.*

Axiom AIV now states that for every tuple, there is some finite number n so that player II has a winning strategy in n moves.

AIV: For each $a \in \mathbb{M}$, there is a number $n < \omega$ such that for any countable model $\mathcal{A}$ and any finite subset $A \subset \mathcal{A}$, player II has a winning strategy in $GI(a, A, \mathcal{A})$ in n moves.

As an example, consider a model class of a first order ω-stable theory. There, types have only finitely many free extensions, so player II can always enlarge the set A'_n to some A_n such that $t(a_n/A_n)$ has a unique free extension. After this, player I has no choice but to play some a_{n+1} and A'_{n+1} so that $MR(a_{n+1}/A_{n+1}) < MR(a_n/A_n)$ (here, MR stands for Morley rank). Hence, AIV is satisfied (with $n = MR(a/A)$).

We now apply AIV to prove that any tuple is free over a model from the model itself and that the number of weak types over a model equals the cardinality of the model.

Lemma 21. *Let $a \in \mathbb{M}$ be arbitrary, and let $\mathcal{A}$ be a model. Then, $a \downarrow^{ns}_{\mathcal{A}} \mathcal{A}$.*

Proof. It suffices to show that there is a finite $A \subseteq \mathcal{A}$ such that $t(a/\mathcal{A})$ does not split over A. Suppose not. Assume first that $\mathcal{A}$ is countable. We claim that then player I can survive ω moves in $GI(a, A, \mathcal{A})$ for any finite subset $A \subset \mathcal{A}$, which contradicts AIV. Suppose we are at move n and that $t(a_n/\mathcal{A})$ splits over every finite subset of $\mathcal{A}$ containing A_n. In particular, it splits over A_n. Let b, c be tuples witnessing this splitting. Let $f \in \mathrm{Aut}(\mathbb{M}/A_n)$ be such that $f(b) = c$ and $f(\mathcal{A}) = \mathcal{A}$. Now player I chooses $a_{n+1} = f(a_n)$ and $A_{n+1} = A_n \cup \{c\}$. Then, $t(a_n/A_n) = t(a_{n+1}/A_n)$ but $t(a_{n+1}c/A_n) = t(a_n b/A_n) \neq t(a_n c/A_n)$ and thus $t(a_{n+1}/A_{n+1}) \neq t(a_n/A_{n+1})$. As $t(a_n/\mathcal{A})$ splits over every finite subset of $\mathcal{A}$ containing A_n, the same is true for $t(a_{n+1}/\mathcal{A})$.

Let now $\mathcal{A}$ be arbitrary and suppose that $t(a/\mathcal{A})$ splits over every finite $A \subset \mathcal{A}$. Let $\mathcal{B}$ be a countable submodel of $\mathcal{A}$. Then, $\mathcal{B}$ contains only countably many finite subsets. For each finite $B \subset \mathcal{B}$, we find some tuples $b, c \in \mathcal{A}$ witnessing the splitting of $t(a/\mathcal{A})$ over B. We now enlarge $\mathcal{B}$ into a countable submodel of $\mathcal{A}$ containing all these tuples. After repeating the process ω many times we have obtained a countable counterexample. □

Lemma 22. *For all models $\mathcal{A}$, the number of weak types $t(a/\mathcal{A})$ for $a \in \mathbb{M}$, is $|\mathcal{A}|$.*

Proof. We prove this first for countable models. Suppose, for the sake of contradiction, that there is a countable model $\mathcal{A}$ and elements $a_i \in \mathbb{M}$, $i < \omega_1$ so that $t(a_i/\mathcal{A}) \neq t(a_j/\mathcal{A})$ if $i \neq j$. As countable models are s-saturated, there are only countably many types over a finite set. In particular, by the pigeonhole principle, we find an uncountable set $J \subseteq \omega_1$ so that $t(a_i/\emptyset)$ is constant for $i \in J$. After relabeling, we may set $J = \omega_1$. For each i, there is a number $n < \omega$ such that player II wins $GI(a_i, \emptyset, \mathcal{A})$ in n moves. Using again the pigeonhole principle, we may assume that the number n is constant for all $i < \omega_1$.

Now we start playing $GI(a_i, \emptyset, \mathcal{A})$ simultaneously for all $i < \omega_1$. Since the a_i have different weak types over $\mathcal{A}$, for each i of the form $i = 2\alpha$ for some $\alpha < \omega_1$, we can find a finite set $A_\alpha \subset \mathcal{A}$ such that $t(a_{2\alpha}/A_\alpha) \neq t(a_{2\alpha+1}/A_\alpha)$. We write $A_0^i = A_\alpha$ for $i = 2\alpha$ and $i = 2\alpha + 1$. As there are only countably many finite subsets of $\mathcal{A}$, we find an uncountable $I \subseteq \omega_1$ so that for all $i \in I$, $A_0^i = A$ for some fixed, finite $A \subset \mathcal{A}$. In $GI(a_i, \emptyset, \mathcal{A})$ for $i \in I$, on his first move player I plays $a_{2\alpha+1}$ and A if $i = 2\alpha$ for some $\alpha < \omega_1$, and $a_{2\alpha}$ and A if $i = 2\alpha + 1$ for some $\alpha < \omega_1$. All the rest of the games he gives up. Now, in each game $GI(a_i, \emptyset, \mathcal{A})$ player II plays some finite $A_1^i \subset \mathcal{A}$ such that $A \subseteq A_1^i$. Again, there is an uncountable $I_1' \subseteq I$ such that for $i \in I_1'$, we have $A_1^i = A_1$ for some fixed, finite A_1. As there are only countably many types over A_1, we find an uncountable $I_1 \subset I_1'$ so that $t(a_i/A_1) = t(a_j/A_1)$ for all $i, j \in I_1$. Again, player I gives up on all the games except for those indexed by elements of I_1. Continuing like this, he can survive more than n moves in uncountably many games. This contradicts AIV.

Suppose now $\mathcal{A}$ is arbitrary. Denote $X = \mathcal{P}_{<\omega}(\mathcal{A})$. Then, $|X| = |\mathcal{A}|$. For each $A \in X$, choose a countable model $\mathcal{A}_A \preccurlyeq \mathcal{A}$ such that $A \subset \mathcal{A}_A$. By Lemma 21, for each weak type $p = t(a/\mathcal{A})$, there is some $A_p \in X$ so that $a \downarrow^{ns}_{A_p} \mathcal{A}$, and hence also $a \downarrow^{ns}_{\mathcal{A}_{A_p}} \mathcal{A}$. By Lemma 11, $t(a/\mathcal{A}_{A_p})$ determines $t(a/\mathcal{A})$ uniquely. As there are only countably many types over countable models, the number of weak types over $\mathcal{A}$ is

$$|X| \cdot \omega = |\mathcal{A}|.$$

□

Lemma 23. *For any $a \in \mathbb{M}$ and any model $\mathcal{A}$, the weak type $t(a/\mathcal{A})$ determines the Galois type $t^g(a/\mathcal{A})$.*

Proof. Suppose $t(a/\mathcal{A}) = t(b/\mathcal{A})$. By Lemma 21, we can find a countable submodel $\mathcal{B}$ of $\mathcal{A}$ so that $a \downarrow^{ns}_{\mathcal{B}} \mathcal{A}$ and $b \downarrow^{ns}_{\mathcal{B}} \mathcal{A}$. By Lemma 14, there is some $f \in \mathrm{Aut}(\mathbb{M}/\mathcal{B})$ such that $f(\mathcal{B}[a]) = \mathcal{B}[b]$ and $f(a) = b$. Moreover, by Lemma 17, $\mathcal{B}[a] \downarrow^{ns}_{\mathcal{B}} \mathcal{A}$ and $\mathcal{B}[b] \downarrow^{ns}_{\mathcal{B}} \mathcal{A}$. Now the map $g = (f \restriction \mathcal{B}[a]) \cup \mathrm{id}_{\mathcal{A}}$ is weakly elementary. For this, it suffices to show that $t(c/\mathcal{A}) = t(f(c)/\mathcal{A})$ for every $c \in \mathcal{B}[a]$. But $t(c/\mathcal{B}) = t(f(c)/\mathcal{B})$, $c \downarrow^{ns}_{\mathcal{B}} \mathcal{A}$, and $f(c) \downarrow^{ns}_{\mathcal{B}} \mathcal{A}$. Thus, by Lemma 11, $t(c/\mathcal{A}) = t(f(c)/\mathcal{A})$.

By AIII, there are unique s-prime models $\mathcal{D}_a$ and $\mathcal{D}_b$, over $\mathcal{B}[a] \cup \mathcal{A}$ and $\mathcal{B}[b] \cup \mathcal{A}$, respectively. The map g extends to an automorphism $h \in \mathrm{Aut}(\mathbb{M}/\mathcal{A})$ so that $h(\mathcal{D}_a) \subseteq \mathcal{D}_b$. The s-prime models are unique and preserved by automorphisms, thus we must have $h(\mathcal{D}_a) = \mathcal{D}_b$. Since $h(a) = b$, we have $t^g(a/\mathcal{A}) = t^g(b/\mathcal{A})$. □

Note that it follows from lemmas 21 and 23 that the class $\mathcal{K}$ is ω-stable (in the sense of AECs).

Axiom AV is a weak form of symmetry (over models). We will use it to show that symmetry actually holds over models.

AV: If $\mathcal{A}$ and $\mathcal{B}$ are countable models, $\mathcal{A} \subseteq \mathcal{B}$ and $a \in \mathbb{M}$, and $\mathcal{B} \downarrow^{ns}_{\mathcal{A}} a$, then $a \downarrow^{ns}_{\mathcal{A}} \mathcal{B}$.

Lemma 24. *Let $A, C \subseteq \mathbb{M}$ and let $\mathcal{B} \subseteq A \cap C$ be a model. If $A \downarrow^{ns}_{\mathcal{B}} C$, then $C \downarrow^{ns}_{\mathcal{B}} A$.*

Proof. We note first that for any finite tuples $a, c \in \mathbb{M}$, and for any countable model $\mathcal{B}$ it holds that if $a \downarrow^{ns}_{\mathcal{B}} c$, then $c \downarrow^{ns}_{\mathcal{B}} a$. Indeed, then by dominance in s-primary models, it holds that $\mathcal{B}[a] \downarrow^{ns}_{\mathcal{B}} c$, and thus by AV, $c \downarrow^{ns}_{\mathcal{B}} \mathcal{B}[a]$, and in particular, $c \downarrow^{ns}_{\mathcal{B}} a$.

Let now $\mathcal{B}$ be arbitrary, and suppose $a \downarrow^{ns}_{\mathcal{B}} c$ but $c \not\downarrow^{ns}_{\mathcal{B}} a$. Then, there is some finite $B \subset \mathcal{B}$ so that $t(a/\mathcal{B}c)$ does not split over B. However, $t(c/\mathcal{B}a)$ splits over B. Let $b, d \in \mathcal{B}a$ be tuples witnessing this. If $\mathcal{B}' \preccurlyeq \mathcal{B}$ is a countable model containing B, $b \cap \mathcal{B}$ and $d \cap \mathcal{B}$, then $a \downarrow^{ns}_{\mathcal{B}'} c$ but $c \not\downarrow^{ns}_{\mathcal{B}'} a$, which contradicts what we have just proved.

Suppose now $A \downarrow^{ns}_{\mathcal{B}} C$ but $C \not\downarrow^{ns}_{\mathcal{B}} A$. Then, there is some $c \in C$ so that $c \not\downarrow^{ns}_{\mathcal{B}} A$, and this is witnessed by some finite $a \in A$, i.e. $c \not\downarrow^{ns}_{\mathcal{B}} a$. But we have $a \downarrow^{ns}_{\mathcal{B}} C$ and hence $a \downarrow^{ns}_{\mathcal{B}} c$, a contradiction. □

Remark 25. *Note that from Lemma 24 it follows that for any $a, b \in \mathbb{M}$ and any model $\mathcal{A}$, it holds that $a \downarrow^{ns}_{\mathcal{A}} b$ if and only if $b \downarrow^{ns}_{\mathcal{A}} a$.*

Axiom AVI states the existence of free extensions.

AVI: For all models $\mathcal{A}, \mathcal{B}$ and $\mathcal{D}$ such that $\mathcal{A} \subseteq \mathcal{B} \cap \mathcal{D}$, there is a model $\mathcal{C}$ such that $t(\mathcal{C}/\mathcal{A}) = t(\mathcal{B}/\mathcal{A})$ and $\mathcal{C} \downarrow^{ns}_{\mathcal{A}} \mathcal{D}$.

It follows that AVI holds also without the assumption that $\mathcal{B}$ and $\mathcal{D}$ are models, as we can always find models extending these sets.

We now show that a form of transitivity holds for non-splitting independence.

Lemma 26. *If $\mathcal{B}$ is a model, $A \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq C$, then $a \downarrow^{ns}_{A} C$ if and only if $a \downarrow^{ns}_{A} \mathcal{B}$ and $a \downarrow^{ns}_{\mathcal{B}} C$.*

Proof. If $a \downarrow^{ns}_{A} C$, then $a \downarrow^{ns}_{A} \mathcal{B}$ and $a \downarrow^{ns}_{\mathcal{B}} C$ follow by monotonicity.

Suppose now $a \downarrow^{ns}_{A} \mathcal{B}$ and $a \downarrow^{ns}_{\mathcal{B}} C$. Let $A_0 \subset A$ and $B_0 \subset \mathcal{B}$ be finite sets so that $A_0 \subseteq B_0$, $t(a/\mathcal{B})$ does not split over A_0 and $t(a/C)$ does not split over B_0. Suppose $a \not\downarrow^{ns}_{A} C$. Then, $t(a/C)$ splits over A_0. Let $b, c \in C$ witness the splitting, i.e. $t(b/A_0) = t(c/A_0)$ but $t(ab/A_0) \neq t(ac/A_0)$. By AI, there are $b', c' \in \mathcal{B}$ so that $t(b'/B_0) = t(b/B_0)$ and $t(c'/B_0) = t(c/B_0)$. Since $t(a/C)$ does not split over B_0, we have $t(ab'/B_0) = t(ab/B_0)$ and $t(ac'/B_0) = t(ac/B_0)$. Thus,

$$t(ab'/A_0) = t(ab/A_0) \neq t(ac/A_0) = t(ac'/A_0),$$

a contradiction since $t(a/\mathcal{B})$ does not split over A_0. □

Next, we prove a stronger version of Lemma 12.

Lemma 27. *Suppose $\mathcal{A}$ is a model, $t(a/\mathcal{A})$ does not split over some finite $A \subset \mathcal{A}$ and B is such that $\mathcal{A} \subseteq B$. Then, there is some b such that $t(b/\mathcal{A}) = t(a/\mathcal{A})$ and $b \downarrow^{ns}_{A} B$.*

Proof. Let $\mathcal{B}$ be a model such that $B \subseteq \mathcal{B}$. Let $\mathcal{C}$ be a model containing $\mathcal{A}a$. By AVI, there is a model $\mathcal{C}'$ such that $t(\mathcal{C}/\mathcal{A}) = t(\mathcal{C}'/\mathcal{A})$ and $\mathcal{C}' \downarrow^{ns}_{\mathcal{A}} \mathcal{B}$. In particular, there is some $b \in \mathcal{C}'$ such that $t(b/\mathcal{A}) = t(a/\mathcal{A})$ and $b \downarrow^{ns}_{\mathcal{A}} \mathcal{B}$. Let $A' \subseteq \mathcal{A}$ be a finite set such that $A \subseteq A'$ and $b \downarrow^{ns}_{A'} B$. Then, by Lemma 26, $b \downarrow^{ns}_{A} B$. □

We now apply AIV to show that there are no infinite descending chains of models. This will guarantee that U-ranks (see Definition 29) will be finite.

Lemma 28. *For all $a \in \mathbb{M}$, there is a number $n < \omega$ such that there are no models $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq ... \subseteq \mathcal{A}_n$ so that for all $i < n$, $a \not\downarrow^{ns}_{\mathcal{A}_i} \mathcal{A}_{i+1}$.*

Proof. Suppose models $\mathcal{A}_i$, $i \leq n$, as in the statement of the lemma, exist. We note first that if $a \not\downarrow^{ns}_{\mathcal{A}_i} \mathcal{A}_{i+1}$, then $a \not\downarrow^{ns}_{\mathcal{A}'_i} \mathcal{A}_{i+1}$ for every countable submodel $\mathcal{A}_{i'} \subset \mathcal{A}_i$. Since a countable set only has countably many finite subsets, all the tuples witnessing the splittings are contained in some countable submodel $\mathcal{A}'_{i+1} \subset \mathcal{A}_{i+1}$. Then, $a \not\downarrow_{\mathcal{A}'_i} \mathcal{A}'_{i+1}$. Thus, we may assume each $\mathcal{A}_i$ is countable. We will show that player I can survive n moves in $GI(a, \emptyset, \mathcal{A}_0)$. Then, the lemma will follow from AIV.

On the first move, player I chooses some finite $B_1 \subset \mathcal{A}_0$ so that $t(a/\mathcal{A}_0)$ does not split over B_1. Then, there is some finite set $C_1 \subset \mathcal{A}_1$ so that $B_1 \subseteq C_1$ and $t(a/C_1)$ splits over B_1 and some $f_1 \in \text{Aut}(\mathbb{M}/B_1)$ such that $f(\mathcal{A}_1) = \mathcal{A}_0$. Now player I plays $a_1 = f(a)$ and $A'_1 = f_1(C_1)$. As $t(a/f_1(C_1))$ does not split over B_1 and $t(f_1(a)/f_1(C_1))$ splits over B_1, we have $t(a/f_1(C_1)) \neq t(f_1(a)/f_1(C_1))$, and this is indeed a legitimate move.

On her move, player II chooses some finite $A_1 \subset \mathcal{A}_0$ such that $A'_1 \subseteq A_1$. On his second move, player I chooses some finite $B_2 \subset \mathcal{A}_0 = f_1(\mathcal{A}_1)$ so that $A_1 \subset B_2$ and $t(a_1/\mathcal{A}_0)$ does not split over B_2. Now there is some finite set $C_2 \subset f_1(\mathcal{A}_2)$ so that $t(a_1/C_2)$ splits over B_2 and some automorphism $f_2 \in \text{Aut}(\mathbb{M}/B_2)$ so that $f_2(f_1(\mathcal{A}_2)) = \mathcal{A}_0$. Player I plays $a_2 = f_2(a_1)$ and $A'_2 = f_2(C_2)$. Continuing in this manner, he can survive n many moves. □

We now define U-ranks over models and finite sets.

Definition 29. *For a and a model $\mathcal{A}$, we define the* U-rank *of a over $\mathcal{A}$, denoted $U(a/\mathcal{A})$, as follows:*

- *$U(a/\mathcal{A}) \geq 0$ always;*
- *$U(a/\mathcal{A}) \geq n+1$ if there is some model $\mathcal{B}$ so that $\mathcal{A} \subseteq \mathcal{B}$, $a \not\downarrow^{ns}_{\mathcal{A}} \mathcal{B}$ and $U(a/\mathcal{B}) \geq n$;*
- *$U(a/\mathcal{A})$ is the largest n such that $U(a/\mathcal{A}) \geq n$.*

For finite A we write $U(a/A)$ for $max(\{U(a/\mathcal{A}) \mid \mathcal{A}$ is a model s.t. $A \subset \mathcal{A}\})$.
Note that by Lemma 28, $U(a/A)$ is finite for finite A.

Later, we will define U-ranks over arbitrary sets, and it will turn out that they are always finite. Thus, we call a class that satisfies the axioms AI-AVI a *FUR-class*, for "Finite U-Rank". Eventually, we will show that FUR-classes have a perfect theory of independence (Theorem 73).

Definition 30. *We say that an abstract elementary class $\mathcal{K}$ is a FUR-class if $\mathcal{K}$ has AP and JEP, $LS(\mathcal{K}) = \omega$, $\mathcal{K}$ has arbitrarily large structures and does not contain finite models, and $\mathcal{K}$ satisfies the axioms AI-AVI.*

As the last result of this section, we show that non-splitting over models can be expressed in terms of preserving U-ranks.

Lemma 31. *Let $\mathcal{A} \subseteq \mathcal{B}$ be models. Then $a \downarrow^{ns}_{\mathcal{A}} \mathcal{B}$ if and only if $U(a/\mathcal{B}) = U(a/\mathcal{A})$.*

Proof. From right to left the claim follows from the definition of U-rank.

For the other direction, suppose $a \downarrow^{ns}_{\mathcal{A}} \mathcal{B}$. It follows from the definition of U-rank that $U(a/\mathcal{B}) \leq U(a/\mathcal{A})$. We will prove $U(a/\mathcal{A}) \leq U(a/\mathcal{B})$.

Let $n = U(a/\mathcal{A})$, and choose models $\mathcal{A}'_i$, $i \leq n$ so that $\mathcal{A}'_0 = \mathcal{A}$ and for each $i < n$, $\mathcal{A}'_i \subseteq \mathcal{A}'_{i+1}$ and $a \not\downarrow^{ns}_{\mathcal{A}'_i} \mathcal{A}'_{i+1}$. Choose a model $\mathcal{C}$ so that $\mathcal{A}'_n a \subseteq \mathcal{C}$. By AVI, there is a model $\mathcal{B}'$ so that $t(\mathcal{B}'/\mathcal{A}) = t(\mathcal{B}/\mathcal{A})$ and $\mathcal{B}' \downarrow^{ns}_{\mathcal{A}} \mathcal{C}$. Let $f \in \mathrm{Aut}(\mathbb{M}/\mathcal{A})$ be such that $f(\mathcal{B}') = \mathcal{B}$. Denote $f(a) = b$ and $f(\mathcal{A}'_i) = \mathcal{A}_i$ for $i \leq n$. Then, $\mathcal{A}_0 = \mathcal{A}$, $t(b/\mathcal{A}) = t(a/\mathcal{A})$ and $b \not\downarrow^{ns}_{\mathcal{A}_i} \mathcal{A}_{i+1}$ for all $i < n$, and $\mathcal{B} \downarrow^{ns}_{\mathcal{A}} \mathcal{A}_n b$.

Let $\mathcal{B}_1$ be the unique s-prime model over $\mathcal{B} \cup \mathcal{A}_1$ (It exists by AIII since $\mathcal{A} \subseteq \mathcal{B} \cap \mathcal{A}_1$ and $\mathcal{B} \downarrow^{ns}_{\mathcal{A}} \mathcal{A}_1$). Suppose now that for $1 \leq i < n$, $\mathcal{B}_{i-1} \downarrow^{ns}_{\mathcal{A}_{i-1}} \mathcal{A}_i$, and that we have defined $\mathcal{B}_i$ as the unique s-prime model over $\mathcal{B}_{i-1} \cup \mathcal{A}_i$ (taking $\mathcal{B}_0 = \mathcal{B}$). Then, we let $\mathcal{B}_{i+1}$ be the unique s-prime model over $\mathcal{B}_i \cup \mathcal{A}_{i+1}$. It exists, since from the "Furthermore" part in AIII it follows that $\mathcal{B}_i \downarrow^{ns}_{\mathcal{A}_i} \mathcal{A}_{i+1}$.

By Lemma 11, $t(b/\mathcal{B}) = t(a/\mathcal{B})$. Thus, to show that $U(a/\mathcal{B}) \geq U(a/\mathcal{A})$, it is enough that $b \not\downarrow^{ns}_{\mathcal{B}_i} \mathcal{B}_{i+1}$ for all $i < n$. Suppose for the sake of contradiction that $b \downarrow^{ns}_{\mathcal{B}_i} \mathcal{B}_{i+1}$ for some $i < n$. Using induction and the "Furthermore" part in AIII, we get that $\mathcal{B}_i \downarrow^{ns}_{\mathcal{A}_i} \mathcal{A}_n b$, and hence by monotonicity and AV, $b \downarrow^{ns}_{\mathcal{A}_i} \mathcal{B}_i$. On the other hand, the counterassumption and monotonicity give $b \downarrow^{ns}_{\mathcal{B}_i} \mathcal{A}_{i+1}$. But from these two and Lemma 26, it follows that $b \downarrow^{ns}_{\mathcal{A}_i} \mathcal{A}_{i+1}$, a contradiction. □

3.1. **Indiscernible and Morley sequences.** In the next section, we will define Lascar types, an analogue to first order strong types. We then define our main independence notion in terms of Lascar splitting. However, there we will need technical tools to prove some properties of Lascar types. For this purpose, we now define strongly indiscernible and Morley sequences. Morley sequences will be strongly indiscernible.

Definition 32. *We say that a sequence $(a_i)_{i<\alpha}$ is* indiscernible *over A if every permutation of the sequence $\{a_i | \ i < \alpha\}$ extends to an automorphism $f \in Aut(\mathbb{M}/A)$.*

We say that a sequence $(a_i)_{i<\alpha}$ is weakly indiscernible *over A if every permutation of a finite subset of the sequence $\{a_i | \ i < \alpha\}$ extends to an automorphism $f \in Aut(\mathbb{M}/A)$.*

We say a sequence $(a_i)_{i<\alpha}$ is strongly indiscernible *over A if for all cardinals κ, there are a_i, $\alpha \leq i < \kappa$, such that $(a_i)_{i<\kappa}$ is indiscernible over A.*

Let $\mathcal{A}$ be a model. We say a sequence $(a_i)_{i<\alpha}$ is Morley *over $\mathcal{A}$, if for all $i < \alpha$, $t(a_i/\mathcal{A}) = t(a_0/\mathcal{A})$ and $a_i \downarrow^{ns}_{\mathcal{A}} \cup_{j<i} a_j$.*

In the rest of this chapter, we will assume that all indiscernible sequences and Morley sequences that we consider are non-trivial, i.e. they do not just repeat the same element.

Applying Fodor's lemma, we now show that every uncountable sequence contains a Morley sequence as a subsequence. This will be extremely useful in many places later on.

Lemma 33. *Let A be a finite set and κ a cardinal such that $\kappa = cf(\kappa) > \omega$. For every sequence $(a_i)_{i<\kappa}$, there is a model $\mathcal{A} \supset A$ and some $X \subset \kappa$ cofinal so that $(a_i)_{i \in X}$ is Morley over $\mathcal{A}$.*

Proof. For $i < \kappa$, choose models $\mathcal{A}_i$ so that for each i, $A \subset \mathcal{A}_i$, $a_i \in \mathcal{A}_{i+1}$, $\mathcal{A}_j \subset \mathcal{A}_i$ for $j < i$, $\mathcal{A}_\gamma = \bigcup_{i<\gamma} \mathcal{A}_i$ for a limit γ, and $|\mathcal{A}_i| = |i| + \omega$. Then, for each limit i, there is some

$\alpha_i < i$ so that $a_i \downarrow^{ns}_{\mathcal{A}_{\alpha_i}} \mathcal{A}_i$ (By Lemma 21, there is some finite $A_i \subset \mathcal{A}_i$ so that $a_i \downarrow^{ns}_{A_i} \mathcal{A}_i$; just choose α_i so that $A_i \subset \mathcal{A}_{\alpha_i}$). By Fodor's Lemma, there is some $X' \subset \kappa$ cofinal and some $\alpha < \kappa$ so that $\alpha_i = \alpha$ for all $i \in X'$. Choose $\mathcal{A} = \mathcal{A}_\alpha$. By Lemma 22, there are at most $|\mathcal{A}| < \kappa$ many weak types over $\mathcal{A}$, and thus by the pigeonhole principle, there is some cofinal $X \subseteq X'$ so that $t(a_i/\mathcal{A}) = t(a_j/\mathcal{A})$ for all $i, j \in X$. □

Lemma 34. *If $(a_i)_{i<\alpha}$ is Morley over a countable model $\mathcal{A}$, then for all $i < \alpha$, $a_i \downarrow^{ns}_{\mathcal{A}} \cup\{a_j|\ j < \alpha,\ j \neq i\}$.*

Proof. See [4]. □

Lemma 35. *If $\mathcal{A}$ is a countable model, then Morley sequences over $\mathcal{A}$ are strongly indiscernible over $\mathcal{A}$.*

Proof. See [4]. □

3.2. **Lascar types and the main independence notion.** In this section, we will present our main independence notion and prove that it has all the usual properties of non-forking. The notion will be based on independence in the sense of Lascar splitting. The key here is that over models, our main independence notion will agree with non splitting independence (Lemma 45), so we will be able to make use of the properties that we proved in the beginning of section 3.

We start by giving the definition for Lascar types. These can be seen as an analogue for first order strong types, and we will eventually show that Lascar types are stationary. We will see that Lascar types imply weak types but also that over models weak types imply Lascar types.

Definition 36. *We say that a set A is* bounded *if $|A| < \delta$, where δ is the number such that $\mathbb{M}$ is δ- model homogeneous.*

Definition 37. *Let A be a finite set, and let E be an equivalence relation on M^n, for some $n < \omega$. We say E is* A-invariant *if for all $f \in Aut(\mathbb{M}/A)$ and $a, b \in \mathbb{M}$, it holds that if $(a, b) \in E$, then $(f(a), f(b)) \in E$. We denote the set of all A-invariant equivalence relations that have only boundedly many equivalence classes by $E(A)$.*

We say that a and b have the same Lascar type *over a set B, denoted $Lt(a/B) = Lt(b/B)$, if for all finite $A \subseteq B$ and all $E \in E(A)$, it holds that $(a, b) \in E$.*

Lemma 38. *If $(a_i)_{i<\omega}$ is strongly indiscernible over B, then $Lt(a_i/B) = Lt(a_0/B)$ for all $i < \omega$*

Proof. For each κ, there are a_i, $\omega \le i < \kappa$, so that $(a_i)_{i<\kappa}$ is indiscernible over B. If $E \in E(A)$ for some finite $A \subset B$, then E has only boundedly many classes, and thus, for a large enough κ, there must be some indices $i < j < \kappa$ so that $(a_i, a_j) \in E$. But this implies that $(a_i, a_j) \in E$ for all $i, j < \kappa$, and the lemma follows. □

Lemma 39. *Let $\mathcal{A}$ be a model and let $t(a/\mathcal{A}) = t(b/\mathcal{A})$. Then, $Lt(a/\mathcal{A}) = Lt(b/\mathcal{A})$.*

Proof. Since the equality of Lascar types is determined locally (i.e. it depends on finite sets only), we may without loss assume that $\mathcal{A}$ is countable.

Since $t(a/\mathcal{A}) = t(b/\mathcal{A})$, there is a sequence $(a_i)_{i<\omega}$ such that $(a) \frown (a_i)_{i<\omega}$ and $(b) \frown (a_i)_{i<\omega}$ are Morley over $\mathcal{A}$. Because Morley sequences are strongly indiscernible, $Lt(a/\mathcal{A}) = Lt(b/\mathcal{A})$ by Lemma 38. □

In particular, by Lemma 22, for any finite set A, the number of Lascar types $Lt(a/A)$ is countable. It follows that every equivalence relation $E \in E(A)$ has only countably many equivalence classes.

Lemma 40. *Let $\mathcal{A}$ be a countable model, A a finite set such that $A \subset \mathcal{A}$ and $b \in \mathbb{M}$. Then, there is some $a \in \mathcal{A}$ such that $Lt(a/A) = Lt(b/A)$.*

Proof. Since there are only countably many Lascar types over A, there is some countable model $\mathcal{B}$ containing A and realizing all Lascar types over A. By AI, we can construct an automorphism $f \in \mathrm{Aut}(\mathbb{M}/A)$ such that $f(\mathcal{B}) = \mathcal{A}$. Let $b' = f^{-1}(b)$. Then, there is some $a' \in \mathcal{B}$ such that $Lt(a'/A) = Lt(b'/A)$. Let $a = f(a')$. Then, $a \in \mathcal{A}$ and $Lt(a/A) = Lt(f(b')/A) = Lt(b/A)$. □

Lemma 41. *Let A be a finite set and let $a, b \in \mathbb{M}$. Then, $Lt(a/A) = Lt(b/A)$ if and only if there are $n < \omega$ and strongly indiscernible sequences I_i over A, $i \leq n$, such that $a \in I_0$, $b \in I_n$ and for all $i < n$, $I_i \cap I_{i+1} \neq \emptyset$.*

Proof. The implication from right to left follows from Lemma 38 and the fact that all the strongly indiscernible sequences intersect each other.

For the other direction, we note that "there are $n < \omega$ and strongly indiscernible sequences I_i over A, $i \leq n$, such that $a \in I_0$, $b \in I_n$ and for all $i < n$, $I_i \cap I_{i+1} \neq \emptyset$" is an A-invariant equivalence relation. Since we assume that $Lt(a/A) = Lt(b/A)$, it is enough to prove that this equivalence relation has only boundedly many classes.

Suppose, for the sake of contradiction, that it has unboundedly many classes. Then, there is a sequence $(a_i)_{i<\omega_1}$ where no two elements are in the same class. By Lemma 33, there is some $X \subseteq \omega_1$, $|X| = \omega_1$, and a model $\mathcal{A} \supset A$ such that $(a_i)_{i \in X}$ is a Morley sequence over $\mathcal{A}$ and thus strongly indiscernible over A. But now by the definition of our equivalence relation, all the elements a_i, $i \in X$ are in the same equivalence class, a contradiction. □

Now we are ready to introduce our main independence notion.

Definition 42. *Let $A \subset B$ be finite. We say that $t(a/B)$* Lascar splits *over A, if there are $b, c \in B$ such that $Lt(b/A) = Lt(c/A)$ but $t(ab/A) \neq t(ac/A)$.*

We say a is free from C over B, denoted $a \downarrow_B C$, if there is some finite $A \subset B$ such that for all $D \supseteq B \cup C$, there is some b such that $t(b/B \cup C) = t(a/B \cup C)$ and $t(b/D)$ does not Lascar split over A.

Remark 43. *Note that it follows from the above definition that if $ab \downarrow_A B$, then $a \downarrow_A B$.*

Also, the independence notion is monotone, i.e. if $A \subseteq B \subseteq C \subseteq D$ and $a \downarrow_A D$, then $a \downarrow_B C$.

Lemma 44. *If $Lt(a/A) = Lt(b/A)$, then $t(a/A) = t(b/A)$.*

Proof. By Lemma 22, the equivalence relation "$t(x/A) = t(y/A)$" has only boundedly many classes. □

Lemma 45. *Let $a \in \mathbb{M}$, let $\mathcal{A}$ be a model and let $B \supseteq \mathcal{A}$. The following are equivalent:*

(i) $a \downarrow_{\mathcal{A}} B$,
(ii) $a \downarrow^{ns}_{\mathcal{A}} B$,
(iii) $t(a/B)$ does not Lascar split over some finite $A \subseteq \mathcal{A}$.

Proof. "(i) ⇒ (iii)" follows from Definition 42 by choosing $D = B$.

For "$(ii) \Rightarrow (i)$", suppose $a \downarrow^{ns}_{\mathcal{A}} B$. Then, there is some finite $A \subset \mathcal{A}$ so that $t(a/B)$ does not split over A, and in paricular $t(a/\mathcal{A})$ does not split over A. Let $D \supset B$ be arbitrary. By Lemma 27, there is some b such that $t(b/\mathcal{A}) = t(a/\mathcal{A})$ and $t(b/D)$ does not split over A. Since $a \downarrow^{ns}_{\mathcal{A}} B$ and $b \downarrow^{ns}_{\mathcal{A}} B$, we have by Lemma 11 that $t(b/B) = t(a/B)$. Now, $t(b/D)$ does not Lascar split over A. Indeed, if it would Lascar split, then we could find $c, d \in D$ so that $Lt(c/A) = Lt(d/A)$ but $t(bc/A) \neq t(bd/A)$. By Lemma 44, this implies that $t(b/D)$ would split over A, a contradiction.

For $(iii) \Rightarrow (ii)$, suppose that $t(a/B)$ does not Lascar split over A. We may without loss assume that $t(a/\mathcal{A})$ does not split over A (just enlarge A if necessary). We claim that $t(a/B)$ does not split over A. If it does, then there are $b, c \in B$ witnessing the splitting. Let $\mathcal{B} \subseteq \mathcal{A}$ be a countable model containing A. By Lemma 40, we find $(b', c') \in \mathcal{B}$ so that $Lt(b', c'/A) = Lt(b, c/A)$. Since $Lt(b/A) = Lt(b'/A)$ and $Lt(c/A) = Lt(c'/A)$, we must have $t(ab/A) = t(ab'/A)$ and $t(ac/A) = t(ac'/A)$ (otherwise $t(a/B)$ would Lascar split over A). But since $t(ab/A) \neq t(ac/A)$, we have

$$t(ab'/A) = t(ab/A) \neq t(ac/A) = t(ac'/A),$$

which means that $t(a/\mathcal{A})$ splits over A, a contradiction. □

Remark 46. *Note that from the proof of "(ii) ⇒ (i)" for Lemma 45, it follows that if $\mathcal{A}$ is a model such that $\mathcal{A} \subseteq B$ and $A \subset \mathcal{A}$ is a finite set so that $a \downarrow^{ns}_{\mathcal{A}} B$, then $a \downarrow_A B$. In particular, for all models $\mathcal{A}$ and all $a \in \mathbb{M}$, there is some finite $A \subset \mathcal{A}$ such that $a \downarrow_A \mathcal{A}$.*

Lemma 47. *Suppose $\mathcal{A}$ is a model, $A \subseteq \mathcal{A}$ finite, and $t(a/\mathcal{A})$ does not Lascar split over A. Then, $a \downarrow_A \mathcal{A}$.*

Proof. Choose a finite set B such that $A \subseteq B \subset \mathcal{A}$ and $t(a/\mathcal{A})$ does not split over B. For an arbitrary $D \supseteq \mathcal{A}$, there is some b so that $t(b/\mathcal{A}) = t(a/\mathcal{A})$ and $t(b/D)$ does not split over B. We will show that $t(b/D)$ does not Lascar split over A. Suppose it does. Then, we can find $c \in \mathcal{A}$ and $d \in D$ such that $Lt(c/A) = Lt(d/A)$ but $t(bc/A) \neq t(bd/A)$. By Lemma 40, there is some $d' \in \mathcal{A}$ such that $Lt(d'/B) = Lt(d/B)$. Then, either $t(d'b/A) \neq t(cb/A)$ or $t(d'b/A) \neq t(db/A)$. In the first case $t(b/\mathcal{A})$ Lascar splits over A, and in the second case, $t(b/D)$ splits over B. Both contradict our assumptions. □

We now show that in certain special cases, preservation of U-ranks implies independence. This lemma will be applied later, after we have defined U-ranks over arbitrary sets and show that independence can be expressed as preservation of U-ranks.

Lemma 48. *Suppose $\mathcal{A}$ is a model, $A \subseteq \mathcal{A}$ is finite and $U(a/A) = U(a/\mathcal{A})$. Then $a \downarrow_A \mathcal{A}$.*

Proof. By Lemma 47, it is enough to show that $t(a/\mathcal{A})$ does not Lascar split over A. Suppose for the sake of contradiction, that $t(a/\mathcal{A})$ does Lascar split over A. We enlarge the model $\mathcal{A}$ as follows. First we go through all pairs $b, c \in \mathcal{A}$ so that $Lt(b/A) = Lt(c/A)$. For each such pair, we find finitely many strongly indiscernible sequences over A of length ω_1 as in Lemma 41. We enlarge $\mathcal{A}$ to contain all these sequences. After this, we repeat the process ω many times. Then, for every permutation of a sequence of length ω_1 that is strongly indiscernible over A and contained in the model, we choose some automorphism $f \in \mathrm{Aut}(\mathbb{M}/A)$ that extends the permutation. We close the model under all the chosen automorphisms. Next, we start looking again at pairs in the model that have same Lascar type over A and adding A-indiscernible sequences of length ω_1 witnessing this. After repeating the whole process sufficiently long, we have obtained a model $\mathcal{A}^* \supseteq \mathcal{A}$ such that for any $b, c \in \mathcal{A}^*$ with $Lt(b/A) = Lt(c/A)$, $\mathcal{A}^*$ contains A-indiscernible sequences witnessing this, and moreover every permutation of a sequence of length ω_1 that is strongly indiscernible over A and contained in $\mathcal{A}^*$ extends to an automorphism of $\mathcal{A}^*$.

Choose now an element a^* so that $t(a^*/\mathcal{A}) = t(a/\mathcal{A})$ and $a^* \downarrow^{ns}_{\mathcal{A}} \mathcal{A}^*$. Then, $U(a^*/\mathcal{A}^*) = U(a^*/A)$ by Lemma 31. Let $f \in \mathrm{Aut}(\mathbb{M}/\mathcal{A})$ be such that $f(a^*) = a$, and denote $\mathcal{A}' = f(\mathcal{A}^*)$. Now, $U(a/\mathcal{A}') = U(a/A)$ and $t(a/\mathcal{A}')$ Lascar splits over A.

Let $b, c \in \mathcal{A}'$ witness the splitting. Then, $Lt(b/A) = Lt(c/A)$ and inside $\mathcal{A}'$ there are for some $n < \omega$, strongly indiscernible sequences I_i, $i \leq n$, over A of length ω_1 so that $b \in I_0$, $c \in I_n$ and $I_i \cap I_{i+1} \neq \emptyset$ for $i < n$. Since $t(ab/A) \neq t(ac/A)$, in at least one of these sequences there must be two elements that have different weak types over Aa. Since there are only countably many weak types over Aa, this implies that there is inside $\mathcal{A}'$ a sequence $(a_i)_{i<\omega_1}$ strongly indiscernible over A such that $t(aa_0/A) \neq t(aa_1/A)$ but $t(aa_1/A) = t(aa_i/A)$ for all $0 < i < \omega_1$. Moreover, every permutation of $(a_i)_{i<\omega_1}$ extends to an automorphism $f \in Aut(\mathcal{A}'/A)$.

For each $i < \omega_1$, let $f_i \in Aut(\mathbb{M}/A)$ be an automorphism permuting the sequence $(a_i)_{i<\omega_1}$ so that $f_i(a_0) = a_i$ and $f_i(\mathcal{A}') = \mathcal{A}'$. Denote $b_i = f_i(a)$ for each $i < \omega_1$. Then, $U(b_i/\mathcal{A}') = U(b_i/A)$ and for all $j < i < \omega_1$, $t(b_i/A) = t(b_j/A)$, but $t(b_i/\mathcal{A}') \neq t(b_j/\mathcal{A}')$ since

$$t(b_i a_i/A) = t(f_i(a)f_i(a_0)/A) = t(aa_0/A) \neq t(af_j^{-1}(a_i)/A) = t(f_j(a)a_i/A) = t(b_j a_i/A).$$

Let $\mathcal{B} \subseteq \mathcal{A}$ be countable model such that $A \subseteq \mathcal{B}$. Then for all $i < \omega$,

$$U(b_i/\mathcal{A}') = U(b_i/\mathcal{B}),$$

so $b_i \downarrow^{ns}_{\mathcal{B}} \mathcal{A}'$ by Lemma 31. Thus, for all $i < j < \omega_1$, $t(b_i/\mathcal{B}) \neq t(b_j/\mathcal{B})$, a contradiction by Lemma 11 since there are only countably many types over $\mathcal{B}$. □

Corollary 49. *For every $a \in \mathbb{M}$, every finite set A and every $B \supseteq A$, there is some $b \in \mathbb{M}$ such that $t(a/A) = t(b/A)$ and $b \downarrow_A B$.*

Proof. Let $\mathcal{A}$ be a model such that $U(a/A) = U(a/\mathcal{A})$. Let $\mathcal{B}$ be a model such that $\mathcal{A} \cup B \subseteq \mathcal{B}$, and let b be such that $t(b/\mathcal{A}) = t(a/\mathcal{A})$ and $b \downarrow^{ns}_{\mathcal{A}} \mathcal{B}$. Then, by Lemma 31,

$$U(b/\mathcal{B}) = U(b/\mathcal{A}) = U(a/\mathcal{A}) = U(a/A) = U(b/A).$$

By Lemma 48, $b \downarrow_A \mathcal{B}$, and thus $b \downarrow_A B$. □

We now prove a weak form of transitivity. We will apply it to prove some properties of the independence notion that we will need before we can prove full transitivity.

Lemma 50. *Suppose $A \subseteq \mathcal{A} \subseteq B$. Then $a \downarrow_A B$ if and only if $a \downarrow_A \mathcal{A}$ and $a \downarrow_{\mathcal{A}} B$.*

Proof. "$\Rightarrow$": $a \downarrow_A \mathcal{A}$ is clear and $a \downarrow_{\mathcal{A}} B$ follows from Lemma 45.

"$\Leftarrow$": Since $a \downarrow_A \mathcal{A}$, there is by definition some finite $A_0 \subseteq A$ and some b such that $t(b/\mathcal{A}) = t(a/\mathcal{A})$ and $t(b/B)$ does not Lascar split over A_0. By Lemma 45, $b \downarrow^{ns}_{\mathcal{A}} B$ and $a \downarrow^{ns}_{\mathcal{A}} B$. Thus, by Lemma 11, $t(b/B) = t(a/B)$. Hence $a \downarrow_A B$, as wanted. □

Next, we prove symmetry over finite sets. Later, this result will be applied when proving full symmetry.

Lemma 51. *Let A be finite. Then, $a \downarrow_A b$ if and only if $b \downarrow_A a$.*

Proof. Suppose $a \downarrow_A b$. Let $\mathcal{A}_0$ be a model such that $A \subset \mathcal{A}_0$. By Corollary 49, there exists some b' such that $t(b'/A) = t(b/A)$ and $b' \downarrow_A \mathcal{A}_0$. Let $f \in \mathrm{Aut}(\mathbb{M}/A)$ be such that $f(b') = b$, and denote $\mathcal{A} = f(\mathcal{A}_0)$. Then, $A \subset \mathcal{A}$ and $b \downarrow_A \mathcal{A}$. Since $a \downarrow_A b$, there is by Definition 42 (take $D = \mathcal{A}b$), some a' such that $t(a'/Ab) = t(a/Ab)$ and $t(a'/\mathcal{A}b)$ does not

Lascar split over A. It now follows from Lemma 45, that $a' \downarrow_A \mathcal{A}b$ and thus $a' \downarrow_A b$. By Lemma 45 and Remark 25, $b \downarrow_A a'$. By Lemma 50, $b \downarrow_A a'$, and thus $b \downarrow_A a$. □

Now, we prove a weak form of extension: that types over finite sets have free extensions. We will apply the lemma when proving full extension.

Lemma 52. *For every a, every finite set A and every $B \supseteq A$, there is b such that $Lt(b/A) = Lt(a/A)$ and $b \downarrow_A B$.*

Proof. Let $\mathcal{A}_0$ be a countable model such that $A \subset \mathcal{A}_0$. By Corollary 49, there is some element a' so that $t(a'/A) = t(a/A)$ and $a' \downarrow_A \mathcal{A}_0$. Let $f \in \mathrm{Aut}(\mathbb{M}/A)$ be such that $f(a') = a$. Denote $\mathcal{A} = f(\mathcal{A}_0)$. Now, $A \subset \mathcal{A}$ and $a \downarrow_A \mathcal{A}$.

Choose now b so that $t(b/\mathcal{A}) = t(a/\mathcal{A})$ and $b \downarrow^{ns}_{\mathcal{A}} B$. Then, $b \downarrow_A B$. By Lemma 50, $b \downarrow_A B$. Moreover, by Lemma 39, $Lt(b/A) = Lt(a/A)$. □

Now, we are ready to prove stationarity and transitivity.

Lemma 53 (Stationarity). *If $A \subseteq B$, $a \downarrow_A B$, $b \downarrow_A B$ and $Lt(a/A) = Lt(b/A)$, then $Lt(a/B) = Lt(b/B)$.*

Proof. Clearly it is enought to prove this under the assumption that A and B are finite (if $Lt(a/B_0) = Lt(b/B_0)$ for every finite $B_0 \subset B$, then $Lt(a/B) = Lt(b/B)$). Suppose the claim does not hold. We will construct countable models $\mathcal{A}_a$ and $\mathcal{A}_b$ so that $Aa \subset \mathcal{A}_a$, $Ab \subset \mathcal{A}_b$, $B \downarrow_A \mathcal{A}_a$ and $B \downarrow_A \mathcal{A}_b$. Let $\mathcal{A}$ be a model such that $Aa \subset \mathcal{A}$. By Lemma 51, we have $B \downarrow_A a$. Thus, by Definition 42, there is some B' such that $t(B'/Aa) = t(B/Aa)$ and $t(B'/\mathcal{A})$ does not Lascar split over A. Let $f \in \mathrm{Aut}(M/Aa)$ be such that $f(B') = B$, and denote $\mathcal{A}_a = f(\mathcal{A})$. Then, $Aa \subset \mathcal{A}_a$ and $t(B/\mathcal{A}_a)$ does not Lascar split over A. By Lemma 47, $B \downarrow_A \mathcal{A}_a$. Similarly, we find a suitable model $\mathcal{A}_b$. Now by Lemma 52, there is some c such that $Lt(c/A) = Lt(a/A)$ and $c \downarrow_A \mathcal{A}_a \cup \mathcal{A}_b \cup B$. By monotonicity, we have $c \downarrow_{\mathcal{A}_a} B$, and thus by Lemma 51, $B \downarrow_{\mathcal{A}_a} c$. Hence, by Lemma 50, $B \downarrow_A \mathcal{A}_a c$ and so $ac \downarrow_A B$.

By the counterassumption, we may without loss assume that $Lt(c/B) \neq Lt(a/B)$. Choose a model $\mathcal{B} \supseteq B$ so that $ac \downarrow_A \mathcal{B}$. By Lemma 39, $t(a/\mathcal{B}) \neq t(c/\mathcal{B})$. So there is some $b' \in \mathcal{B}$ that withesses this, i.e. $t(ab'/A) \neq t(cb'/A)$. As $Lt(c/A) = Lt(a/A)$, this means $t(b'/Aac)$ Lascar splits over A, a contradiction since $b' \downarrow_A ac$. □

Lemma 54 (Transitivity). *Suppose $A \subseteq B \subseteq C$, $a \downarrow_A B$ and $a \downarrow_B C$. Then $a \downarrow_A C$.*

Proof. Clearly it is enough to prove this for finite A. Choose b so that $Lt(b/A) = Lt(a/A)$ and $b \downarrow_A C$. Then, by monotonicity, $b \downarrow_A B$, and thus by Lemma 53, $Lt(b/B) = Lt(a/B)$. Again by monotonicity, $b \downarrow_B C$, and by Lemma 53, $Lt(b/C) = Lt(a/C)$. The claim follows. □

We don't yet have all the results needed for proving finite character, but we prove the following special case that we will need when proving other properties.

Lemma 55. *Suppose $A \subset B$, A is finite, and $a \not\downarrow_A B$. Then there is some $b \in B$ such that $a \not\downarrow_A b$.*

Proof. By Lemma 52, there is some c such that $Lt(c/A) = Lt(a/A)$ and $c \downarrow_A B$. We have $a \not\downarrow_A B$, and thus $t(c/B) \neq t(a/B)$. Hence, there is some $b \in B$ so that $t(cb/A) \neq t(ab/A)$. Since $c \downarrow_A B$, we have $c \downarrow_A b$. Now, $a \not\downarrow_A b$. Indeed, otherwise Lemma 53 would imply $Lt(c/Ab) = Lt(a/Ab)$ and thus $t(c/Ab) = t(a/Ab)$, a contradiction against the fact that $t(cb/A) \neq t(ab/A)$. □

We will now start working towards a more comprehensive definition of U-rank that will allow characterizing independence in terms of U-ranks. For this, we need the notion of strong automorphism.

Definition 56. *Let A be a finite set and let $f \in Aut(\mathbb{M}/A)$. We say that f is a* strong automorphism *over A if it preserves Lascar types over A, i.e. if for any a, $Lt(a/A) = Lt(f(a)/A)$. We denote the set of strong automorphisms over A by $Saut(\mathbb{M}/A)$.*

Lemma 57. *Suppose A is finite and $Lt(a/A) = Lt(b/A)$. Then there is $f \in Saut(\mathbb{M}/A)$ such that $f(a) = b$.*

Proof. Choose a countable model $\mathcal{A}$ such that $A \subseteq \mathcal{A}$ and $ab \downarrow_A \mathcal{A}$. In particular, by Remark 43, $a \downarrow_A \mathcal{A}$ and $b \downarrow_A \mathcal{A}$. By Lemma 53, $Lt(a/\mathcal{A}) = Lt(b/\mathcal{A})$. Thus, there is some $f \in \mathrm{Aut}(\mathbb{M}/\mathcal{A})$ such that $f(a) = b$. By Lemma 39, $f \in \mathrm{Saut}(\mathbb{M}/\mathcal{A})$. □

In order to give a general definition of U-rank, we will show that if $\mathcal{A}$ is a model and $a \in \mathbb{M}$, then the rank $U(a/\mathcal{A})$ equals the minimum of ranks $U(a/A)$, where A ranges over finite subsets of $\mathcal{A}$. This is obtained as a corollary of the following lemma.

Lemma 58. *Suppose $\mathcal{A}$ is a model, $A \subseteq \mathcal{A}$ is finite and $t(a/\mathcal{A})$ does not split over A. Then $U(a/A) = U(a/\mathcal{A})$.*

Proof. Suppose not. If we choose some countable model $\mathcal{A}'$ such that $A \subset \mathcal{A}' \subseteq \mathcal{A}$, then $a \downarrow^{ns}_{\mathcal{A}'} \mathcal{A}$, and thus, by Lemma 31, $U(a/\mathcal{A}') = U(a/\mathcal{A})$. Hence, we may assume that $\mathcal{A}$ is countable.

Choose a countable model $\mathcal{B}$ such that $A \subset \mathcal{B}$ and $U(a/\mathcal{B}) = U(a/A)$. Now, there is some $f \in \mathrm{Aut}(\mathbb{M}/A)$ so that $f(\mathcal{B}) = \mathcal{A}$. Let $a' = f(a)$. We have

$$U(a/\mathcal{A}) \neq U(a/A) = U(a/\mathcal{B}) = U(a'/\mathcal{A}),$$

and thus $t(a/\mathcal{A}) \neq t(a'/\mathcal{A})$. Hence there is some $c \in \mathcal{A}$ such that $t(ac/A) \neq t(a'c/A)$. Let $b \in \mathcal{B}$ be such that $f(b) = c$ (and thus $t(b/A) = t(c/A)$). Then, $t(a'c/A) = t(ab/A)$, so $t(ac/A) \neq t(ab/A)$. Let $c' \in \mathcal{A}$ be such that $Lt(c'/A) = Lt(b/A)$, and thus $t(c'/A) = t(b/A) = t(c/A)$. Since $t(a/\mathcal{A})$ does not split over A, we have $t(ac'/A) \neq t(ab/A)$.

We note that since $a \downarrow^{ns}_A \mathcal{A}$, we have by Remark 46 $a \downarrow_A \mathcal{A}$, and thus in particular $a \downarrow_A c'$. Choose $g \in Saut(\mathbb{M}/A)$ so that $g(b) = c'$. Let $a'' = g(a)$. By Lemma 48, we have $a \downarrow_A b$, and thus $a'' \downarrow_A c'$. But now $Lt(a''/A) = Lt(a/A)$, $a \downarrow_A c'$ and $a'' \downarrow_A c'$, yet

$$t(ac'/A) \neq t(a'c'/A) = t(ab/A) = t(a''c'/A),$$

so in particular $t(a/Ac') \neq t(a''/Ac')$, a contradiction to Lemma 53. □

Corollary 59. *Let $\mathcal{A}$ be a model. Then,*

$$U(a/\mathcal{A}) = min(\{U(a/B) \mid B \subset \mathcal{A} \text{ finite }\}.$$

Proof. By Definition 29, for each finite $B \subset \mathcal{A}$, it holds that $U(a/B) \geq U(a/\mathcal{A})$. On the other hand, by Lemma 21, there is some finite $A \subset \mathcal{A}$ so that $t(a/\mathcal{A})$ does not split over A. By Lemma 58, $U(a/\mathcal{A}) = U(a/A)$. □

Corollary 59 allows us to define $U(a/A)$ for arbitrary A as follows. By Definition 29, for finite A it holds that $U(a/A_0) \geq U(a/A)$ for all $A_0 \subseteq A$. Thus, the following definition corresponds to Definition 29 also in the case that A is finite.

Definition 60. *Let A be arbitrary. We define $U(a/A)$ to be the minimum of $U(a/B)$, $B \subseteq A$ finite.*

Now we can finally characterize independence in terms of U-ranks. As corollaries, we will get local character and extension for the independence notion.

Lemma 61. *For all $A \subseteq B$ and a, $a \downarrow_A B$ if and only if $U(a/A) = U(a/B)$.*

Proof. Suppose first B is finite.

"$\Leftarrow$": Choose a model $\mathcal{A} \supseteq B$ such that $U(a/\mathcal{A}) = U(a/B)$. Then $U(a/\mathcal{A}) = U(a/A)$ and thus by Lemma 48, $a \downarrow_A \mathcal{A}$, and in particular $a \downarrow_A B$.

"$\Rightarrow$": Choose a model $\mathcal{A} \supseteq A$ such that $U(a/\mathcal{A}) = U(a/A)$ and a model $\mathcal{B} \supseteq \mathcal{A}B$. By Lemma 12, there is some a' such that $t(a'/\mathcal{A}) = t(a/\mathcal{A})$ and $a' \downarrow^{ns}_{\mathcal{A}} \mathcal{B}$. Then, by Lemma 31,

$$U(a'/A) = U(a'/\mathcal{A}) = U(a'/\mathcal{B}),$$

so by Lemma 48, $a' \downarrow_A \mathcal{B}$. By Lemma 39, $Lt(a'/A) = Lt(a/A)$, and thus by Lemma 53, $t(a'/B) = t(a/B)$. Thus

$$U(a/B) = U(a'/B) = U(a'/A) = U(a/A).$$

We now prove the general case. Let A, B be arbitrary such that $A \subseteq B$.

"$\Rightarrow$": Suppose $a \downarrow_A B$. There are some finite sets $A_0, A_0' \subseteq A$ such that $a \downarrow_{A_0} B$ and some $U(a/A) = U(a/A_0')$. We may without loss assume that $A_0 = A_0'$. Indeed, this follows from monotonicity and the fact that if A_0'' is any set such that $A_0' \subseteq A_0'' \subseteq A$, then $U(a/A_0'') = U(a/A)$. Let $B_0 \subseteq B$ be a finite set such that $U(a/B_0) = U(a/B)$. By similar argument as above, we may without loss suppose that $A_0 \subseteq B_0$. Thus, since the result holds for finite sets, we have

$$U(a/A) = U(a/A_0) = U(a/B_0) = U(a/B).$$

"$\Leftarrow$": Suppose $U(a/A) = U(a/B)$, but $a \not\downarrow_A B$. Let $A_0 \subseteq A$ and $B_0 \subseteq B$ be finite sets such that

$$U(a/A_0) = U(a/A) = U(a/B) = U(a/B_0).$$

By monotonicity, we have $a \not\downarrow_{A_0} B$, and by Lemma 55, there is some $b \in B$ such that $a \not\downarrow_{A_0} b$. Then, also $a \not\downarrow_{A_0} B_0 b$. But

$$U(a/B_0 b) = U(a/B) = U(a/A_0),$$

a contradiction. □

Corollary 62 (Local character). *For all A and a there is finite $B \subseteq A$ such that $a \downarrow_B A$.*

Corollary 63 (Extension). *For all a and all sets $A \subseteq B$, there is some b such that $Lt(b/A) = Lt(a/A)$ and $b \downarrow_A B$.*

Proof. Let $A_0 \subseteq A$ be a finite set such that $U(a/A_0) = U(a/A)$. Then, $a \downarrow_{A_0} A$ by Lemma 61. By Lemma 52, there is some b such that $Lt(b/A_0) = Lt(a/A_0)$ and $b \downarrow_{A_0} B$. By Lemma 53, $Lt(b/A) = Lt(a/A)$. □

Now it is easy to prove also finite character and symmetry.

Lemma 64 (Finite character). *Suppose $A \subset B$, and $a \not\downarrow_A B$. Then there is some $b \in B$ such that $a \not\downarrow_A b$.*

Proof. Choose a finite $C \subseteq A$ such that $a \downarrow_C A$ and an element c such that $Lt(c/C) = Lt(a/C)$ and $c \downarrow_C A \cup B$ (they exist by Corollary 62 and Lemma 52). Then, by Lemma 53, $Lt(c/A) = Lt(a/A)$. We have $a \not\downarrow_C B$, and thus $t(c/B) \neq t(a/B)$. Hence, there is some $b \in B$ so that $t(cb/C) \neq t(ab/C)$. By monotonicity, we have $c \downarrow_A B$, and in particular $c \downarrow_A b$. If $a \downarrow_A b$, then $a \downarrow_C b$ by Lemma 54. Since $t(a/Cb) \neq t(c/Cb)$, this contradicts Lemma 53. □

Lemma 65 (Symmetry). *Let A be arbitrary. If $a \downarrow_A b$, then $b \downarrow_A a$.*

Proof. Suppose not. Choose some finite $B \subseteq A$ such that $a \downarrow_B Ab$ and $b \downarrow_B A$ (such a set can be found by Corollary 62). Since $b \not\downarrow_A a$, we have $b \not\downarrow_B Aa$. By Lemma 64, there is some finite set C such that $B \subseteq C \subseteq A$ and $b \not\downarrow_B Ca$. By transitivity, $b \not\downarrow_C a$. On the other hand, $a \downarrow_B Ab$, and thus $a \downarrow_B Cb$, so $a \downarrow_C b$, which contradicts Lemma 51. □

We now show that ranks can be added together in the usual way.

Lemma 66. *For any a, b and A, it holds that*

$$U(ab/A) = U(a/bA) + U(b/A).$$

Proof. We first note that it suffices to prove the lemma in case A is finite. Indeed, by definition 60, we find finite $A_1, A_2, A_3 \subset A$ so that $U(ab/A) = U(ab/A_1)$, $U(a/bA) = U(a/bA_2)$ and $U(b/A) = U(b/A_3)$. Denote $A_0 = A_1 \cup A_2 \cup A_3$. Since the above ranks are minimal, we have $U(ab/A) = U(ab/A_0)$, $U(a/bA) = U(a/bA_0)$ and $U(b/A) = U(b/A_0)$. Thus it suffices to show that the lemma holds for A_0, a finite set.

Next, we show that for any c and any finite set B, $U(c/B)$ is the maximal number n such that there are sets B_i, $i \leq n$ so that $B_0 = B$, and for all $i < n$, $B_i \subseteq B_{i+1}$ and $c \not\downarrow_{B_i} B_{i+1}$. By Lemma 61, $U(c/B_i) > U(c/B_{i+1})$ for all $i < n$, and thus, $U(c/B) \geq n$. On the other hand, by the definition of U-rank (Definition 29), there are models $\mathcal{B}_i$, $i \leq m = U(c/B)$, so that $B \subset \mathcal{B}_0$, and for each $i < m$, $\mathcal{B}_i \subset \mathcal{B}_{i+1}$ and $c \not\downarrow_{\mathcal{B}_i} \mathcal{B}_{i+1}$. Write $B_0 = B$. By Lemma 64, for each $1 \leq i < m$, we find some finite $B_i \subset \mathcal{B}_i$ so that $c \not\downarrow_{B_{i-1}} B_i$. Thus, $n \geq m = U(c/B)$.

To show $U(ab/A) \leq U(a/bA) + U(b/A)$, we let $n = U(ab/A)$ and A_i, $i \leq n$ be as above for $U(ab/A)$. Then, for each $i < n$, we must have either $a \not\downarrow_{bA_i} A_{i+1}$ or $b \not\downarrow_{A_i} A_{i+1}$. Indeed, if we would have both $a \downarrow_{bA_i} A_{i+1}$ and $b \downarrow_{A_i} A_{i+1}$, then by Lemma 51, we would have $A_{i+1} \downarrow_{A_i} b$ and $A_{i+1} \downarrow_{bA_i} a$, and thus by applying first Lemma 54 and monotonicity, then Lemma 51 again, we would get $ab \downarrow_{A_i} A_{i+1}$. Thus, $U(a/bA) + U(b/A) \geq n$.

Let now $U(b/A) = m$ and let A'_i, $i \leq m$ be the sets witnessing this (here $A'_0 = A$). Choose a' so that $t(a'/Ab) = t(a/Ab)$ and $a' \downarrow_{bA} A'_m$. Using a suitable automorphism, we find A_i, $i \leq m$, also witnessing $U(b/A) = m$ so that $a \downarrow_{bA} A_m$. Thus, by Lemma 61, $U(a/bA_m) = U(a/bA)$. Let $U(a/bA_m) = k$ and choose B_i, $i \leq k$ witnessing this. Now, $A = A_0, \ldots, A_{m-1}, B_0, \ldots, B_k$ witness that $U(ab/A) \geq m + k$ (note that we may without loss assume that $A_m = B_0$). □

Next, we give our analogue to first order algebraic closure: bounded closure. We then show that models are closed in terms of the bounded closure, that the bounded closure operator has finite character, and that it really is a closure in the sense that the closure of a closed set is the set itself. In section 4, we will present quasiminimal classes. This setting is analoguous to the first order strongly minimal setting. In strongly minimal classes, the algebraic closure operator yields a pregeometry, and ranks can be calculated as pregeometry dimensions. Similarly, in quasiminimal classes, a pregeometry is obtained from the bounded closure operator, and U-ranks are given as pregeometry dimensions.

Definition 67. *We say a is in the* bounded closure *of A, denoted $a \in bcl(A)$, if $t(a/A)$ has only boundedly many realizations.*

Lemma 68. *Let $\mathcal{A}$ be a model. Then, $bcl(\mathcal{A}) = \mathcal{A}$.*

Proof. Clearly $\mathcal{A} \subseteq \mathrm{bcl}(\mathcal{A})$. For the converse, suppose towards a contradiction that $a \in \mathrm{bcl}(\mathcal{A}) \setminus \mathcal{A}$. By Lemma 21, there is some finite $A \subset \mathcal{A}$ so that $a \downarrow_A^{ns} \mathcal{A}$. Choose now an element a' such that $t(a'/\mathcal{A}) = t(a/\mathcal{A})$ and $a' \downarrow^{ns}_{A} \mathrm{bcl}(\mathcal{A})$. Then, $a' \in \mathrm{bcl}(\mathcal{A})$. By Axiom I, there is some $b \in \mathcal{A}$ such that $t(b/A) = t(a'/A)$ and thus $b \neq a'$. In particular, $t(a'a'/A) \neq t(ba'/A)$. Thus, a' and b witness that $t(a'/\mathrm{bcl}(\mathcal{A}))$ splits over A, a contradiction. □

Lemma 69. *If $a \in bcl(A)$, then there is some finite $B \subseteq A$ so that $a \in bcl(B)$.*

Proof. There is some finite $B \subseteq A$ such that $a \downarrow_B A$. We claim that $a \in \mathrm{bcl}(B)$. Suppose not. Let $\mathcal{A}$ be a model such that $A \subseteq \mathcal{A}$. Now there is some a' so that $Lt(a'/A) = Lt(a/A)$ and $a' \downarrow_B \mathcal{A}$. By Lemma 68, $a' \in \mathrm{bcl}(A) \subseteq \mathrm{bcl}(\mathcal{A}) = \mathcal{A}$. Since $a \notin \mathrm{bcl}(B)$, the weak type $t(a/B)$ has unboundedly many realizations. Hence, by Lemma 33, there is a Morley sequence $(a_i)_{i<\omega}$ over some model $\mathcal{B} \supset B$ so that $a_0 = a'$ (just use a suitable automorphism to obtain this). By Axiom AI, there is an element $a'' \in \mathcal{A}$ so that $t(a''/a'B) = t(a_1/a'B)$, and by Lemma 38 , $Lt(a_1/B) = Lt(a'/B)$. Thus, there is an automorphism $f \in \mathrm{Aut}(\mathbb{M}/B)$ such that $f(a'') = a_1$ and $f(a') = a'$. Using Lemma 41, one sees that automorphisms preserve equality of Lascar types. Hence, the fact that $Lt(a_1/B) = Lt(a'/B)$ implies $Lt(a''/B) = Lt(a'/B)$. But we have $a' = a_0 \neq a_1$, and thus also $a'' \neq a'$, so $t(a'a'/B) \neq t(a'a''/B)$, which contradicts Lemma 53 since we assumed $a' \downarrow_B \mathcal{A}$. □

Lemma 70. *For every A, $bcl(bcl(A)) = bcl(A)$.*

Proof. By Lemma 69, we may assume that A is finite. Suppose now $a \in \mathrm{bcl}(\mathrm{bcl}(A)) \setminus \mathrm{bcl}(A)$. By Lemma 69, there is some $b \in \mathrm{bcl}(A)$ so that $a \in \mathrm{bcl}(Ab)$. Let κ be an uncountable cardinal such that $\kappa > |\mathrm{bcl}(\mathrm{bcl}(A))|$. Since $a \notin \mathrm{bcl}(A)$, there are a_i, $i < \kappa$ so that $a_i \neq a_j$ when $i \neq j$ and $t(a_i/A) = t(a/A)$ for all $i < \kappa$. For each i, there is some $b_i \in \mathrm{bcl}(A)$ such that $t(b_i a_i/A) = t(ba/A)$. By the pigeonhole principle, there is some b' and some $X \subseteq \kappa$ so that $|X| = \kappa$ and $b_i = b'$ for $i \in X$. Hence, for any $i \in X$, $t(a_i/Ab')$ has unboundedly many realizations, a contradiction since $a_i \in \mathrm{bcl}(Ab')$. □

Lemma 71. *Let $A \subset B$. If $a \in bcl(A)$, then $a \downarrow_A B$.*

Proof. By Lemma 69, we may assume that A is finite. Choose a' so that $Lt(a'/A) = Lt(a/A)$ and $a' \downarrow_A B$. Then, $a' \in \mathrm{bcl}(A)$. Consider the equivalence relation E defined so that $(x, y) \in E$ if either $x, y \notin \mathrm{bcl}(A)$ or $x = y \in \mathrm{bcl}(A)$. This is an A-invariant equivalence relation. Moreover, since A is finite, we may choose a countable model $\mathcal{A}$ so that $A \subset \mathcal{A}$. By Lemma 68, $\mathrm{bcl}(A) \subset \mathcal{A}$, so E has boundedly many classes, and thus $(a, a') \in E$. It follows that $a = a'$. □

Lemma 72 (Reflexivity)**.** *If $a \in bcl(B) \setminus bcl(A)$, then $a \not\downarrow_A B$.*

Proof. Suppose $a \downarrow_A B$. Choose a model $\mathcal{A}$ so that $B \subseteq \mathcal{A}$ and a' so that $t(a'/B) = t(a/B)$ and $a' \downarrow_A \mathcal{A}$. By Lemma 68, $a' \in \mathcal{A}$. Now we proceed as in the proof of Lemma 69 to obtain a contradiction. □

Now we have shown that our main independence notion $\downarrow$ has all the properties of non-forking.

Theorem 73. *Let $\mathcal{K}$ be a FUR-class, let $\mathbb{M}$ be a monster model for $\mathcal{K}$, and suppose $A \subseteq B \subseteq C \subseteq D \subset \mathbb{M}$. Then, the following hold.*

(i) Local character: For each a, there is some finite $A_0 \subseteq A$ such that $a \downarrow_{A_0} A$.
(ii) Finite character: If $a \not\downarrow_A B$, then there is some $b \in B$ so that $a \not\downarrow_A b$.
(iii) Stationarity: Suppose that $Lt(a/A) = Lt(b/A)$, $a \downarrow_A B$ and $b \downarrow_A B$. Then, $Lt(a/B) = Lt(b/B)$.

(iv) *Extension: For every* a*, there is some* b *such that* $Lt(b/A) = Lt(a/A)$ *and* $b \downarrow_A B$.
(v) *Monotonicity: If* $a \downarrow_A D$*, then* $a \downarrow_B C$.
(vi) *Transitivity: If* $a \downarrow_A B$ *and* $a \downarrow_B C$*, then* $a \downarrow_A C$.
(vii) *Symmetry: If* $a \downarrow_A b$*, then* $b \downarrow_A a$.
(viii) *U-ranks:* $a \downarrow_A B$ *if and only if* $U(a/B) = U(a/A)$.
(ix) *Finiteness of U-rank: For all* a*,* $U(a/\emptyset) < \omega$.
(x) *Addition of ranks: For all* a, b*,* $U(ab/A) = U(a/bA) + U(b/A)$.
(xi) *Independence of bcl: If* $a \in bcl(A)$*, then* $a \downarrow_A B$.
(xii) *Reflexivity: If* $a \in bcl(B) \setminus bcl(A)$*, then* $a \not\downarrow_A B$.
(xiii) *Local character of bcl: If* $a \in bcl(A)$*, then there is some finite* $A_0 \subseteq A$ *so that* $a \in bcl(A_0)$.
(xiv) *Closure:* $bcl(bcl(A)) = bcl(A)$.
(xv) *Models are closed: If* $\mathcal{A}$ *is a model, then* $bcl(\mathcal{A}) = \mathcal{A}$.

4. Quasiminimal classes

As we have pointed out, the model class of Example 3 is an example of a FUR-class. But it is also an example of a quasiminimal class, and in fact a quasiminimal class is always a FUR-class, given that it only contains infinite-dimensional models. It follow from Theorem 73 that such classes have a perfect theory of independence.

Quasiminimal classes are AECs that arise from a quasiminimal pregeometry structure. Quasiminimal pregeometry structures can be seen as an analogue to strongly minimal structures. They are defined as structures equipped with a pregeometry that has similar properties as the pregeometry obtained from the algebraic closure operator in the strongly minimal case. In fact, it turns out that this pregeometry is actually obtained from the bounded closure operator, and that U-ranks are given as pregeometry dimensions, so the situation resembles that of the strongly minimal context more than one might expect at the first glance.

In [1], a quasiminimal pregeometry structure and a quasiminimal class are defined as follows.

Definition 74. *Let* M *be an* L*-structure for a countable language* L*, equipped with a pregeometry* cl *(or* cl_M *if it is necessary to specify* M*). We say that* M *is a* quasiminimal pregeometry structure *if the following hold (tp denotes first order quantifier free type):*

(1) *(QM1) The pregeometry is determined by the language. That is, if* a *and* a' *are singletons and* $tp(a, b) = tp(a', b')$*, then* $a \in cl(b)$ *if and only if* $a' \in cl(b')$.
(2) *(QM2)* M *is infinite-dimensional with respect to cl.*
(3) *(QM3) (Countable closure property) If* $A \subseteq M$ *is finite, then* $cl(A)$ *is countable.*
(4) *(QM4) (Uniqueness of the generic type) Suppose that* $H, H' \subseteq M$ *are countable closed subsets, enumerated so that* $tp(H) = tp(H')$*. If* $a \in M \setminus H$ *and* $a' \in M \setminus H'$ *are singletons, then* $tp(H, a) = tp(H', a')$ *(with respect to the same enumerations for* H *and* H'*).*
(5) *(QM5) (*$\aleph_0$*-homogeneity over closed sets and the empty set) Let* $H, H' \subseteq M$ *be countable closed subsets or empty, enumerated so that* $tp(H) = tp(H')$*, and let* b, b' *be finite tuples from* M *such that* $tp(H, b) = tp(H', b')$*, and let* a *be a singleton such that* $a \in cl(H, b)$*. Then there is some singleton* $a' \in M$ *such that* $tp(H, b, a) = tp(H', b', a')$.

We say M *is a* weakly quasiminimal pregeometry structure *if it satisfies all the above axioms except possibly QM2.*

It is easy to see that the class from Example 3 satisfies the axioms. Another example of a quasiminimal pregeometry structure is the cover of an algebraically closed field (see Definition 1). It satisfies the axioms of Definition 74 if you take $cl(A) = log(alg(exp(A)))$, where alg stands for the field theoretic algebraic closure and $log(X)$ is defined in the usual way, i.e. $log(X) = \{u \in V \mid exp(u) \in X\}$ (see [3] for details).

Definition 75. *Suppose M_1 and M_2 are weakly quasiminimal pregeometry L-structures. Let θ be an isomorphism from M_1 to some substructure of M_2. We say that θ is a* closed embedding *if $\theta(M_1)$ is closed in M_2 with respect to cl_{M_2}, and cl_{M_1} is the restriction of cl_{M_2} to M_1.*

Given a quasiminimal pregeometry structure M, let $\mathcal{K}^-(M)$ be the smallest class of L-structures which contains M and all its closed substructures and is closed under isomorphisms, and let $\mathcal{K}(M)$ be the smallest class containing $\mathcal{K}^-(M)$ which is also closed under taking unions of chains of closed embeddings.

From now on, we suppose that $\mathcal{K} = \mathcal{K}(\mathbb{M})$ for some quasiminimal pregeometry structure $\mathbb{M}$, and that we have discarded all the finite-dimensional structures from $\mathcal{K}$. We will call such a class a *quasiminimal class.* For $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, we define $\mathcal{A} \preccurlyeq \mathcal{B}$ if $\mathcal{A}$ is a closed submodel of $\mathcal{B}$. It is well known that $(\mathcal{K}, \preccurlyeq)$ is an AEC with $LS(\mathcal{K}) = \omega$. We may without loss assume that $\mathbb{M}$ is a monster model for $\mathcal{K}$. In [1], it is shown that $\mathcal{K}$ is totally categorical and has arbitrarily large models (Theorem 2.2). It is easy to see that $\mathcal{K}$ has AP and JEP. It is also relatively easy to see that it is in fact a FUR-class (see [4]).

We note that we may reformulate the conditions QM4 and QM5 so that the concept of Galois type is used instead of the concept of quantifier-free type. Indeed, for QM4, let $H, H' \subset \mathbb{M}$ be countable and closed, let $t^g(H) = t^g(H')$, and let a, a' be singletons such that $a \notin \mathrm{cl}(H)$ and $a' \notin \mathrm{cl}(H')$. As H and H' are closed, they are models. Since H and H' are countable, there is some isomorphism $f : H \to H'$. Using QM4, we may extend f to a map $g_0 : Ha \to H'a'$ that preserves quantifier-free formulae. Let $\mathcal{A} = \mathrm{cl}(Ha)$ and $\mathcal{B} = \mathrm{cl}(H'a')$. We will extend g_0 to an isomorphism $g : \mathcal{A} \to \mathcal{B}$. Indeed, if $b \in \mathcal{A} = \mathrm{cl}(Ha)$, then by QM5 and QM1, there is some $b' \in \mathcal{B} = \mathrm{cl}(H'a')$ such that $\mathrm{tp}(H, a, b) = \mathrm{tp}(H, a', b')$, so f_0 extends to a map $f_1 : H, a, b \to H', a', b'$ preserving quantifier-free formulae. Since both $\mathcal{A}$ and $\mathcal{B}$ are countable, we can do a back-and-forth construction to obtain an isomorphism $g : \mathcal{A} \to \mathcal{B}$. Then, $g(H, a) = (H', a')$ and g extends to an automorphism of $\mathbb{M}$, so $t^g(H, a) = t^g(H', a')$, as wanted.

For QM5, suppose $H, H' \subset \mathbb{M}$ are either countable and closed or empty, let $t^g(H) = t^g(H')$, and let $b, b' \in \mathbb{M}$ be such that $t^g(H, b) = t^g(H', b')$ and let $a \in \mathrm{cl}(H, b)$. Again, there is a map f such that $f(H) = H'$, $f(b) = b'$ and f preserves quantifier-free formulae. As in the case of QM4, we may extend f to an isomorphism $g : \mathrm{cl}(Hb) \to \mathrm{cl}(H'b')$. If $a \in \mathrm{cl}(Hb)$, then $t^g(H, b, a) = t^g(H', b', g(a))$.

It's relatively easy to see that the following holds.

Lemma 76. *If $\mathcal{K}$ is a quasiminimal class, then $\mathcal{K}$ is a FUR-class.*

Proof. See [4]. □

It turns out that in a quasiminimal pregeometry structure, the operator bcl gives a pregeometry, just like the model theoretic algebraic closure operator in the case of strongly minimal structures. Moreover, ranks can be calculated as pregeometry dimensions (just like Morley rank in the strongly minimal case).

Lemma 77. *Let $\mathbb{M}$ be a monster model for a quasiminimal class, $A \subset \mathbb{M}$. Then, $cl(A) = bcl(A)$.*

Proof. Suppose first $a \in \mathrm{cl}(A)$. Then, $a \in \mathrm{cl}(A_0)$ for some finite $A_0 \subseteq A$. Let a' be such that $t(a/A) = t(a'/A)$. By QM1, $a' \in \mathrm{cl}(A_0)$. By QM3, $\mathrm{cl}(A_0)$ is countable, so $t(a/A)$ only has countably many realizations. Thus, $\mathrm{cl}(A) \subseteq \mathrm{bcl}(A)$.

On the other hand, suppose $a \in \mathrm{bcl}(A)$. By Theorem 73 (xiii), there is some finite $A_0 \subseteq A$ such that $a \in \mathrm{bcl}(A_0)$. We claim $a \in \mathrm{cl}(A_0)$. Suppose not. By QM4, $t(a/\mathrm{cl}(A_0))$ has uncountably many realizations, and thus $a \notin \mathrm{bcl}(\mathrm{cl}(A_0))$, a contradiction.

Thus, $\mathrm{cl}(A) = \mathrm{bcl}(A)$. □

It is now easy to prove that in a quasiminimal pregeometry structure, $U(a/A) = \dim_{\mathrm{bcl}}(a/A)$.

References

[1] Bays, M., Hart, B., Hyttinen, T., Kesälä, M. and Kirby, J.: *Quasiminimal structures and excellence.* Bull. London Math. Soc. (2014) 46 (1): 155-163. doi: 10.1112/blms/bdt076 First published online: October 24, 2013

[2] Jónsson, B.: *Homogeneous universal relational systems.* Mathematica Scandinavica, 8: 137-142, 1960.

[3] Hyttinen, T. and Kangas, K.: *On model theory of covers of algebraically closed fields.* AASF Mathematica 40:2, 2015, 507-533.

[4] Hyttinen, T. and Kangas, K.: *Quasiminimal structures, groups and Zariski-like geometries.* APAL 167 (2016), 457-505.

[5] Kangas, K.: *Finding groups in Zariski-like structures.* Licentiate's Thesis. University of Helsinki, 2014. arXiv:1404.6811

[6] Kangas, K.: *Finding a field in a Zariski-like structure.* Submitted. http://arxiv.org/abs/1502.03225

[7] Kirby, J.: *On quasiminimal excellent classes.* J. Symbolic Logic, 75(2): 551-564, 2010.

[8] Pillay, A.: *Geometric Stability Theory.* Clarendon Press, Oxford, 1996.

[9] Shelah, S.: *Classification of nonelementary classes II, abstract elementary classes.* In Baldwin, J.T. (ed.): *Classification Theory*, Chigaco 1985. Volume 1292 of *Lecture Notes in Mathematics*, 419497. Springer-Verlag, Berlin, 1987.